



Technical Report
RAL-TR-96-025

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March 1996

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ISSN 1358-6254

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Rudiments of Dual Feynman Rules for Yang-Mills Monopoles in Loop Space

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Abstract

Dual Feynman rules for Dirac monopoles in Yang-Mills fields are obtained by the Wu-Yang (1976) criterion in which dynamics result as a consequence of the constraint defining the monopole as a topological obstruction in the field. The usual path-integral approach is adopted, but using loop space variables of the type introduced by Polyakov (1980). An antisymmetric tensor potential $L_{\mu\nu}[\xi|s]$ appears as the Lagrange multiplier for the Wu-Yang constraint which has to be gauge-fixed because of the “magnetic” \tilde{U} -symmetry of the theory. Two sets of ghosts are thus introduced, which subsequently integrate out and decouple. The generating functional is then calculated to order g^0 and expanded in a series in \tilde{g} . It is shown to be expressible in terms of a local “dual potential” $\tilde{A}_\mu(x)$ found earlier, which has the same propagator and the same interaction vertex with the monopole field as those of the ordinary Yang-Mills potential A_μ with a colour charge, indicating thus a certain degree of dual symmetry in the theory. For the abelian case the Feynman rules obtained here are the same as in QED to all orders in g , as expected by dual symmetry.

1 Introduction

It has long been known that monopoles in gauge theories acquire through their definition as topological obstructions in the gauge field an intrinsic interaction with the field. In fact, in an inspiring paper of 1976, Wu and Yang [1] first showed by a beautiful line of argument how the standard (dual) Lorentz equation for a classical point magnetic charge could be derived as a consequence of its definition as a monopole of the Maxwell field. Since electromagnetism is dual symmetric, it follows that the ordinary Lorentz equation for a point electric charge can also be derived by considering the latter as a monopole of the dual Maxwell field. Moreover, it can be seen that this approach for deriving the interactions of monopoles, which we shall henceforth refer to as the Wu-Yang criterion, is in principle not restricted alone to electromagnetism. Indeed, having been supplemented by some technical development necessary for its implementation, the method has since been generalized to monopole charges in nonabelian Yang-Mills theories [2, 3, 4], not only for classical point particles but also for Dirac particles, giving respectively the Wong and the Yang-Mills-Dirac equations or their respective generalized duals as the result. [5, 6, 7]

All this work so far on the Wu-Yang criterion, however, has been restricted to the classical field level. The purpose of the present paper is to begin exploring the dynamics of nonabelian monopoles at the quantum field level as implied by the same Wu-Yang criterion. We shall start by attempting to derive some rudiments of the “dual Feynman rules” in this approach.

One purpose of this exercise is to compare the Feynman rules so derived for (colour) monopoles with those for (colour source) charges of the standard approach. Although it has recently been shown that nonabelian Yang-Mills theory possesses a generalized dual symmetry in which monopoles and sources play exact dual roles [8], so that the dynamics of (colour) charges derived using the Wu-Yang criterion when they are considered as monopoles of the field is the same as that of the usual Yang-Mills dynamics when these charges are considered as sources, this result is again known to hold so far only at the field equation level. On the other hand, the exciting fully quantum investigation program on duality initiated by Seiberg and Witten and extended by many others [9, 10, 11, 12] applies at present strictly only to supersymmetric theories in a framework in which the Wu-Yang criterion plays no role, and is for these reasons not yet very helpful to the questions raised in the present paper. The crucial point is the existence of the dual potential which is guaranteed only by the equation of motion obtained by extremizing the action and thus need no longer hold in the quantum theory when the field variables move off-shell. It is therefore interesting to explore whether this generalized dual symmetry breaks down at the quantum field level and if so in what way. Furthermore, even if the presently known generalized dual symmetry

is eventually seen to apply also at the quantum field level, as seems to us possible, we believe that our investigation here is still likely to prove useful in future for attacking the ultimate problem of both (colour) electric and magnetic charges interacting together with the Yang-Mills field.

Another purpose of this work is mainly of technical interest, namely to examine how Feynman integrals work in loop space. As is well-known, the loop space approach to gauge theory is attractive in that it gives in principle a gauge independent description in terms of physical observables, in contrast to the standard description in terms of the gauge potential $A_\mu(x)$. A grave drawback of the loop space approach, however, is the high degree of redundancy of loop variables which necessitates the imposition on them of an infinite number of constraints to remove this redundancy, making thus the whole approach rather unwieldy. For the problem of nonabelian monopoles, on the other hand, it turns out that it pays for various reasons to work in loop space, and a set of useful tools has been developed for the purpose. [5, 6, 7] In fact, it was only by means of these loop space tools that the results quoted above on nonabelian monopoles at the classical field level have so far been derived. We are therefore keen to investigate how these tools apply to Feynman integrals at the quantum field level, the understanding of which, we think, may contribute towards the future utilization of the loop space technique as a whole.

That the definition of a charge as a topological obstruction in a field should imply already an interaction between the charge and the field is intuitively clear, because the presence of a charge at a point x in space means that the field around that point will have a certain topological configuration. When that point moves, therefore, the field around it will have to re-adapt itself so as to give the same topological configuration around the new point. Hence, it follows that there must be a coupling between the coordinates of the charge and the variables describing the field, or in other words in physical language, an “interaction” between the charge and the field.

The Wu-Yang criterion enframes the above intuitive assertion as follows. One starts with the free action of the field and the particle, which one may write symbolically as:

$$\mathcal{A}^0 = \mathcal{A}_F^0 + \mathcal{A}_M^0, \quad (1.1)$$

where \mathcal{A}_F^0 depends on only the field variables and \mathcal{A}_M^0 on only the particle variables. If the variables are regarded as independent, then the field is completely decoupled from the particle. However, by specifying that the particle is a topological obstruction of the field, one has imposed a constraint on the system in the form of a condition relating the field variables to the particle variables. Hence, for example, if one extremizes the free action (1.1) subject to this constraint, one obtains not free equations any more but equations with interactions between the

particle and the field. Indeed, it was in this way that the Wu-Yang criterion has been shown to lead to the Lorentz-Wong and Dirac-Yang-Mills equations for respectively the classical and Dirac charge. [1, 5, 7]

For the quantum theory, the equations of motion will not be enough. One will need instead to calculate Feynman integrals over the field and particle variables with the exponential of the action (1.1) above as a weight factor. If the variables are regarded as independent and integrated freely with respect to one another, then we have again a free decoupled system, but since the particle and field variables are here related by the constraint specifying that the particle carries a monopole charge, the resulting Feynman integrals will involve interactions between the particle and the field. Our aim in this paper then is just to evaluate some such Feynman integrals to see what sort of interactions will emerge.

Let us now be specific and consider an $su(2)$ Yang-Mills field with a Dirac particle carrying a (colour) magnetic charge. The free action in that case is:¹

$$\mathcal{A}_F^0 = -\frac{1}{16\pi} \int d^4x \text{Tr}\{F_{\mu\nu}(x)F^{\mu\nu}(x)\}, \quad (1.2)$$

with:

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ig[A_\mu(x), A_\nu(x)], \quad (1.3)$$

for the field in terms of the gauge potential $A_\mu(x)$ as variable, and:

$$\mathcal{A}_M^0 = \int d^4x \bar{\psi}(x)(i\partial_\mu \gamma^\mu - m)\psi(x), \quad (1.4)$$

for the particle in terms of the wave function $\psi(x)$ as variable.

