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Proceedings of the School for Young High Energy Physicists

Dr D C Dunbar Dr E W N Glover Dr S King and Dr T R Morris

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Proceedings of the School for Young High Energy Physicists

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HEP SUMMER SCHOOL FOR YOUNG HIGH ENERGY PHYSICISTS

RUTHERFORD APPLETON LABORATORY/THE COSENER'S HOUSE, ABINGDON:

01-13 SEPTEMBER 1996

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RAL Summer School for Young Experimental High Energy Physicists

Cosener's House, 01-13 September 1996

Preface

Fifty-four young experimental particle physicists attended the 1996 Summer School, held as normal, at Coseners House in Abingdon in September. This year twenty of the students were women, double the previous highest number. The weather was fine allowing tutorials and private study to take place in the relaxed atmosphere of the lovely gardens.

The material was as usual, intellectually challenging, not least to the experimentalist tutors and the Director! The lectures reproduced here were given by David Dunbar (Quantum Field Theory), Steve King (Relativistic Quantum Mechanics), Tim Morris (The Standard Model) and Nigel Glover (Phenomenology). They were all of a very high standard and thoroughly enjoyable.

Sarah Unger (RAL) gave an interesting seminar on the ISO Project and Ian Corbett (PPARC) delivered an upbeat after dinner speech. Mike Whalley (Durham) introduced the new generation of students to the Durham HEP database.

The students each gave a ten minute seminar in the evening sessions. The quality of the talks was very impressive and the time keeping excellent. The broad range of activities covered from front-line physics results to preparations for the next generations of machines gave a clear indication of the breadth of particle physics activities in the UK.

The tutors, Paul Dauncey (RAL), Jeff Forshaw (Manchester), Stephen Haywood (RAL), Ken Long (IC) and Julia Segebeer (IC) worked tremendously hard and their efforts were well appreciated by the students.

The organisation by Ann Roberts and Coseners staff was efficient and effective and I am personally indebted to them for leading me successfully through my first year as Director. I would also like to thank Dave Kelsey and Gareth Smith for providing me with computer support. I wish Steve King good luck for his year at CERN. I hope he will be able to return for the School in 1998. On behalf of myself and my predecessor, Ken Peach, I would like to thank Paul Dauncey for his support over the three years he has tutored at the School.

The School was physically and intellectually demanding but very satisfying and I wish all the students who attended all the very best for the future.

Steve Lloyd (Director)
Department of Physics
Queen Mary & Westfield College

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LECTURE COURSES

INTRODUCTION TO QUANTUM FIELD THEORY AND GUAGE THEORIES

By Dr D C Dunbar
University of Wales, Swansea

Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1996

Introduction to Quantum Field Theory and Gauge Theories

David C. Dunbar

University of Wales Swansea

Contents

Acknowledgments:

In preparing these lectures I have extensively "borrowed" ideas from the equivalent courses given by previous speakers especially those of Ian Halliday and Ken Barnes. In places, this "borrowing" is close to complete. These notes are more extensive in places than what was actually discussed during the lecture course. In particular the issue of Gauge fixing was not mentioned during the lectures although I include it here for completeness. I have also taken into account the "consensus" which was reached on some of the signs.

Finally, I would like to thank Steve Lloyd for his huge efforts in running the school sucessfully, Ann Roberts for organising things impeccably again and the students for "hanging in" through the rather fast schedule and for still finding time for post-midnight aquatic excursions.

Feb 5th 1997

Introduction

The purpose of this course is twofold.

Firstly, it is provide a simple introduction to quantum field theory starting from, roughly, your undergraduate quantum mechanics course. Since you no doubt come from a very varied background this is not particularly easy and I guess the beginning material will be fairly familiar to many of you. To ensure a level "playing field" I will assume only that you are all familiar with the distributed prerequisites. I hope you are! The intended endpoint will be to enable you to take a general field theory and write down the appropriate Feynman rules which are used to evaluate scattering amplitudes. There are two formalisms commonly used for this. The simplest *for a simple theory* is the "Canonical quantisation" whereas the more modern approach is to use the "Path Integral Formulation". I will cover both during the course although the Path Integral Formulation will be done rather heuristically.

