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# A Multimode Description of the Non-Linear Evolution of Modulational Instabilities in Plasmas

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A MULTIMODE DESCRIPTION OF THE NON-LINEAR EVOLUTION OF MODULATIONAL INSTABILITIES IN PLASMAS

by R Bingham and C N Lashmore-Davies*

ABSTRACT.

A general discussion of modulational instabilities in plasmas is given. The basic mechanism is a four wave interaction and examples include the Langmuir modulational instability, the oscillating two-stream instability and the filamentation of laser light in plasmas. The single envelope model which usually leads to the nonlinear Schrodinger equation is contrasted with the more general multimode description of this article. General equations for the modulation of finite amplitude, high frequency waves in unmagnetised plasmas are given. The stability properties of the linearised equations are briefly discussed and the conservation relations of the fully nonlinear equations are obtained.

The filamentation of an electromagnetic wave in a plasma is discussed in more detail. A physical argument is given for restricting the analysis to the initial wave and two sidebands. This is then put on a firmer footing with the aid of a recent theorem of Thyagaraja on the effective number of modes carrying the wave energy. Finally, exact analytic solutions of the fully nonlinear equations are obtained, and the resulting filamentation length compared with experiment.

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## 1. INTRODUCTION.

The Langmuir modulational instability, Zhakharov(1972), the oscillating two stream instability, Nishikawa(1968), and self-modulation, Kaw et al(1973), of electromagnetic waves in plasmas are all examples of modulational instabilities. In all cases the basic mechanism is a four wave interaction, Bingham and Lashmore-Davies(1979a), involving two pump "quanta" and two sidebands. All these processes can be described by the same general form of equations. A solution of any one of these problems can easily be generalized to any of the others. In this treatment we shall restrict ourselves to unmagnetized plasmas although similar processes occur in magnetized plasmas.

In the usual treatment of this problem the nonlinear Schrödinger equation is used as the model equation. This involves an averaging procedure over the pump and sideband amplitudes to describe the resultant wave envelope. This method is appropriate to cases where the sidebands have wave numbers close to the pump wave number and where the pump and sidebands belong to the same wave branch. In cases where the pump and sidebands belong to different wave branches or the wave numbers are significantly different a multimode treatment is necessary. We have developed a multimode nonlinear theory c.f. Bingham and Lashmore-Davies (1979a,b), of these processes in which the initial pump wave $\underline{k}_o$ generates pairs of sidebands $\underline{k}_o \pm n\underline{k}_s$ and their associated density perturbations $n\underline{k}_s$, where n is an integer. In this treatment each high frequency wave is described by a separate non-linear equation.

We have previously restricted our multimode description to an analysis of the pump wave and one pair of sidebands, Bingham and Lashmore-Davies(1979a,b). The sidebands chosen were those corresponding to maximum growth. This allowed exact analytic solutions to be obtained. In this article we shall consider the justification for this approximation in much greater depth. First we shall give a physical argument for the procedure. We shall then put the method on a firmer theoretical foundation using a recent result of Thyagaraja [1979] which allows us to estimate the effective number of modes actively involved in the interaction process.

This result of Thyagaraja [1979] provides the mathematical justification for the physical argument and shows under what conditions a given physical situation can be approximated in this way.

The plan of the article is as follows. In section 2, the general equations describing the four wave interaction for the various cases are derived from the two fluid model and Maxwell's equations, by expanding these equations about the linearized solution. A linear stability analysis is carried out to obtain the initial growth rate as a function of the wavenumber of the low frequency density perturbation. In Section 3 conservation relations are derived which illustrate the fact that the basic instability results from a four wave interaction. In section 4 the specific example of the filamentation of an electromagnetic wave in a plasma is considered. Finally, in section 5 exact nonlinear solutions for the filamentation case are obtained and the non-linear filamentation length is calculated and compared with experiment.

## 2. THE MODEL.

We shall consider a uniform, infinite plasma in which a small but finite amplitude pump wave is propagating. The initial pump wave is described through its electric field

$$\underline{E}_o(\underline{x},t) = A_o(\underline{x},t) \exp i(\underline{k}_o.\underline{x} - \omega_o t) \qquad (2.1)$$

where $\omega_o$ amd $k_o$ are related by the linear dispersion relation

$$\omega^2_o = \omega^2_{pe} + v^2k^2_o \qquad (2.2)$$

where $\omega_{pe}$ is the plasma frequency and v represents the electron thermal velocity if the wave is a Langmuir wave or the speed of light, c, if the wave is a transverse electromagnetic wave.