In the presence of monopoles, however, $A_\mu(x)$ has to be patched, which makes it rather clumsy to use in this problem. For this reason, it was found convenient in all previous work on the classical theory [5, 6, 7] to employ as field variable instead the Polyakov variable $F_\mu[\xi|s]$ [13] defined as:

$$F_\mu[\xi|s] = \frac{i}{g} \Phi[\xi]^{-1} \delta_\mu(s) \Phi[\xi], \quad (1.5)$$

for:

$$\Phi[\xi] = P_s \exp ig \int_0^{2\pi} ds A_\mu(\xi(s)) \dot{\xi}^\mu(s), \quad (1.6)$$

where $\Phi[\xi]$ is the holonomy element for the loop parametrized by the function ξ of s for $s = 0 \rightarrow 2\pi$ with $\xi(0) = \xi(2\pi) = P_0$, or in other words, maps of the circle into space-time beginning and ending at the fixed reference point P_0 , and

¹Although given explicitly only for $su(2)$, our results are trivially generalizable to all $su(N)$ theories. In our convention for $su(2)$, $B = B^i T_i$, $T_i = \tau_i/2$, $\text{Tr} B = 2 \times \text{sum of diagonal elements}$, so that $\text{Tr}(T_i T_j) = \delta_{ij}$. Our metric is $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

$\delta_\mu(s) = \delta/\delta\xi^\mu(s)$ is the functional derivative with respect to ξ^μ at s . In terms of $F_\mu[\xi|s]$ as variable, the free action of the field now reads as:

$$\mathcal{A}_F^0 = \int \delta\xi ds a_\xi(s) \text{Tr}\{F_\mu[\xi|s] F^\mu[\xi|s]\}, \quad (1.7)$$

where:

$$a_\xi(s) = -\frac{1}{4\pi\bar{N}} \dot{\xi}(s)^{-2}, \quad (1.8)$$

with $\dot{\xi}^\mu(s)$ being the tangent to the loop ξ at s and \bar{N} an (infinite) normalization factor defined as:

$$\bar{N} = \int_0^{2\pi} ds \int \prod_{s' \neq s} d^4\xi(s'), \quad (1.9)$$

and where the integral is to be taken over all parametrized loops² and over all points s on each loop.

The action (1.1), with \mathcal{A}_F^0 as given in (1.7) and \mathcal{A}_M^0 as given in (1.4), is subject to constraints on two counts. First, the variables $F_\mu[\xi|s]$, as already noted, are highly redundant as all loop variables are and have to be constrained so as to remove this redundancy. Second, the stipulation that the particle represented by $\psi(x)$ should correspond to a monopole of the field implies that $\psi(x)$ must be related to the field variable $F_\mu[\xi|s]$ by a topological condition representing this fact. The beauty of the loop space formalism is that both these constraints are contained in the single statement:

$$G_{\mu\nu}[\xi|s] = -4\pi J_{\mu\nu}[\xi|s], \quad (1.10)$$

where:

$$G_{\mu\nu}[\xi|s] = \delta_\nu(s) F_\mu[\xi|s] - \delta_\mu(s) F_\nu[\xi|s] + ig[F_\mu[\xi|s], F_\nu[\xi|s]], \quad (1.11)$$

is the loop space curvature with $F_\mu[\xi|s]$ as connection, and $J_{\mu\nu}[\xi|s]$ is essentially just the (colour) magnetic current carried by $\psi(x)$, only expressed in loop space terms, the explicit form of which will be given later but need not at present bother us.³

That being the case, the Wu-Yang criterion then says that the dynamics of the monopole interacting with the field is already contained in the constraint

²We note that parametrized loops ξ being by definition just functions of s , integrals over ξ are just ordinary functional integrals, which is in fact one reason why we prefer to work with parametrized loops rather than the actual loops in space-time.

³Strictly speaking, to remove completely their redundancy, the variables $F_\mu[\xi|s]$ are required to have vanishing components along the direction of the loop ξ , which “transversality condition” has in principle to be treated as an additional constraint on the system. [5] This constraint is however easily handled though giving added complications. The calculations reported in this paper have actually been done taking full account of transversality but since the result is the same, the arguments are not given here for the sake of a simpler presentation. For details, see ref. [14, 15].

(1.10). Indeed, it was by extremizing the ‘free’ action (1.1) under this constraint (1.10) that in our earlier work the (dual) Yang-Mills equations of motion for the monopole have been derived. To extend now the considerations to the quantum theory, we shall need to evaluate Feynman integrals over the variables $F_\mu[\xi|s]$ and $\psi(x)$, but subject again to the constraint (1.10). Thus, the partition function of the quantum theory would be of the form:

$$Z = \int \delta F \delta \psi \delta \bar{\psi} \exp i \mathcal{A}^0 \prod_{\mu, \nu, [\xi|s]} \delta \{G_{\mu\nu}[\xi|s] + 4\pi J_{\mu\nu}[\xi|s]\}. \quad (1.12)$$

Equivalently, writing the δ -functions representing the constraint as Fourier integrals, we have:

$$Z = \int \delta F \delta \psi \delta \bar{\psi} \delta L \exp i \mathcal{A}, \quad (1.13)$$

with:

$$\mathcal{A} = \mathcal{A}^0 + \text{Tr} \{L^{\mu\nu}[\xi|s] (G_{\mu\nu}[\xi|s] + 4\pi J_{\mu\nu}[\xi|s])\}. \quad (1.14)$$

Since basically the only functional integral we can do is the Gaussian, the standard procedure is to expand into a power series all terms of higher order in the exponent of the integrand and perform the integral power by power in the expansion. We shall follow here the same procedure. However, in contrast to the usual cases met with in quantum field theory, there are in the exponent of the integrand in (1.13) two terms of order higher than the quadratic, namely one coming from the commutator term of $G_{\mu\nu}[\xi|s]$ in (1.11) which is proportional to the Yang-Mills coupling g , and the other coming from $J_{\mu\nu}[\xi|s]$ which is proportional to the colour magnetic charge \tilde{g} . The result of the expansion would thus be a double power series in g and \tilde{g} . In view of the fact that g and \tilde{g} are related by the Dirac quantization condition which means usually that if one is small then the other will be large, we can normally regard such a double series only as a formal and not as a perturbation expansion. Only in certain special circumstances can one see it leading possibly to an approximate perturbative method. For example, for gauge group $SU(N)$, the Dirac condition reads as:

$$g\tilde{g} = 1/2N, \quad (1.15)$$

with an additional factor N compared with the standard Dirac condition for electromagnetism. Thus if the effective gauge symmetry is continually enlarged so that $N \rightarrow \infty$ as energy is increased as some believe it may, then in principle both g and \tilde{g} can be asymptotically small. A case of perhaps more practical interest is quantum chromodynamics with $N = 3$ where for Q ranging from 3 to 100 GeV, phenomenological values quoted for α_s run from about .25 to .115 [16]. This corresponds to g , say, running from about $1/2$ to $1/3$ ⁴ and implies by (1.15) that

⁴Notice that the coupling g occurring in the Dirac condition (1.15) is the so-called unratinalized coupling related to α_s by $g^2 = \alpha_s$, without a factor 4π .

the dual coupling \tilde{g} runs also in the same range, namely from $1/3$ to $1/2$. Thus, if we accept, as is at present generally accepted, that the expansion in g gives a reasonable approximation, then it is not excluded that a parallel expansion in \tilde{g} can also do so. However, as far as this paper is concerned, we treat the double expansion in g and \tilde{g} merely as a formal means of generating Feynman diagrams, the study of which only is our immediate purpose.

The expansion having been made, the evaluation of the remaining Gaussian integrals then proceeds along more or less conventional lines apart from two complications. First, as was shown in an earlier work [7], the theory possesses now an enlarged gauge symmetry, from the original $SU(N)$ doubled to an $SU(N) \times SU(N)$ where the second $SU(N)$ has a parity opposite to that of the first and is associated with the phase of the monopole wave function $\psi(x)$. Under this second $SU(N)$ symmetry the Lagrange multiplier $L_{\mu\nu}[\xi|s]$ occurring in the integral (1.13) transforms as an antisymmetric tensor potential of the Freedman-Townsend type [17] and has thus to be gauge-fixed using the technology given in the literature for such tensor potentials. [18, 19, 20] Second, the field variables $F_\mu[\xi|s]$ and $L_{\mu\nu}[\xi|s]$ being themselves functionals (i.e. functions of the parametrized loops ξ which are functions of s), extra care has to be used in defining functional operations, such as the Fourier transform, of the field quantities. Apart from these complications, the calculations are otherwise fairly straightforward.

In this paper, we have carried the calculation only to order g^0 . Although there is in principle no great difficulty apart from complication to carry some of the calculation to higher orders in g , and we have done so for exploration, the expansion cannot yet be carried out systematically until some basic questions are resolved. Nevertheless, even the simple examples we have calculated are sufficient to demonstrate several interesting facts. First, that it is possible, though unwieldy, to calculate Feynman diagrams in loop space. Secondly, that the Wu-Yang criterion does yield specific rules for evaluating Feynman diagrams of monopoles interacting with the field. Thirdly, that the result so far is dual symmetric to the standard interaction of a colour (electric or source) charge. We are therefore hopeful that these, albeit yet strictly limited, results will give at least a foothold to serve as a base for extending the exploration further.

2 Preliminaries, Gauge-Fixing and Ghosts

We begin by quoting from earlier work the form of the monopole (or colour magnetic) current expressed in loop space terms: [7]

$$J_{\mu\nu}[\xi|s] = \tilde{g} \epsilon_{\mu\nu\rho\sigma} \dot{\xi}^\sigma(s) [\bar{\psi}(\xi(s)) \gamma^\rho T^i \psi(\xi(s))] \Omega_\xi^{-1}(s, 0) \tau_i \Omega_\xi(s, 0), \quad (2.1)$$

which is to be substituted into the topological constraint (1.10) defining the monopole charge at $\xi(s)$. Here,

$$\Omega_\xi(s, 0) = \omega(\xi(s_+))\Phi_\xi(s_+, 0), \quad (2.2)$$

where $\Phi_\xi(s, 0)$ is the parallel phase transport from the reference point P_0 to the point $\xi(s)$, and $\omega(x)$ is a local transformation matrix which rotates from the frame in which the field is measured to the frame in which the “phase” of the monopole is measured. An important point here is the appearance of s_+ in the argument of $\Phi_\xi(s_+, 0)$ which represents $s + \epsilon/2$ with $\epsilon > 0$ where ϵ is taken to zero after the functional differentiation and integration in ξ have been performed. [7, 8] As a result, $\Omega_\xi(s, 0)$ satisfies, for example,

$$\Omega_\xi^{-1}(s, 0) \frac{\delta}{\delta \xi^\mu(s)} \Omega_\xi(s, 0) = -ig F_\mu[\xi|s]. \quad (2.3)$$

The occurrence in $J_{\mu\nu}[\xi|s]$ of the factors $\Omega_\xi(s, 0)$ and its inverse, both depending on the point $\xi(s)$, will make the integrations we have to do rather awkward. For this reason, we prefer to recast the whole problem in terms of a new set of rotated, “hatted” variables:

$$\hat{F}_\mu[\xi|s] = \Phi[\xi] F_\mu[\xi|s] \Phi^{-1}[\xi], \quad (2.4)$$

and

$$\hat{L}_{\mu\nu}[\xi|s] = \Phi[\xi] L_{\mu\nu}[\xi|s] \Phi^{-1}[\xi]. \quad (2.5)$$

Note that these hatted variables no longer depend on the early part of ξ from $s' = 0$ to $s' = s$ as the original variables $F_\mu[\xi|s]$ and $L_{\mu\nu}[\xi|s]$ do, but rather on the later part of the loop with $s_- \leq s' \leq 2\pi$, where $s_- = s - \epsilon/2$, with ϵ being a positive infinitesimal quantity. This can be seen by observing that $F_\mu = \frac{i}{g} \Phi^{-1} \delta_\mu \Phi$ and therefore $\hat{F}_\mu = \frac{i}{g} (\delta_\mu \Phi) \Phi^{-1}$, which is a function of $\xi(s')$ with $s_- \leq s' \leq 2\pi$. In terms of the hatted variables, we have for (1.14):

$$\begin{aligned} \hat{\mathcal{A}} = & \int \delta\xi ds a_\xi \text{Tr} \left\{ \hat{F}_\mu \hat{F}^\mu \right\} + \int d^4x \bar{\psi} (i\partial_\mu \gamma^\mu - m) \psi \\ & + \int \delta\xi ds \text{Tr} \left\{ \hat{L}_{\mu\nu} \left[\delta^\nu \hat{F}^\mu - \delta^\mu \hat{F}^\nu - ig \left[\hat{F}^\mu, \hat{F}^\nu \right] + 4\pi \hat{J}^{\mu\nu} \right] \right\}, \end{aligned} \quad (2.6)$$

where $\hat{J}_{\mu\nu}[\xi|s]$ differs from $J_{\mu\nu}[\xi|s]$ only by having $\Omega_\xi(s, 0)$ in (2.1) replaced by

$$\hat{\Omega}_\xi(s, 0) = \Omega_\xi(s, 0) \Phi^{-1}[\xi], \quad (2.7)$$

which is independent of the early part of the loop up to and including the point $\xi(s)$ since by (2.3), for $s' < s_+$:

$$\begin{aligned} \delta_\mu(s') \hat{\Omega}_\xi(s, 0) &= \delta_\mu(s') \left[\omega(\xi(s_+)) \Phi_\xi(s_+, 0) \Phi^{-1}[\xi] \right] \\ &= \omega(\xi(s_+)) \delta_\mu(s') [\Phi_\xi(2\pi, s_+)] = 0. \end{aligned} \quad (2.8)$$

As we shall see, this property of $\hat{\Omega}_\xi$ in $\hat{J}_{\mu\nu}[\xi|s]$ will make our task in evaluating Feynman integrals much easier.

In terms of the hatted variables, the partition function Z appears now as:⁵

$$\hat{Z} = \int \delta\hat{F} \delta\hat{L} \delta\psi \delta\bar{\psi} \exp i\hat{\mathcal{A}}. \quad (2.9)$$

Since we shall be working exclusively with these hatted variables from now on, we shall henceforth drop the “hat” in our notation, assuming it now to be understood. We shall also suppress the arguments of the field variables unless this should lead to ambiguities.

We shall try now to evaluate the integral (2.9) to order 0 in g starting with the integral in F . To this order, \mathcal{A} is quadratic in F so that the integral in F is Gaussian and can be evaluated just by completing squares. This brings about a term of the form $\delta_\alpha L^{\mu\alpha} \delta^\rho L_{\mu\rho}$ in the exponent of the resulting integrand which is thus again quadratic in the variable $L_{\mu\nu}$. To evaluate next the integral in L , we encounter a problem in completing the square for $L_{\mu\nu}$, due to the noninvertibility of the projection operator involved in the quadratic term. This is a reflection of the fact that in L there is a gauge redundancy. Although $L_{\mu\nu}$, like the Polyakov variable F_μ , is by construction gauge invariant (apart from an unimportant x -independent gauge rotation at the reference point P_0) under the original Yang-Mills gauge transformation, there is another gauge symmetry of the theory [7] under which $L_{\mu\nu}$ transforms like an antisymmetric tensor potential of the Freedman-Townsend type [17]. Thus, in order to complete the square for $L_{\mu\nu}$ and integrate this field out, we need to impose a gauge-fixing condition on $L_{\mu\nu}$ by taking advantage of this new \tilde{U} -symmetry of the theory.

We propose then to impose on $L_{\mu\nu}$ the following gauge condition: [14, 20]

$$C_\mu(L) = \epsilon_{\mu\nu\rho\sigma} \delta^\nu(s) L^{\rho\sigma} [\xi|s] = 0. \quad (2.10)$$

In this gauge, which can be shown to be always possible [14], the transverse degrees of freedom of $L_{\mu\nu}$ do not propagate and only the longitudinal ones are physical. Following the standard procedures, we introduce then the suppression factor C^2 and the Faddeev-Popov determinant Δ_1 as follows:

$$C^2 = \int \delta\xi \, ds \, \frac{1}{2\alpha(s)} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\alpha\beta\gamma} \text{Tr} [\delta^\nu L^{\rho\sigma} \delta_\alpha L_{\beta\gamma}], \quad (2.11)$$

⁵There can in principle be a Jacobian of transformation in the integral depending on which variable one chooses originally to quantize in, whether $\hat{F}_\mu[\xi|s]$, $F_\mu[\xi|s]$ or $A_\mu(x)$. However, working to order g^0 as we do here we need not bother, the Jacobian between any pair of these variables being then just a constant factor. To higher orders, it will matter. In fact our inability as yet to handle the Jacobian is one main reason preventing us from going to higher orders in g at present.