The plasma model we use to analyse the problem is the two-fluid isothermal approximation. We have chosen this model in the interests of simplicity. It gives an adequate description of the phenomena, at least for the initial stages of the non-linear behaviour. The equations are as follows

$$\frac{\partial}{\partial t} n_j + \underline{\nabla}.(n_j \underline{v}_j) = 0 \qquad (2.3)$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_j . \underline{\nabla}\right) \underline{v}_j + \frac{\kappa T_j}{n_j m_j} \underline{\nabla} n_j + \nu_j \underline{v}_j = \frac{e_j}{m_j} \left(\underline{E} + \underline{v}_j \times \underline{B}\right) \qquad (2.4)$$

$$\underline{\nabla}.\underline{E} - \frac{1}{\varepsilon_o} \sum_j n_j q_j = 0 \qquad (2.5)$$

$$\underline{\nabla} \times \underline{E} = -\mu_o \frac{\partial \underline{H}}{\partial t} \qquad (2.6)$$

$$\underline{\nabla} \times \underline{H} = \underline{J} + \varepsilon_o \frac{\partial \underline{E}}{\partial t} \qquad (2.7)$$

where $\underline{J} = \sum_j n_j q_j \underline{v}_j$ , $j = i, e$ and $\underline{v}_j$, $n_j$, $q_j$, $m_j$, $\nu_j$, and $T_j$ are, respectively the fluid velocity, density, charge, mass, phenomenological damping coefficients and the temperature of the $j^{th}$ species. $\underline{E}$ and $\underline{H}$ are the electric and magnetic fields and $\kappa$, $\mu_o$ and $\varepsilon_o$ are Boltzmanns constant and the magnetic permeability and dielectric coefficients for a vacuum (we use M.K.S. units). Equations (2.3)-(2.7) are fully non-linear and contain all the fields and perturbations of interest.

We now use the above set of equations to generate the non-linear equations describing the coupling of a high frequency pump wave (Langmuir, or electromagnetic) to two other high frequency sidebands which can also be either Langmuir or electromagnetic fields. We assume there is a low frequency density perturbations of frequency and wavenumber ($\Omega$, $\underline{k}_s$), which are as yet unspecified. This low-frequency density perturbation, which involves both ions and electrons, will beat with the initial pump wave to produce high frequency sidebands with wavenumbers $\underline{k}_o - \underline{k}_s$ the Stokes wave and $\underline{k}_o + \underline{k}_s$ the anti-Stokes wave. These two sidebands can couple to the pump wave to regenerate the initial low frequency density perturbation at $\underline{k}_s$ ; they can also couple to each other to generate another low frequency density perturbation at $2\underline{k}_s$. This new density perturbation can beat with the pump to produce another pair of Stokes and anti-Stokes waves at $\underline{k}_o \pm 2\underline{k}_s$ which beat together to produce another density perturbation at $4\underline{k}_s$ which, in turn, will also generate new Stokes and anti-Stokes waves, etc. A hierarchy of low frequency density perturbations together with higher order sidebands are thus generated.

This process of producing higher order sidebands can be halted at the first two by choosing the initial density perturbation as the one corresponding to maximum growth $\underline{k}_{sm}$, Bingham and Lashmore-Davies (1979a,b). With this choice, $2\underline{k}_{sm}$ lies outside the unstable band and therefore, $\underline{E}_3$ and $\underline{E}_4$ are stable together with all the other higher order sidebands. This is the physical justification for considering only the first two sidebands.

We now write the sideband waves and the density perturbation as

$$\underline{E}_j(x,t) = A_j(\underline{x},t)\, e^{i(\underline{k}_j.\underline{x}-\omega_j t)} \qquad (2.8)$$

where $A_j$ is a slowly varying amplitude and $j = 1,2$

$$n_{es}(\underline{x},t) = N_1(\underline{x},t)e^{i\underline{k}_s.\underline{x}} + N_2(x,t)e^{i2\underline{k}_s.\underline{x}} + \ldots \qquad (2.9)$$

where the slow amplitude variation of the electromagnetic wave is determined by the linear dissipation and the non-linear interaction and the phase factor is due to the linear dispersion. For the density perturbation the time variation is dominated by the non-linear coupling. In the present analysis we will consider the dominant force which drives the density perturbation to be due to the ponderomotive force. Other mechanisms, such as relativistic and thermal effects result in similar equations.

We now use a perturbation procedure to obtain the equation for the pump wave, sidebands and the density perturbation. The details are given elsewhere, Bingham and Lashmore-Davies (1979a,b). The method consists in expanding the equations for the high frequency waves about their linear solutions $(\omega_j, k_j)$, $j = 0, 1, 2$ .. where

$$\omega^2_j = \omega^2_{pe} + v^2k^2_j$$

We also assume perfect $\underline{k}$-matching of the interacting waves but allow a small frequency mis-match such that

$$\underline{k}_o = \underline{k}_{1,2} \pm \underline{k}_s, \quad \underline{k}_o = \underline{k}_{3,4} \pm 2\underline{k}_s \text{ .. etc.}$$

and $\omega_o \approx \omega_n$ , $n = 1,2 \ldots$

The equations for the pump wave, the first two sidebands and the density perturbation are

$$\left(\frac{\partial}{\partial t} + \underline{v}_o \cdot \frac{\partial}{\partial \underline{x}} + \gamma_o\right) A_o(\underline{x},t) = -ic_{so}\left(\frac{\omega_1^2}{\omega_o} N_{s1} A_1 e^{i\delta_1 t} + \frac{\omega_2^2}{\omega_o} N^*_{s1} A_2 e^{i\delta_2 t}\right) \quad (2.10)$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}} + \gamma_1\right) A_1(\underline{x},t) = -ic_{so}\left(\frac{\omega_o^2}{\omega_1} N^*_{s1} A_o e^{-i\delta_1 t} + \frac{\omega_2^2}{\omega_1} N^*_{s2} A_2\right) \quad (2.11)$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_2 \cdot \frac{\partial}{\partial \underline{x}} + \gamma_2\right) A_2(\underline{x},t) = -ic_{so}\left(\frac{\omega_o^2}{\omega_2} N_{s1} A_o e^{-i\delta_2 t} + \frac{\omega_1^2}{\omega_2} N_{s2} A_1\right) \quad (2.12)$$

$$\left(\frac{\partial^2}{\partial t^2} + \gamma_s \frac{\partial}{\partial t} + k_s^2 c_s^2\right) N_{s1}(\underline{x},t) = - c_{01}\left(\omega_0\omega_1 A_0 A_1^* e^{-i\delta_1 t} + \omega_0\omega_2 A_0^* A_2 e^{i\delta_2 t}\right) \quad (2.13)$$

$$\left(\frac{\partial^2}{\partial t^2} + \gamma_s \frac{\partial}{\partial t} + 4k_s^2 c_s^2\right) N_{s2}(\underline{x},t) = -4c_{01}\, \omega_1\omega_2\, A_1^* A_2 \varepsilon^{-i(\delta_1-\delta_2)t} \quad (2.14)$$

where $c_{s0}$, $c_{01}$, are coupling coefficients which are determined by the particular interaction process under consideration, $\delta_{1,2} = \omega_0 - \omega_{1,2}$ are the frequency mismatch terms and $v_0$, $v_1$, $v_2$, $c_s$ are the group velocities and ion sound speed respectively.

In order to solve equations (2.10) - (2.14) we follow other authors, e.g. Zhakharov (1972) in making the static approximation. Thus assuming that $\gamma \frac{\partial}{\partial t}, \frac{\partial^2}{\partial t^2} \ll k_s^2\, c_s^2$ we can obtain expressions for $N_{s1}$ and $N_{s2}$

directly.

$$N_{s1} = -\frac{c_{01}}{k_s^2 c_s^2}\left(\omega_0\omega_1 A_0 A_1^* e^{-i\delta_1 t} + \omega_0\omega_2 A_0^* A_2 e^{i\delta_2 t}\right) \tag{2.15}$$

$$N_{s2} = -\frac{c'_{01}}{4k_s^2 c_s^2}\,\omega_1\omega_2\, A_1^* A_2\, e^{-i(\delta_1-\delta_2)t} \tag{2.16}$$

Substituting these expressions for $N_{s1}$, $N_{s2}$ into equations (2.10)-(2.13) leads to the following three equations for $A_0$, $A_1$ and $A_2$

$$\left(\frac{\partial}{\partial t} + \underline{v}_o\cdot\frac{\partial}{\partial \underline{x}} + \gamma_o\right)A_o(\underline{x},t) = i\Gamma_o\Big[\omega_1^3 A_o\left|A_1\right|^2 + \omega_2^3 A_o\left|A_2\right|^2 + \omega_1\omega_2(\omega_1+\omega_2)A_o^* A_1 A_2 e^{i(\delta_1+\delta_2)t}\Big] \tag{2.17}$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_1\cdot\frac{\partial}{\partial \underline{x}} + \gamma_1\right)A_1(\underline{x},t) = i\Gamma_1\Big[\omega_o^3\left|A_o\right|^2 A_1 + \omega_o^3\frac{\omega_2}{\omega_1}A_o^2 A_2^* e^{-i(\delta_1+\delta_2)t} + \alpha_1\,\omega_2^3 A_1\left|A_2\right|^2\Big] \tag{2.18}$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_2\cdot\frac{\partial}{\partial \underline{x}} + \gamma_2\right)A_2(\underline{x},t) = i\Gamma_1\Big[\omega_o^3\frac{\omega_1}{\omega_2}A_o^2 A_1^* e^{-i(\delta_1+\delta_2)t} + \omega_o^3\left|A_o\right|^2 A_2 + \alpha_1\,\omega_1^3\left|A_1\right|^2 A_2\Big] \tag{2.19}$$

where the coupling coefficients $\Gamma_o$ and $\Gamma_1$ for the different types of modulational instabilities are as follows