and:

$$\Delta_1 = \int \delta\eta \delta\bar{\eta} \exp i \int \delta\xi ds Tr \left\{ \bar{\eta}_\mu[\xi|s] \left(\frac{\delta C'^\mu(L^\Lambda)}{\delta \Lambda^\nu} \right) \eta^\nu[\xi|s] \right\}, \quad (2.12)$$

where η and $\bar{\eta}$ are two independent vector-valued Grassmann variables depending on the later part of the loop, and $C'^\mu(L^\Lambda)$ is obtained by applying to (2.10) a \tilde{U} -transformation with gauge parameter $\Lambda_\beta[\xi|s]$ [7, 14], thus:

$$C'_\mu(L^\Lambda) = C_\mu(L) + \epsilon_{\mu\nu\rho\sigma} \delta^\nu(s) \tilde{\Delta} L^{\rho\sigma}[\xi|s], \quad (2.13)$$

for $\tilde{\Delta} L^{\rho\sigma} = \epsilon^{\rho\sigma\alpha\beta} \delta_\alpha \Lambda_\beta$. The path-integral in (2.9) then becomes:

$$\begin{aligned} Z = & \int \delta L \delta F \delta\eta \delta\bar{\eta} \delta\psi \delta\bar{\psi} \exp i \int d^4x \bar{\psi} (i\partial_\mu \gamma^\mu - m) \psi \\ & \exp i \int \delta\xi ds Tr \{ a_\xi F_\mu F^\mu + L_{\mu\nu} [\delta^\nu F^\mu - \delta^\mu F^\nu - ig [F^\mu, F^\nu] + 4\pi J^{\mu\nu}] \\ & + (2\alpha)^{-1} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\alpha\beta\gamma} \delta^\nu L^{\rho\sigma} \delta_\alpha L_{\beta\gamma} + 2\bar{\eta}_\mu (g^{\mu\nu} \square_\xi - \delta^\mu \delta^\nu) \eta_\nu \}, \end{aligned} \quad (2.14)$$

where \square_ξ denotes:

$$\square_\xi(s) = \frac{\delta^2}{\delta\xi^\mu(s) \delta\xi_\mu(s)}. \quad (2.15)$$

Here, $\alpha(s) = 2a_\xi(s)$ is chosen such that the gauge-fixing term for $L_{\mu\nu}$ cancels the term $\delta_\alpha L^{\mu\alpha} \delta^\rho L_{\mu\rho}$ brought about by completing the square for F_μ .

In (2.14), we see again the appearance of a non-invertible operator $g^{\mu\nu} \square_\xi - \delta^\mu \delta^\nu$. In order to integrate out the fields η and $\bar{\eta}$ and eliminate the off-diagonal term $\bar{\eta}_\mu \delta^\mu \delta^\nu \eta_\nu$, we must fix the gauge for a second time, by finding a second gauge-symmetry of the action. This is accomplished by writing the last term in (2.14) as:

$$\int \delta\xi ds Tr [\bar{\eta}_\mu (g^{\mu\nu} \delta_\rho \delta^\rho - \delta^\mu \delta^\nu) \eta_\nu] = - \int \delta\xi ds Tr [(\delta_\nu \bar{\eta}_\mu - \delta_\mu \bar{\eta}_\nu) (\delta^\nu \eta^\mu - \delta^\mu \eta^\nu)], \quad (2.16)$$

which we notice is invariant under:

$$\eta^\mu[\xi|s] \rightarrow \eta^\mu[\xi|s] + \delta^\mu \lambda[\xi|s]. \quad (2.17)$$

This symmetry allows us to “fix the gauge” for η by choosing λ such that:

$$\delta_\mu \eta'^\mu = \delta_\mu \eta^\mu + \square_\xi \lambda = 0, \quad (2.18)$$

which is always possible, given initial conditions for λ .

Including the suppression factor:

$$D^2 = \int \delta\xi ds \frac{1}{2\beta} Tr [(\delta_\mu \bar{\eta}^\mu) (\delta_\nu \eta^\nu)], \quad (2.19)$$

as well as the Faddeev-Popov determinant:

$$\Delta_2 = \int \prod_{i=1,2} \delta\phi_i \delta\bar{\phi}_i \exp i \int \delta\xi ds \frac{1}{2\beta} \text{Tr} \sum_{i=1,2} \{ \bar{\phi}_i \square_\xi \phi_i \}, \quad (2.20)$$

in (2.14) yields:

$$\begin{aligned} Z = & \int \delta L \delta F \delta \eta \delta \bar{\eta} \delta \psi \delta \bar{\psi} \prod_{i=1,2} \delta\phi_i \delta\bar{\phi}_i \exp i \int d^4x \bar{\psi} (i\partial_\mu \gamma^\mu - m) \psi \\ & \exp i \int \delta\xi ds \text{Tr} \{ a_\xi F_\mu F^\mu + 2 \bar{\eta}_\mu \square_\xi \eta^\mu \\ & + L_{\mu\nu} [\delta^\nu F^\mu - \delta^\mu F^\nu - ig [F^\mu, F^\nu] + 4\pi J^{\mu\nu}] - \frac{1}{4} \sum_{i=1,2} \bar{\phi}_i \square_\xi \phi_i \\ & + (4a_\xi)^{-1} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\alpha\beta\gamma} \delta^\nu L^{\rho\sigma} \delta_\alpha L_{\beta\gamma} \}. \end{aligned} \quad (2.21)$$

where $\phi_i[\xi|s]$ and $\bar{\phi}_i[\xi|s]$ for $i = 1, 2$, are four independent commuting fields depending on the later part of the loop ξ [18, 19], and we have chosen $\beta = -1/2$, in order for the second gauge-fixing condition to cancel the off-diagonal term in $\bar{\eta}_\mu \delta^\mu \delta^\nu \eta_\nu$.

The ghosts η , $\bar{\eta}$, ϕ_i and $\bar{\phi}_i$ can easily be integrated out and decouple from the theory. The effective action after integrating out the ghost fields is:

$$\begin{aligned} \mathcal{A}_{eff} = & \int \delta\xi ds \text{Tr} \{ a_\xi F_\mu F^\mu + 2 \bar{\psi} (i\partial_\mu \gamma^\mu - m) \psi \\ & + L_{\mu\nu} [\delta^\nu F^\mu - \delta^\mu F^\nu - ig [F^\mu, F^\nu] + 4\pi J^{\mu\nu}] \\ & + (4a_\xi)^{-1} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\alpha\beta\gamma} \delta^\nu L^{\rho\sigma} \delta_\alpha L_{\beta\gamma} \}. \end{aligned} \quad (2.22)$$

The decoupling of the ghosts can be explained as follows: although the theory itself is nonabelian in character, the \tilde{U} -transformation on $L_{\mu\nu}$ does not involve a term coupling Λ_μ to $L_{\mu\nu}$ [7], in contrast to the usual Yang-Mills “ U ”-transformation on the gauge potential $A_\mu(x)$. In the following, we shall assume that the ghost fields have all been integrated out, and drop henceforth the subscript *eff* to \mathcal{A} from our notation.

3 Generating Functional, Propagators, and Vertex

To manage the perturbation expansion we shall adopt the usual generating functional method. One starts by considering in the action only the “free field” terms of order 2 and lower in the fields, denoted generically say by H , and ignoring all “interaction” terms of higher order. External current terms are then added to this action. On integrating out the fields H by completing squares one obtains

the free-field generating functional $Z^{(2)}[\mathcal{J}]$ which depends on the external current \mathcal{J} only. The propagator for any field H say, will then be given by the expression:

$$\langle H(x)H(y) \rangle = \frac{1}{i} \frac{\delta}{\delta \mathcal{J}(x)} \frac{1}{i} \frac{\delta}{\delta \mathcal{J}(y)} Z^{(2)}[\mathcal{J}]|_{\mathcal{J}=0}, \quad (3.1)$$

where \mathcal{J} here denotes the external current which corresponds to H . Next, collecting the higher order “interaction” terms of the action, say, $\mathcal{A}_I[H]$, one can write the full generating functional formally as:

$$Z[\mathcal{J}] = \exp [i\mathcal{A}_I[-i\delta/\delta \mathcal{J}]] Z^{(2)}[\mathcal{J}]. \quad (3.2)$$

Any term in the perturbation expansion can then be obtained by taking the appropriate derivative of $Z[\mathcal{J}]$ with respect to \mathcal{J} .