Langmuir Modulational Instability $\quad \Gamma_o = \Gamma_1 \equiv \dfrac{\varepsilon_0\,\omega_0^3}{8n_o\gamma_e\kappa T_e\omega_{pe}^2}$ ; $\alpha_1 = 1$.

Filamentation and Self-Modulation of E.M. Waves $\quad \Gamma_o \equiv \dfrac{e^2\omega_{pe}^2}{8m_e m_i \omega_0^2\omega_1^4}$, $\Gamma_1 \equiv \Gamma_o \dfrac{\omega_1^2}{\omega_o^2}$, $\alpha_1 = \dfrac{\omega_o^4}{\omega_1^4}$

Oscillating Two Stream Instability $\quad \Gamma_o \equiv \dfrac{e^2}{8m_e m_i c^2\omega_s^2{}_{pe}^2\omega_0\omega_1}$ ; $\Gamma_1 = \Gamma_o \dfrac{k_s^2}{k_{L1}^2} \dfrac{\omega_{pe}^2\omega_1^2}{\omega_o^4}$ ;

where $\alpha_1 = \dfrac{\omega_o^5}{\omega_{pe}^4} \dfrac{k_{L1}^2}{\omega_1 k_s^2}$ and $k_{L1}^2 = k_s^2 + k_o^2$.

The above equations appear somewhat more complicated than they are in each of the above specific cases because they have been written in a general form. For any of the above cases the equations take on a more symmetric form. Note that for filamentation and the oscillating two stream instability $\omega_1 = \omega_2$.

Other non-linear terms which could be included in equations (2.17)-(2.19) are the self energy terms $|A_j|^2 A_j$ which in the present analysis could only arise from second harmonic terms. These terms are very much smaller than the terms derived from the ponderomotive force. If the dominant non-linear term is due to relativistic effects where the amplitude of the wave is so large that the oscillating velocity of electrons in the wave field approaches c, the velocity of light, the non-linear coupling involves only high frequency modes and the self-interaction term is comparable to the other terms and cannot be neglected. The equations describing modulational instabilities due to relativistic effects (cf. Bingham (1983)) thus include the self interaction terms $|A_j|^2 A_j$ on the right hand side. In this case the equations resemble more closely those describing the modulation of deep water waves [cf. Benjamin and Feir (1967)].

Equations (2.17)-(2.19), although similar in appearance to equations (11) - (13) in Bingham and Lashmore-Davies (1979a,b) now contain the additional non-linear terms $i\,\Gamma\,\omega_2^3 |A_2|^2 A_1 + i\,\Gamma\,\omega_1^3 |A_1|^2 A_2$. These two new terms are generated in the following manner. The two sideband waves at $\underline{k}_0 \pm \underline{k}_s$, beat together to excite the density perturbation at $2\underline{k}_s$ which in turn beats with each sideband wave to produce the other.

Before considering the non-linear solutions of equations (2.17) - (2.19) let us first obtain the initial behaviour of the perturbations assuming the pump wave amplitude $A_0$ remains constant. Using $A_1^* \exp(-i\delta_1 t)$, $A_2 \exp(i\delta_2 t)$ as the amplitude variables, equations (2.18) - (2.19) become a set of linear differential equations with constant coefficients and can be solved in the usual way assuming a variation $\exp i(px-\Omega t)$. The resulting dispersion relation is a quadratic in $\Omega$. If we also take $p = 0$ then the dispersion relation for $\Omega$ reduces to the simpler form

$$(\Omega-\delta_1+i\gamma_1)(\Omega+\delta_2+i\gamma_2) - \tfrac{1}{2}(\delta_1+\delta_2)\omega_0 K = 0 \qquad (2.20)$$

where $K = \dfrac{2\Gamma\left|A_0\right|^2}{\omega_o^2}$

Solving this equation we obtain the following threshold for instability:

$$K = -\left[\gamma^2 + \Delta^2\right]/\omega_0\Delta \qquad (2.21)$$

where $\Delta = \frac{1}{2}(\delta_1 + \delta_2)$, i.e. we have instability only when $\Delta < 0$. However, from the definitions of $\delta_1$ and $\delta_2$ given by $\delta_{1,2} = \pm \underline{k}_s \cdot \underline{v}_o - \dfrac{v^2 k_s^2}{2\omega_o}$ we find that $\Delta \approx \dfrac{-k_s^2 v^2}{\omega_0}$ and so $\Delta$ is negative definite. When the threshold is exceeded, the real part of $\Omega$ is given by

$$\mathrm{Re}\Omega = \tfrac{1}{2}(\delta_1 - \delta_2) = \frac{\underline{k}_s \cdot \underline{k}_0 v^2}{\omega_o} \qquad (2.22)$$

$\mathrm{Re}\Omega$ is the frequency of oscillation of the low-frequency density perturbation. In the limit of an infinite wavelength pump wave or a transverse perturbation $\underline{k}_s \perp \underline{k}_o$, the density perturbation is purely growing. In the case of a longitudinal perturbation $\underline{k}_s \parallel \underline{k}_o$ the instability excites a low-frequency wave of frequency given by (22) and two high frequency sideband waves whose frequencies are shifted from their unperturbed values $\omega_{1,2}$ to $\omega_0 \mp \frac{1}{2}(\omega_1 - \omega_2)$. For a transverse perturbation $\underline{k}_s \perp \underline{k}_o$ the frequency of density perturbation is zero and the frequencies of the sidebands are all equal and locked to the pump frequency $\omega_o$.

The growth rate resulting from (20) can be expressed as

$$\frac{\gamma}{\omega_o} = -\frac{\gamma_1}{\omega_o} + \kappa_s(K - \kappa_s^2)^{1/2} \tag{2.23}$$

where $\kappa_s = \dfrac{k_s v}{\sqrt{2\omega_o}}$ and we have assumed $\gamma_1 = \gamma_2$. The relation (23) gives the growth ($\gamma$) versus $k_s^2$ stability diagram, known in fluid mechanics as the Benjamin-Feir (1967) stability diagram. We can see from the expression for the growth rate that there will be a wavenumber $\underline{k}_{sm}$ at which the growth is a maximum. The maximum growth rate occurs for $\kappa_{sm} = (K/2)^{1/2}$ and is given by

$$\frac{\gamma_m}{\omega_o} = \frac{K}{2} - \frac{\gamma_1}{\omega_o} \tag{2.24}$$

We also note that the condition for the growth rate to be a maximum is also the condition for the instability threshold to be a minimum.

## 3. CONSERVATION RELATIONS.

Conservation relations for the wave energy density can be obtained from equations (17) - (19) taking into account the fact that the sideband wave frequencies are shifted by the interaction from $\omega_{1,2}$ to $\omega_o \mp \frac{1}{2}(\omega_2 - \omega_1)$. When this is done the expression for the total energy density for the sideband waves becomes

$$\varepsilon_{1,2} = \frac{\varepsilon_o}{2} \left|E_{1,2}\right|^2 \left[ 1 - \frac{(\delta_1+\delta_2)}{2\omega_o} \right] \tag{3.25}$$

for electromagnetic waves and

$$\varepsilon_{1,2} = \frac{\varepsilon_o}{2} \left|E_{1,2}\right|^2 \frac{\omega_{1,2}^2}{\omega_{pe}^2} \left[ 1 - \frac{(\delta_1+\delta_2)}{2\omega_{pe}} \right] \tag{3.26}$$

for Langmuir waves. Using these new expressions for the wave energy density the conservation of wave energy density for electromagnetic waves becomes

$$\frac{\partial}{\partial t}\{\left|\alpha_o\right|^2 + (1 - \frac{(\delta_1+\delta_2)}{2\omega_o}) \left|\alpha_1\right|^2 + (1 - \frac{(\delta_1+\delta_2)}{2\omega_o}) \left|\alpha_2\right|^2\} = 0 \tag{3.27}$$

where the $\alpha_j$'s$(j = 0,1,2)$ are normalized so that $\left|\alpha_j\right|^2$ is the total energy density. Using (27) we can obtain the equation for conservation of wave action density,

$$\frac{\partial}{\partial t}\{\frac{\left|\alpha_o\right|^2}{\omega_o} + \frac{\left|\alpha_1\right|^2}{\omega_1'}(1 - (\frac{\delta_1+\delta_2}{2\omega_o})) + \frac{\left|\alpha_2\right|^2}{\omega_2'}(1 - (\frac{\delta_1+\delta_2}{2\omega_o}))\} = 0 \tag{3.28}$$

$\omega_1'$, $\omega_2'$ are the perturbed frequencies defined above. Equation (3.28) is analogous to the Manley-Rowe relations for a three wave interaction. We see clearly from this equation that the basic mechanism of modulational instabilities is a four wave interaction in which two pump "quanta" create two excited wave "quanta" or vice versa.