In our problem here formulated in loop space, the terms of the action \mathcal{A} in (1.14) which are second order or lower in the fields ψ , F_μ and $L_{\mu\nu}$ do not correspond to just the free action \mathcal{A}^0 of (1.1), and the concept of interaction has been replaced by that of a constraint imposed through the Wu-Yang criterion. Nevertheless, the method can still be applied. Writing then the action (1.14) to second order in the fields, including external current terms, we have:

$$\begin{aligned} \mathcal{A}^{(2)}[\mathcal{J}] &= \int \delta\xi ds \left\{ a_\xi F_\mu^i F_i^\mu + L_{\mu\nu}^i (\delta^\nu F_i^\mu - \delta^\mu F_i^\nu) + \mathcal{J}_i^{\mu\nu} L_{\mu\nu}^i + \mathcal{J}_\mu^i F_i^\mu \right. \\ &\quad \left. + (2\alpha)^{-1} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\alpha\beta\gamma} \delta^\nu L_i^{\rho\sigma} \delta_\alpha L_{\beta\gamma}^i \right\} + \int d^4x \left[(\bar{\mathcal{J}}\psi + \bar{\psi}\mathcal{J}) + \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi \right] \\ &= \mathcal{A}_F^{(2)} + \mathcal{A}_M^{(2)}. \end{aligned} \quad (3.3)$$

The generating functional factors into a gauge term and a matter term where the matter term is the same as in ordinary local formulations. For the gauge term written in loop space:

$$Z_F^{(2)} = \int \delta L \delta F \exp i\mathcal{A}_F^{(2)}, \quad (3.4)$$

after completing the squares for F_μ and $L_{\mu\nu}$ and integrating them out, we obtain, up to a multiplicative factor:

$$\begin{aligned} Z_F^{(2)}[\mathcal{J}] &= \exp -i \int \delta\xi ds \frac{1}{4a_\xi} \left[2\Box_\xi^{-1} (\delta^\nu \mathcal{J}_i^\mu - a_\xi \mathcal{J}_i^{\mu\nu})^2 + \mathcal{J}_\mu^i \mathcal{J}_i^\mu \right] \\ &= \exp -i \int \delta\xi ds \left[\frac{1}{2a_\xi} \Box_\xi^{-1} \delta^\nu \mathcal{J}_i^\mu \delta_\nu \mathcal{J}_\mu^i - \Box_\xi^{-1} \delta^\nu \mathcal{J}_i^\mu \mathcal{J}_{\mu\nu}^i \right. \\ &\quad \left. + \frac{a_\xi}{2} \Box_\xi^{-1} \mathcal{J}_i^{\mu\nu} \mathcal{J}_{\mu\nu}^i + \frac{1}{4a_\xi} \mathcal{J}_\mu^i \mathcal{J}_i^\mu \right]. \end{aligned} \quad (3.5)$$

Differentiating functionally with respect to the currents yields for the propagators:

$$\langle F_\mu^i[\xi|s] F_{\mu'}^{i'}[\xi'|s'] \rangle = -\frac{3i}{4a_{\xi'}(s')} \delta^{ii'} g_{\mu\mu'} \delta(s-s') \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})), \quad (3.6)$$

and:

$$\langle L_{\mu\nu}^i[\xi|s] L_{\mu'\nu'}^{i'}[\xi'|s'] \rangle = \frac{i a_{\xi'}(s')}{2} \delta^{ii'} \delta(s-s') \square_{\xi'}^{-1}(s') \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})) (g_{\mu\mu'} g_{\nu\nu'} - g_{\mu\nu'} g_{\nu\mu'}). \quad (3.7)$$

Note that for the functional differentiation above, we have used:

$$\begin{aligned} \frac{\delta X_{\mu}^i[\xi|s]}{\delta X_j^{\alpha}[\xi'|s']} &= \delta^{ij} g_{\mu\alpha} \delta(s-s') \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})), \\ \frac{\delta Y_{\mu\nu}^i[\xi|s]}{\delta Y_j^{\alpha\beta}[\xi'|s']} &= \frac{1}{2} \delta^{ij} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) \delta(s-s') \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})). \end{aligned} \quad (3.8)$$

To order g^0 , the commutator term in the loop space curvature $G_{\mu\nu}$ can be dropped so that there remains only one term in the action \mathcal{A} of (1.14) which is of higher than second order, namely the term coming from the current $J_{\mu\nu}$ in the constraint:

$$\mathcal{A}^{(3)} = 4\pi\tilde{g} \int \delta\xi ds \epsilon_{\mu\nu\sigma\rho} \dot{\xi}^{\sigma}(s) \bar{\psi}(\xi(s)) \Omega_{\xi}(s, 0) T^i \Omega_{\xi}^{-1}(s, 0) \psi(\xi(s)) \gamma^{\rho} L_i^{\mu\nu}[\xi|s]. \quad (3.9)$$

This can be substituted as \mathcal{A}_I in (3.2) to construct the generating functional $Z[\mathcal{J}]$. Since the “interaction” only involves the fields ψ and $L_{\mu\nu}$ and not F_{μ} , we can put \mathcal{J}_{μ}^i , the external current for F_{μ} , equal to zero in (3.5), keeping only the remaining relevant terms. If we denote the propagator of ψ by $S_F(x-y)$ and the propagator (3.7) for $L_{\mu\nu}$ by $\Delta_{\mu\nu, \mu'\nu'}[\xi, \xi'|s, s']$, the free field generating functional up to factors can then be written as:

$$\begin{aligned} Z^{(2)}[\mathcal{J}] &= \exp -i \int d^4x d^4y \bar{\mathcal{J}}(x) S_F(x-y) \mathcal{J}(y) \\ &\quad \exp -\frac{i}{2} \int \delta\xi \delta\xi' ds ds' \mathcal{J}_j^{\mu\nu}[\xi|s] \Delta_{\mu\nu, \mu'\nu'}^{jj'}[\xi, \xi'|s, s'] \mathcal{J}_{j'}^{\mu'\nu'}[\xi'|s']. \end{aligned} \quad (3.10)$$

Applying the operation as indicated in (3.2) to this will give us the full generating functional we want.

For example, suppose we are interested in the “interaction vertex”, we expand (3.2) to first order in \tilde{g} , obtaining after a straightforward calculation, up to numerical factors:

$$\begin{aligned} Z[\mathcal{J}] &= \left[1 + 4\pi i \tilde{g} \int \delta\xi ds \epsilon_{\mu\nu\sigma\rho} \Omega_{\xi}^{kl}(s, 0) T_{lm}^i \Omega_{\xi}^{-1 mn}(s, 0) \gamma^{\rho} \dot{\xi}^{\sigma}(s) \right. \\ &\quad + \int d^4y S_F^{\mu\nu}(\xi(s) - y) \mathcal{J}_l(y) \int d^4x \bar{\mathcal{J}}_j(x) S_F^{jk}(x - \xi(s)) \\ &\quad \left. \int \delta\xi' ds' \Delta_{\mu\nu, \mu'\nu'}^{ii'}[\xi, \xi'|s, s'] \mathcal{J}_{i'}^{\mu'\nu'}[\xi'|s'] \right] Z^{(2)}[\mathcal{J}], \end{aligned} \quad (3.11)$$

where we have dropped the vacuum term $S_F(0)$ and, to avoid confusion, we have written out explicitly the internal symmetry indices. Differentiating with respect

to the appropriate currents then yields:

$$\begin{aligned}
& (-i)^3 \frac{\delta}{\delta \mathcal{J}_{\mu\nu}^l[\xi'|s']} \frac{\delta}{\delta \mathcal{J}^m(x_2)} \frac{\delta}{\delta \bar{\mathcal{J}}^i(x_1)} Z[\mathcal{J}] \Big|_{\mathcal{J}=0} \\
&= -4\pi\bar{g} \int \delta\xi ds S_F^{ik}(x_1 - \xi(s)) \Omega_\xi^{kl'}(s, 0) T_{l'm'}^j \Omega_\xi^{-1 m'n}(s, 0) S_F^{nm}(\xi(s) - x_2) \\
&\quad \epsilon^{\mu'\nu'\rho\sigma} \gamma_\rho \dot{\xi}_\sigma(s) \Delta_{\mu\nu, \mu'\nu'}^{lj}[\xi, \xi'|s, s'], \tag{3.12}
\end{aligned}$$

where the vertex is obtained by eliminating the propagators from the external lines.