It is worth noting that in the conservation relations the unperturbed energy density for the pump wave appears. This is due to the fact that the Stokes, anti-Stokes and density perturbations drive the pump wave resonantly whereas the sideband waves are driven off resonance resulting in their frequency shift.

4. FILAMENTATION.

In order to obtain non-linear solutions to equations (2.17)-(2.19) we shall consider the particular case of electromagnetic filamentation. In this case the equations can be written as

$$\left(\frac{\partial}{\partial t} + v_o \frac{\partial}{\partial x} + \gamma_t\right) A_o(x,t) = \frac{i\Gamma}{\omega_o}\left(|A_1|^2 A_0 + |A_2|^2 A_0 + 2 A_0^* A_1 A_2 e^{i2\delta t}\right) \quad (4.29)$$

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x} + \gamma_t\right) A_1(x,t) = \frac{i\Gamma}{\omega_1}\left(|A_0|^2 A_1 + \frac{\omega_o}{\omega_1}|A_2|^2 A_1 + A_0^2 A_2^* e^{-i2\delta t}\right) \quad (4.30)$$

$$\left(\frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial x} + \gamma_t\right) A_2(x,t) = \frac{i\Gamma}{\omega_1}\left(|A_0|^2 A_2 + \frac{\omega_o}{\omega_1}|A_1|^2 A_2 + A_0^2 A_1^* e^{-i2\delta t}\right) \quad (4.31)$$

where $\Gamma$ corresponding to filamentation is given below equ.(2.19). Note also that $\delta_1 = \delta_2 = \delta$ for this case.

We have presented above a physically intuitive reason for restricting the analysis to a few modes. However, a recent theorem due to Thyagaraja (1979) provides the mathematical justification for this approximation. In order to make full use of the theorem we first need to formulate the problem in terms of the non-linear Schrödinger equation which for a plane polarized electro-magnetic wave in a homogeneous plasma is

$$2i \frac{\omega_o}{c^2} \frac{\partial A_o}{\partial t} + \frac{\partial^2 A_o}{\partial y^2} = -\frac{n_o e^4 \mu_o}{2\omega_o^2 m_e^2 \kappa T_e} |A_o|^2 A_o \quad (4.32)$$

where we have assumed a plane wave initially which becomes modulated in the direction transverse to its propagation direction

i.e. $\quad E_o(x,y,t) = \mathrm{Re}\, A_o(y,t)\, e^{i(k_o x - \omega_o t)}$ and

$\omega_o^2 = \omega_{pe}^2 + c^2 k_o^2$. Transforming to the dimensionless space and time coordinates $\xi$ and $\tau$ given by

$$\tau = \frac{c^2 t}{2\omega_o L^2}, \quad \xi = \frac{y}{L}$$

where L is a scale length determined by the plasma conditions, equation (4.32) becomes

$$i \frac{\partial A_o}{\partial \tau} = - \frac{\partial^2 A_o}{\partial \xi^2} - \mu |A_o|^2 A_o \tag{4.33}$$

where $\mu = \frac{n_o e^4 \mu_o L^2}{2\omega_o^2 m_e^2 \kappa Te}$ .

The non-linear Schrödinger equation describes a continuous wave field which formally possesses an infinite number of modes. Thyagaraja (1979) has obtained the remarkable result that for certain dynamical systems there is an effective number of modes $N_E$ which actively participate in the motion of the system. The non-linear Schrödinger equation represents one such system.

Applying the theorem of Thyagaraja, in the case where $\mu > 0$, to the non-linear Schrödinger equation results in the following expression for the number of modes ($N_E$) carrying the wave energy.

$$N_E = \frac{\mu I_o}{4\pi} \left[ \frac{1}{4\pi} \mu^2 I_o{}^2 + 4\left(\frac{J_o}{I_o} + \frac{\mu}{2} I_o\right) \right] \tag{4.34}$$

$I_o$ and $J_o$ (defined by Thyagaraja (1979)) are constants of the motion for the system and therefore can be determined from the initial state of the electric field amplitude $A_o(\xi,0)$. Since the initial state was assumed to be a plane wave in the x-direction (i.e. no variation in the y direction) $A_o(\xi,0)$ is independent of $\xi$ i.e. $A_o(\xi,0) = A_o$, $\frac{\partial A_o}{\partial \xi}(\xi,0) = 0$. $I_o$ and $J_o$ can now be easily obtained

$$I_o = |A_o|^2 \; ; \; J_o = - \frac{\mu}{2} |A_o|^4$$

Using these expressions in equation (4.34) we obtain

$$N_E = \frac{1}{8\pi} \frac{\omega_{pe}^2}{c^2} \frac{v_o^2}{v_{te}^2} L^2 \tag{4.35}$$

where $v_o$ is the quiver velocity ($v_o = \frac{e|A_o|}{m_e \omega_o}$) and $v_{te}$ is the electron thermal velocity. From equation (7) we note that the number of modes carrying the wave energy increases with increasing pump strength and also as L increases. These results are what one would expect physically since with a larger pump more energy is available to be shared out among the different modes (the unstable band of wavenumbers increases) and the number of modes increases as the physical size of the system increases.