4 The dual potential $\tilde{A}_\mu(x)$

Before we proceed to work out explicitly the loop space formulae for the interaction vertex and generating functional, we shall first introduce a quantity $\tilde{A}_\mu(x)$ found in an earlier paper [7] which will considerably simplify our task:

$$\tilde{A}_\mu(x) = 4\pi \int \delta\xi ds \epsilon_{\mu\nu\rho\sigma} \Omega_\xi(s, 0) T^i \Omega_\xi^{-1}(s, 0) L_i^{\rho\sigma}[\xi|s] \dot{\xi}^\nu(s) \delta^4(x - \xi(s)). \tag{4.1}$$

In the classical theory, it is now known [8] that this quantity $\tilde{A}_\mu(x)$ plays an exactly dual role to the ordinary Yang-Mills potential $A_\mu(x)$, acting as the parallel phase transport for the monopole wave function and giving a complete description of the dual field. This last statement by itself does not necessarily imply that $\tilde{A}_\mu(x)$ will play the same role also in the quantum field theory but, as we shall see, it turns out to do so, at least to order g^0 .

We note first that the gauge fixing condition that we have imposed on the loop variable $L_{\mu\nu}$ is in fact equivalent to the standard Lorentz condition on the local quantity $\tilde{A}_\mu(x)$. This can be seen as follows. Differentiating $\tilde{A}_\mu(x)$ in (4.1) with respect to x and then integrating by parts with respect to ξ with the help of $\delta^4(x - \xi(s))$, we obtain:

$$\partial^\mu \tilde{A}_\mu(x) = -4\pi \int \delta\xi ds \epsilon_{\mu\nu\rho\sigma} \delta^\mu(s) \left[\Omega_\xi(s, 0) L^{\rho\sigma}[\xi|s] \Omega_\xi^{-1}(s, 0) \right] \dot{\xi}^\nu(s) \delta^4(x - \xi(s)). \tag{4.2}$$

Recall now the fact that we are working with what we called “hatted” variables so that by (2.8) the derivative $\delta^\mu(s)$ above commutes with $\Omega_\xi(s, 0)$ and acts only on $L^{\rho\sigma}[\xi|s]$. We see then that the gauge-fixing condition (2.10) that we have imposed on $L_{\mu\nu}[\xi|s]$ is indeed equivalent to the Lorentz condition:

$$\partial^\mu \tilde{A}_\mu(x) = 0. \tag{4.3}$$

Secondly, we note that to order g^0 and in the absence of its interactions with ψ , $\tilde{A}_\mu(x)$ satisfies the free field equation:

$$\square \tilde{A}_\mu(x) = 0, \tag{4.4}$$

which allows for its expansion into the usual plane wave creation/annihilation operators. The loop-space curvature $G_{\mu\nu} = 0$ in the absence of the interaction term. To zeroth order in g , the curvature is given by $G_{\mu\nu} = \delta_\mu F_\nu - \delta_\nu F_\mu$. The equation $F_\mu = a_\xi^{-1} \delta^\nu L_{\mu\nu}$ also holds, since to zeroth order in g , the covariant loop space derivative is the same as the ordinary loop space derivative. For the same reason, the gauge fixing condition implies $\epsilon_{\mu\nu\rho\sigma} \delta^\nu L^{\rho\sigma} = 0$. Inserting F_μ and using this expression we obtain:

$$\square_\xi L_{\mu\nu} = 0. \quad (4.5)$$

On the other hand:

$$\square \tilde{A}_\mu(x) = 4\pi \int \delta\xi ds \epsilon_{\mu\nu\rho\sigma} \square_\xi \left[\Omega_\xi(s, 0) L^{\rho\sigma}[\xi|s] \Omega_\xi^{-1}(s, 0) \right] \dot{\xi}^\nu(s) \delta^4(x - \xi(s)). \quad (4.6)$$

For the same reasons as before, the derivatives could be taken inside the bracket to act on $L_{\rho\sigma}$ only. Using (4.5) we then obtain (4.4) as required.

Finally, we show that (3.2) can in fact be expressed in terms of the dual potential $\tilde{A}_\mu(x)$ and a corresponding local current $\tilde{j}^\mu(x)$ instead of the L -field and its corresponding current in loop space. We note first that the “interaction” \mathcal{A}_I in (3.2) or, in other words, $\mathcal{A}^{(3)}$ of (3.9), can be rewritten as:

$$\mathcal{A}^{(3)} = \tilde{g} \int d^4x \bar{\psi}(x) \tilde{A}_\mu(x) \gamma^\mu \psi(x), \quad (4.7)$$

for $\tilde{A}_\mu(x)$ as given in (4.1), so that it is a function of $L_{\mu\nu}$ only in that particular combination. We need therefore introduce a current really only for this combination of $L_{\mu\nu}$, namely a local current $\tilde{j}^\mu(x)$ corresponding to $\tilde{A}_\mu(x)$, by incorporating in the action a term of the form:

$$\int d^4x \tilde{A}_i^\mu(x) \tilde{j}_\mu^i(x). \quad (4.8)$$

This can be rewritten in the original form given in (3.3):

$$\int \delta\xi ds L_{\rho\sigma}^i[\xi|s] \mathcal{J}_i^{\rho\sigma}[\xi|s], \quad (4.9)$$

provided that $\mathcal{J}_i^{\mu\nu}[\xi|s]$ is of the special form:

$$\mathcal{J}_i^{\mu\nu}[\xi|s] = 4\pi \int d^4x \epsilon^{\mu\nu\rho\sigma} \left[\Omega_\xi(s, 0) T_i \Omega_\xi^{-1}(s, 0) \right]^j \dot{\xi}_\sigma(s) \tilde{j}_\rho^j(x) \delta^4(x - \xi(s)). \quad (4.10)$$

Substituting this $\mathcal{J}_i^{\mu\nu}[\xi|s]$ into $Z^{(2)}[\mathcal{J}]$ of (3.10) and $Z[\mathcal{J}]$ of (3.11), one easily obtains that up to numerical factors:

$$Z^{(2)}[\mathcal{J}] = \exp -\frac{i}{2} \int d^4x d^4y \left[\bar{\mathcal{J}}(x) S_F(x - y) \mathcal{J}(y) + 2 \tilde{j}_j^\mu(x) \left\langle \tilde{A}_\mu^j(x) \tilde{A}_{\mu'}^{j'}(y) \right\rangle \tilde{j}_{j'}^{\mu'}(y) \right], \quad (4.11)$$

and that:

$$Z[\mathcal{J}] = \left[1 + i\tilde{g} \int d^4x d^4y d^4z d^4w \bar{\mathcal{J}}^i(x) S_F^{ik}(x-w) T_j^{kn} S_F^{ni}(w-y) \mathcal{J}^{i'}(y) \gamma^\rho \langle \tilde{A}_\rho^j(x) \tilde{A}_{\rho'}^{j'}(z) \rangle \tilde{j}_{j'}^{\rho'}(z) \right] Z^{(2)}[\mathcal{J}], \quad (4.12)$$

with:

$$\begin{aligned} \langle \tilde{A}_\mu^i(x) \tilde{A}_{\mu'}^{i'}(x') \rangle &= 16\pi^2 \int \delta\xi \delta\xi' ds ds' \epsilon_{\mu\nu\rho\sigma} \epsilon_{\mu'\nu'\rho'\sigma'} \dot{\xi}^\nu(s) \dot{\xi}'^{\nu'}(s') \\ &\quad \left[\Omega_\xi(s, 0) T^j \Omega_\xi^{-1}(s, 0) \right]^i \left[\Omega_{\xi'}(s', 0) T^{j'} \Omega_{\xi'}^{-1}(s', 0) \right]^{i'} \\ &\quad \Delta_{jj'}^{\rho\sigma, \rho'\sigma'}[\xi, \xi'|s, s'] \delta^4(x - \xi(s)) \delta^4(x' - \xi'(s')). \end{aligned} \quad (4.13)$$

The formulae (4.11) and (4.12) are formally the same as those in standard Yang-Mills theory. Indeed, if the quantity $\langle \tilde{A}_\mu^i(x) \tilde{A}_{\mu'}^{i'}(x') \rangle$ can be identified with the standard propagator of the gauge potential in Yang-Mills theory, then, apart from the gauge-boson self-interaction which has been dropped in working only to order g^0 , one would obtain exactly the same perturbation series in \tilde{g} here as one does in g in ordinary Yang-Mills theory. At present, however, $\langle \tilde{A}_\mu^i(x) \tilde{A}_{\mu'}^{i'}(x') \rangle$ is still given in (4.13) as a complicated integral in loop space. That this integral is in fact the same as the standard propagator of the Yang-Mills potential is the subject of the next section.