For small values of $\underline{k}_s$ ($\ll k_{sm}$), where $k_{sm}$ is the wavenumber corresponding to the maximum growth rate, many modes become unstable as a result of the generation of higher order density perturbations. For larger values of $k_s$ ($\sim k_{sm}$) only a few sideband modes become unstable. There is also a lower limit to the value of $\underline{k}_s$ set by

a) the physical dimensions of the plasma
b) the spot size of the laser beam
and c) the threshold condition i.e. $|\delta| = \gamma_t$

Which of these provide the long wevelength limit depends on which is smallest in a given physical situation. The more difficult question is how do we justify the neglect of shorter wavelength modes, which fit more and more easily into the system? On physical grounds we have argued that $k_{sm}$ is the high wave-number cut-off since harmonics of this wavenumber will be stable. It then makes sense to concentrate on $k_{sm}$ since longer wavelengths will tend to generate these faster growing modes, whereas, for the reason given above, these modes will not generate shorter wavelengths (because these shorter wavelength modes are stable). This physical argument can be justified with the aid of Thyagaraja's theorem in the following way. Equation (4.35) gives the effective number of degrees of freedom for a given value of the initial wave amplitude as a function of the scale length. As already noted, $N_E$ decreases as the scale length decreases. If we now substitute for the scale length L the wavelength

$\lambda_{sm}$ (N.B. $\lambda_{sm} \alpha E_o^{-1}$) of the density perturbation corresponding to maximum growth we obtain the result

$$N_E = \frac{1}{2\pi}$$

In other words, Thyagaraja's theorem provides the mathematical justification for treating $k_{sm}$ as the high wavenumber cut-off. For any given application, the effective number of modes will be given by the number of wavelengths $\lambda_{sm}$ corresponding to maximum growth that will fit into the dimension L.

## 5. NON-LINEAR SOLUTIONS.

The method of solving equations (4.29-4.31) has already been described in detail elsewhere, Bingham and Lashmore-Davies (1979a,b). There are two types of solution (a) time dependent, spatially independent case with and without damping; (b) spatially dependent case, stationary solutions. For the time dependent case we find two solutions whereas before we only had one solution. These are periodic and phase jump solutions which were obtained by Bingham and Lashmore-Davies(1979a,b), and will not be discussed here since we are more interested in the case which provides a closer link with experiment i.e., the spatially dependent case. Following the analysis already outlined in the above references, we obtain the following equation

$$\frac{dw}{d\xi} = \pm \frac{2\Gamma}{\omega_1 V_1} (\alpha w^4 + \beta w^3 + \gamma w^2 + \eta w + \varepsilon)^{1/2} \qquad (5.36)$$

where $w = |A_1|^2$, $\alpha = -\frac{V_1}{V_o}\left(2 - \frac{\omega_o^2 V_o}{4\omega_1^2 V_1}\right)$

$$\beta = \frac{4\omega_1 V_1}{\omega_o V_o} \left\{\frac{\omega_1 V_1}{\omega_o V_o} (m_2 - m_1) - A - \frac{4\omega_1 V_1}{\omega_o V_o} N \left(W^1 - \frac{\delta\omega_1}{\Gamma} - 2NA\right)\right\},$$

$$\gamma = A^2 - \frac{4\omega_1 V_1}{\omega_o V_o} A(m_2 - m_1) - \frac{4\omega_1 V_1}{\omega_o V_o} N\Big[\Lambda + \frac{\delta\omega_o V_o}{2\Gamma V_1} A + \frac{N\omega_o V_o}{2\omega_1 V_1} A^2 -$$

$$(W^1 - \frac{\delta\omega_1}{\Gamma} - 2NA)\Big]^2,$$

$$\eta = A^2 (m_2 - m_1) - 2(W^1 - \frac{\delta\omega_1}{\Gamma} - 2NA)(\Lambda + \frac{\delta\omega_o V_o}{2\Gamma V_1} A + \frac{N\ \omega_o V_o}{2\omega_1 V_1} A^2),$$

$$\varepsilon = (\Lambda + \frac{\delta\omega_o V_o}{2\Gamma V_1} A + \frac{N\omega_o V_o}{2\omega_1 V_1} A^2),$$

$$A = (m_o + \frac{2\omega_1 V_1}{\omega_o V_o} m_1), \quad N = 1 + \frac{\omega_o^{\ 2} V_o}{4\omega_1^2 V_1}, \quad W^1 = (1 - \frac{\omega_o^{\ 2} V_o}{2\omega_1^{\ 2} V_1})$$

and $m_0$, $m_1$ and $m_2$ are the values of $|A_n|^2$ at $\xi = 0$ where $n = 0$, 1 or 2. $V_0$, $V_1$, $\Lambda$, $W$ have already been defined in Bingham and Lashmore-Davies (1979a). Equation (5.36) now differs significantly from the one obtained in the above reference. It is exactly the same as equation (60) of Bingham and Lashmore-Davies (1979b) where it was found that different solutions existed depending on the sign of $\alpha$. A full discussion of the different types of solution is contained in this reference. We will only consider the case of physical interest i.e., the one that allows us to calculate the filamentation length. This case corresponds to the one where we have a solitary wave solution given by

$$|A_o|^2 = \frac{ab\ \text{sech}^2(G\xi)}{a-b\ \tanh^2(G\xi)} \tag{5.37}$$

where $G = \frac{1-Z}{Z}\frac{\omega_o\omega_1}{c^2 k_o}$, $Z = \frac{1}{4}\frac{\omega_o^2}{\omega_1^2} + \frac{\omega_o}{2\omega_1}\frac{m_o}{m_1}$, $b = m_1$

and $a = \frac{m_1 (a_o^2)_{max}}{m_o}\left(1 - \frac{2m_o}{(a_o^2)_{max}}\right) / \left[\frac{(a_o^2)_{max}}{m_o}\left(1 - \frac{2m_o}{(a_o^2)_{max}}\right) - m_1\right]$ where

$m_1$ is the maximum intensity of the Stokes wave, $m_o$ is the minimum intensity of the pump wave and $(a_o^2)_{max}$ is its maximum value. The condition for the above solitary wave solution to exist is

$$\frac{1}{2}\frac{\omega_o}{\omega_1} m_1 + 2 m_o - (a_o^2)_{max} = 0 \tag{5.38}$$

If we assume that $A_1$ and $A_2$ grow to the same order as $(A_o)_{max}$ then the pump minimum will be small compared with its maximum and our theory describes a large intensity difference inside and outside the filament.

Using equations (5.37) and (5.38) we can calculate the filamentation length. We define this as the distance required for the Stokes and anti-Stokes wave intensities to increase from a value determined by the thermal noise to their maximum value. With this definition we obtain the following expression for the filamentation length $\ell$

$$\ell = \frac{1}{2G} \ln \left[\left(1 - \frac{2m_o}{(a_0^2)_{max}}\right) \frac{4(a_o^4)_{max}}{m_o \varepsilon_1}\right] \tag{5.39}$$

where $\varepsilon_0 \varepsilon_1$ is the energy in the Stokes (or anti-Stokes) wave in thermal equilibrium and G has been defined below equation (5.37) (for $m_1 >> m_o$, G $\simeq 3k_o$).

Using the experimental values obtained by Donaldson and Spalding (1976) i.e. $\varepsilon_o E_o^2/n_o KT \simeq 10^{24} m^{-3}$, fixes the value of $(a_o^2)_{max}$ which then enables us to calculate $\ell$ for different values of $(a_o^2)_{max}/m_o$. Table (1) shows the dependence of $\ell$ as a function of $(a_o^2)_{max}/m_o$.

Table 1.

| | | | | |
|---|---|---|---|---|
| $(a_o^2)_{max}/m_o$ | 4 | 10 | 100 | 1000 |
| $\ell(\mu m)$ | 28 | 18 | 17 | ~17 |

It is clear that the filamentation length is no longer very sensitive to the parameter $(a_o^2)_{max}/m_o$. The values of $\ell$ are all the correct order of magnitude as observed experimentally. $(a_o^2)_{max}/m_o = 10^3$ corresponds to the case when the pump falls to a minimum value given by the minimum threshold condition for the instability. If we take the focal spot size of the laser beam as a value of L, eq. (5.39) allows us to calculate the effective number of modes involved in the interaction. For Donaldson and Spalding's (1976) experiment $N_E \approx 1$, suggesting that there really are only a few modes involved for a real experiment. We can also calculate the width of a filament using the expression for the wavenumber of the density perturbation $k_{sm}$ $\left(= \frac{\omega_{pe}}{2c} \frac{v_o}{v_{te}}\right)$ corresponding to maximum growth rate. The value for the filament width for Donaldson and Spalding's (1976) experiment is ~ 100 μm which is in very good agreement with the experimental value.


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