5 Loop Space Fourier Transform and the Propagator for \tilde{A}_μ

Inserting the expression for the L -field propagator from (3.7) into (4.13), we have:

$$\begin{aligned} \langle \tilde{A}_\mu^i(x) \tilde{A}_{\mu'}^{i'}(x') \rangle T^i T^{i'} &= 8\pi^2 \int \delta\xi \delta\xi' ds \epsilon_{\mu\nu\rho\sigma} \epsilon_{\mu'\nu'\rho'\sigma'} a_{\xi'}(s) \delta^{jj'} \dot{\xi}^\nu(s) \dot{\xi}'^{\nu'}(s) \\ &\quad \left[\Omega_\xi(s, 0) T^j \Omega_\xi^{-1}(s, 0) \right] (g^{\rho\rho'} g^{\sigma\sigma'} - g^{\rho\sigma'} g^{\sigma\rho'}) \square_{\xi'}^{-1}(s) \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})) \\ &\quad \left[\Omega_{\xi'}(s, 0) T^{j'} \Omega_{\xi'}^{-1}(s, 0) \right] \delta^4(x - \xi(s)) \delta^4(x' - \xi'(s)), \end{aligned} \quad (5.1)$$

where we have performed the s' integration already. This simplifies to:

$$\begin{aligned} \langle \tilde{A}_\mu^i(x) \tilde{A}_{\mu'}^{i'}(x') \rangle T^i T^{i'} &= -16\pi^2 \int \delta\xi \delta\xi' ds a_{\xi'}(s) \dot{\xi}^\nu(s) \dot{\xi}'^{\nu'}(s) \left[\Omega_\xi(s, 0) T^j \Omega_\xi^{-1}(s, 0) \right] \\ &\quad G_{\mu\nu\mu'\nu'} \square_{\xi'}^{-1}(s) \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})) \left[\Omega_{\xi'}(s, 0) T^{j'} \Omega_{\xi'}^{-1}(s, 0) \right] \\ &\quad \delta^{jj'} \delta^4(x - \xi(s)) \delta^4(x' - \xi'(s)), \end{aligned} \quad (5.2)$$

where we have used the abbreviation:

$$G_{\mu\nu\mu'\nu'} = (g_{\mu\mu'}g_{\nu\nu'} - g_{\mu\nu}g_{\nu\mu'}). \quad (5.3)$$

Using the bra-ket notation of Dirac, we now define:

$$\langle x | \Gamma_j^\nu | \xi \rangle = 4\pi i \dot{\xi}^\nu(s) \left[\Omega_\xi(s, 0) T^j \Omega_\xi^{-1}(s, 0) \right] \delta^4(x - \xi(s)), \quad (5.4)$$

$$\langle \xi | \Delta_{\mu\nu\mu'\nu'}^{jj'} | \xi' \rangle = a_{\xi'}(s) G_{\mu\nu\mu'\nu'} \delta^{jj'} \square_{\xi'}^{-1}(s) \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})), \quad (5.5)$$

and write:

$$\langle \tilde{A}_\mu^i(x) \tilde{A}_{\mu'}^{i'}(x') \rangle T^i T^{i'} = \int ds \langle x | \Omega_{\mu\mu'} | x' \rangle, \quad (5.6)$$

with:

$$\Omega_{\mu\mu'} = \Gamma_j^\nu \Delta_{\mu\nu\mu'\nu'}^{jj'} \Gamma_{j'}^{\nu'}. \quad (5.7)$$

Our aim now is to transform this propagator to momentum space so as to compare with the standard propagator of the Yang-Mills potential. Although this propagator is itself an ordinary space-time quantity for which the Fourier transform is well-defined, it is expressed in terms of loop quantities the Fourier transformation of which will require some care. First, if in analogy to $\langle x | p \rangle = \exp(-ipx)$ in ordinary space-time, we define in loop space:

$$\langle \xi | \pi \rangle = \int dt \exp i \xi(t) \pi(t). \quad (5.8)$$

then we can write:

$$\begin{aligned} \langle p | \Gamma_j^\nu | \pi \rangle &= i \int d^4x \delta\xi \langle p | x \rangle \langle x | \Gamma_j^\nu | \xi \rangle \langle \xi | \pi \rangle \\ &= i \int d\xi e^{ip\xi(s)} 4\pi \dot{\xi}^\nu(s) \left[\Omega_\xi T^j \Omega_\xi^{-1} \right] \exp -i \int dt \xi(t) \pi(t), \end{aligned} \quad (5.9)$$

$$\langle \pi' | \Gamma_{j'}^{\nu'} | p' \rangle = i \int d\xi' e^{-ip'\xi'(s)} 4\pi \dot{\xi}'^{\nu'}(s) \left[\Omega_{\xi'} T^{j'} \Omega_{\xi'}^{-1} \right] \exp i \int dt \xi'(t) \pi'(t). \quad (5.10)$$

and:

$$\begin{aligned} \langle \pi | \Delta_{\mu\nu\mu'\nu'}^{jj'} | \pi' \rangle &= \int d\xi d\xi' a_\xi(s) \left[G_{\mu\nu\mu'\nu'} \delta^{jj'} \square_{\xi'}^{-1}(s) \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})) \right] \\ &\quad \left(\exp i \int_0^{2\pi} dt \pi(t) \xi(t) \right) \left(\exp -i \int_0^{2\pi} dt \pi'(t) \xi'(t) \right). \end{aligned} \quad (5.11)$$

However, if we proceed now to evaluate these quantities, we shall find δ -functional ambiguities connected with the definition of the loop derivative $\delta_\mu(s) = \delta/\delta\xi^\mu(s)$ and the tangent to the loop $\dot{\xi}^\mu(s)$ both of which occur in the formulae above. In other words, some regularization procedure is required in order to give these quantities an unambiguous meaning. Our procedure, which we have followed

throughout our program [5, 7, 8], is to replace first the δ -function $\delta(s-s')$ inherent in the definition of the loop derivative by a bump function $\beta_\epsilon(s-s')$ of width ϵ and the tangent $\dot{\xi}^\mu(s)$ to the loop at s by:

$$\dot{\xi}^\mu(s) = \frac{\xi^\mu(s_+) - \xi^\mu(s_-)}{s_+ - s_-}, \quad (5.12)$$

for $s_\pm = s \pm \epsilon/2$, and then take the limit $\epsilon \rightarrow 0$ after the required operations have been performed. We propose to follow the same procedure here.

With these provisos we return to the evaluation of (5.11). The box-operator there acts on the δ -function, but by integrating by parts, it can be made to act on the exponential function and the above expression becomes:

$$\begin{aligned} \langle \pi | \Delta_{\mu\mu'\nu\nu'}^{jj'} | \pi' \rangle &= \int \delta\xi \delta\xi' G_{\mu\nu\mu'\nu'} \delta^{jj'} \prod_{\bar{s}=0}^{2\pi} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})) \exp i \int_0^{2\pi} dt \pi(t) \xi(t) \\ &\quad \left\{ \square_{\xi'}^{-1}(s) \exp -i \int_0^{2\pi} dt \pi'(t) \xi'(t) \right\}. \end{aligned} \quad (5.13)$$

Simplifying further we obtain:

$$\langle \pi | \Delta_{\mu\mu'\nu\nu'}^{jj'} | \pi' \rangle = G_{\mu\nu\mu'\nu'} \prod_{\bar{s}=0}^{2\pi} \delta^4(\pi(\bar{s}) - \pi'(\bar{s})) \int_{s_-}^{s_+} ds' ds'' \frac{\beta_\epsilon(s' - s) \beta_\epsilon(s'' - s)}{\pi_\alpha(s') \pi'^\alpha(s'')}, \quad (5.14)$$

Substituting into (5.7) and performing the π' -integration we obtain:

$$\begin{aligned} \int ds \langle p | \Omega_{\mu\mu'} | p' \rangle &= -16\pi^2 G_{\mu\nu\mu'\nu'} \int \delta\pi \delta\xi \delta\xi' ds e^{ip\xi(s)} e^{-ip'\xi'(s)} \dot{\xi}^\nu(s) \dot{\xi}'^{\nu'}(s) \delta^{jj'} \\ &\quad a_{\xi'}(s) \left[\Omega_\xi T^j \Omega_\xi^{-1} \right] \left[\Omega_{\xi'} T^{j'} \Omega_{\xi'}^{-1} \right] \exp -i \int_0^{2\pi} dt \pi(t) [\xi(t) - \xi'(t)] \\ &\quad \int_{s_-}^{s_+} ds' ds'' \frac{\beta_\epsilon(s' - s) \beta_\epsilon(s'' - s)}{\pi_\alpha(s') \pi^\alpha(s'')}. \end{aligned} \quad (5.15)$$

Integrating then over $\pi(\bar{s})$ for $\bar{s} > s_+$ and $\bar{s} < s_-$ yields up to factors:

$$\begin{aligned} \int ds \langle p | \Omega_{\mu\mu'} | p' \rangle &= \int \delta\xi \delta\xi' ds \prod_{s_- < \bar{s} < s_+} d^4\pi(\bar{s}) G_{\mu\nu\mu'\nu'} e^{ip\xi(s) - p'\xi'(s)} \\ &\quad \frac{[\xi^\nu(s_+) - \xi^\nu(s_-)] [\xi'^{\nu'}(s_+) - \xi'^{\nu'}(s_-)]}{|\xi(s_+) - \xi(s_-)| |\xi'(s_+) - \xi'(s_-)|} \\ &\quad \left[\Omega_\xi T^j \Omega_\xi^{-1} \right] \left[\Omega_{\xi'} T^{j'} \Omega_{\xi'}^{-1} \right] \prod_{\bar{s} < s_-, \bar{s} > s_+} \delta^4(\xi(\bar{s}) - \xi'(\bar{s})) \\ &\quad \int_{s_-}^{s_+} ds' ds'' \frac{\beta_\epsilon(s' - s) \beta_\epsilon(s'' - s)}{\pi_\alpha(s') \pi^\alpha(s'')} \delta^{jj'} \\ &\quad \exp -i \int_{s_-}^{s_+} dt \pi(t) [\xi(t) - \xi'(t)]. \end{aligned} \quad (5.16)$$

We recall next that the Ω_ξ -matrices in the above formula are actually what we called the “hatted” quantities and they depend on the loop only for $\bar{s} \geq s_+$ and

due to the $\prod \delta^4(\xi(\bar{s}) - \xi'(\bar{s}))$ factor, we have $\xi = \xi'$ in this range, so that we can replace the factor involving these Ω_ξ -matrices with:

$$\begin{aligned} [\Omega_\xi T^j \Omega_\xi^{-1}] [\Omega_{\xi'} T^{j'} \Omega_{\xi'}^{-1}] \delta^{jj'} &\rightarrow [\Omega_\xi T^j \Omega_\xi^{-1}] [\Omega_\xi T^{j'} \Omega_\xi^{-1}] \delta^{jj'} \\ &= T^j T^{j'}, \end{aligned} \quad (5.17)$$

where the last equality follows from the fact that $T^j T^{j'}$ is a Casimir operator of the Lie algebra and therefore invariant under rotations in the Lie algebra. The result is a factor independent of Ω_ξ and of ξ being thus constant in the remaining integration, which is in fact the main reason why we changed right in the beginning to these so-called “hatted variables”. Hence, since the exponentials in (5.16) depend only on $\xi(\bar{s})$ and $\xi'(\bar{s})$ for $\bar{s} \in (s_-, s_+)$ and $\xi^\mu(s)$, according to (5.12), only on $\xi(s_+)$ and $\xi(s_-)$, we can perform both the ξ - and the ξ' -integration for $0 \leq \bar{s} \leq s_-, s_+ \leq \bar{s} \leq 2\pi$ by using the relation:

$$\int \delta\xi \dot{\xi}^\nu(s) \dot{\xi}^{\nu'}(s) \propto \frac{1}{4} g^{\nu\nu'} \dot{\xi}^2(s). \quad (5.18)$$

We now write:

$$e^{i[p\xi(s) - p'\xi'(s)]} = \exp i \int_{s_-}^{s_+} dt [p\xi(t) - p'\xi'(t)] \beta_\epsilon(s-t), \quad (5.19)$$

to give:

$$\begin{aligned} \int ds \langle p | \Omega^{\mu\mu'} | p' \rangle &= \int \prod_{s_- < \bar{s} < s_+} d^4\xi(\bar{s}) d^4\xi'(\bar{s}) d^4\pi(\bar{s}) ds T^j T^{j'} \delta^{jj'} \\ &\quad \int_{s_-}^{s_+} ds' ds'' \frac{\beta_\epsilon(s' - s) \beta_\epsilon(s'' - s)}{\pi_\alpha(s') \pi^\alpha(s'')} g^{\mu\mu'} \\ &\quad \exp i \int_{s_-}^{s_+} dt [p\beta_\epsilon(s-t) + \pi(t)] \xi(t) \\ &\quad \exp -i \int_{s_-}^{s_+} dt [p'\beta_\epsilon(s-t) - \pi(t)] \xi'(t). \end{aligned} \quad (5.20)$$

which can be simplified further to obtain:

$$\int ds \langle p | \Omega^{\mu\mu'} | p' \rangle = g^{\mu\mu'} T^j T^{j'} \delta^{jj'} \delta^4(p - p') \frac{1}{p^2}. \quad (5.21)$$

Our result for the \tilde{A} -propagator in momentum space therefore is:

$$\tilde{D}_{\mu\mu'}^{ii'}(p) T^i T^{i'} = g_{\mu\mu'} \delta^{ii'} \frac{1}{p^2} T^i T^{i'}, \quad (5.22)$$

which is exactly what we wanted to prove.

6 Remarks

Although the results we have obtained so far in attempting to extend the discussion of monopole dynamics in Yang-Mills fields to the quantum theory are quite limited, they have, we believe, given us some insight on several points.

Firstly, the Wu-Yang criterion [1] which has been applied in all previous work in the literature only to monopoles in the classical field theory, has now been shown to be extendable to the quantum field level to define their dynamics and to generate Feynman diagrams. In the nonabelian theory, the result cannot be checked, because the dynamics of monopoles is otherwise unknown. However, the same calculation applies of course also to the abelian theory which is expected to be dual symmetric, so that the dynamics of monopoles there should be the same as that of ordinary charges. Further, in the abelian theory, both the gauge boson self interaction term and the Jacobian can be ignored so that our g^0 calculation given above is exact. Hence, the fact that we obtained the same “perturbation series” in \tilde{g} above as the normal expansion in g in ordinary electrodynamics is a check not only on the Wu-Yang criterion but also of the loop space method employed.

Secondly, if the result recently obtained in the classical theory that Yang-Mills theory is dual symmetric [8] is extendable to the quantum theory, one would expect that the dynamics of monopoles dealt with here, in spite of its original loop space formulation, should eventually be expressible in terms only of the *local* dual potential $\tilde{A}_\mu(x)$. It was seen that at least at the g^0 level we were working, this was indeed the case. Whether it may persist at higher orders in g , and in such a way as to restore the dual symmetry, however, is at present unknown.

Thirdly, we have demonstrated that one can indeed do perturbation theory using the loop space techniques already developed. The calculation is a little clumsy but perfectly tractable. Though starting with loop variables which are invariant under the original U -transformation, it turns out that in order to remove the intrinsic redundancy of loop variables, one encounters in the Lagrange multiplier of the constraint a new field $L_{\mu\nu}[\xi|s]$ which depends on the dual (magnetic) \tilde{U} -gauge, so that we had again to gauge-fix. However, it is possible that by imposing the constraint in a different (global) manner [5], one may have a chance of obtaining explicitly gauge invariant results.

For these reasons, in spite of the limited scope of the result obtained so far, we hope that it will serve as a basis for further explorations.

7 Acknowledgements

JF acknowledges the support of the Mihran and Azniv Essefian Foundation (London), the Soudavar Foundation (Oxford) and the Calouste Gulbenkian Foundation (Lisbon), JP is grateful to Studienstiftung d.d. Volkes for financial support and TST thanks the Wingate Foundation for partial support during most of this work.

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