The second theme will be to consider the quantisation of gauge theories. For various reasons this is not completely a trivial application of general quantum field theory methods. Hopefully this will connect up to the other courses at this school.

1. Classical Formulations of Dynamics

There are three "equivalent" but different formulations of classical mechanics which I will consider here,

- Newtonian
- Lagrangian
- Hamiltonian

I will illustrate these formulations with a specific example - the simple pendulum, which approximates to a harmonic oscillator when the perturbations are small. The ideal pendulum which we consider here is an object of mass m described by its positions x and y connected to the point $(0,0)$ by a rigid string. This is an example of a constrained system because x and y are forced to satisfy the constraint $x^2+y^2=L^2$ where L is the length of the string. The object could equivalently be described by the angle θ which is a function of x, y given by $\tan\theta=-x/y$.

• Firstly consider Newtonian Mechanics. Newtonian mechanics are only valid if we consider inertial coordinates. In this case good coordinates are $\underline{x}=(x,y)$ and *not* θ whence we have Newton's equations

$$m\frac{d^2\underline{x}}{dt^2}=\underline{F} \tag{1.1}$$

Newton's equations reduce to a pair of second order coordinates. To these equations we have to explicitly insert the forces applied by the string.

• Next we consider the Lagrangian method. For Lagrange an important difference is that any coordinates will do not merely inertial ones. Thus we are free to describe the pendulum using θ. In general a system will be described by coordinates q_r. We construct the *Lagrangian* from the kinetic (T) and potential (V) energy terms $L=T-V$. Lagrange's equations in terms of L are

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{q}_r}\right]-\frac{\partial}{\partial q_r}L=0 \tag{1.2}$$

For the simple pendulum, if we use θ as a coordinate Lagrange's equation produces a single second order equation. The advantage over Newton's method lies in the simplicity in the way which constraints may be applied.

• We now turn to the Hamiltonian method. The idea is to work with first order differential equations rather than second order equations. Suppose we define

$$p_r\equiv\frac{\partial L}{\partial\dot{q}_r} \tag{1.3}$$

then we can write Lagrange's equations as

$$\frac{dp_r}{dt}=\frac{\partial L}{\partial q_r} \tag{1.4}$$

For a system with Kinetic term

$$T=\sum_r\frac{1}{2}m_r\dot{q}_r^2 \tag{1.5}$$

then p_r is just the normal momentum. The Lagrangian is a function of q_r and $\dot{q}_r$. We wish to change variables from $q, \dot{q}$ to q, p. (This is a very close analogy to what happens in a thermodynamic system when changing variables from V, S to V, T.) Examine the response of L to a small change in q_r and $\dot{q}_r$,

$$\begin{aligned} \delta L &= \sum_r \left(\frac{\partial L}{\partial q_r} \delta q_r + \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r \right) \\ &= \sum_r \left(\dot{p}_r \delta q_r + p_r \delta \dot{q}_r \right) \end{aligned} \tag{1.6}$$

by eqs.(1.3)and (1.4). We can, by adding and subtracting $\sum_r \dot{q}_r \delta p_r$, rewrite this as

$$= \delta \left(\sum_r p_r \dot{q}_r \right) + \sum_r \left(\dot{p}_r \delta q_r - \dot{q}_r \delta p_r \right) \tag{1.7}$$

So that by shuffling terms we obtain

$$\delta \left(-L + \sum_r p_r \dot{q}_r \right) = \sum_r \dot{q}_r \delta p_r - \sum_r \dot{p}_r \delta q_r \tag{1.8}$$

So we have obtained a quantity whose responses are in terms of δp_r and δq_r. This is the Hamiltonian. It is given, in general, in terms of the Lagrangian by

$$H = \sum_r p_r \dot{q}_r - L \tag{1.9}$$

The Hamiltonian is to be thought of as a function of q_r and p_r only. If $T \sim \dot{q}^2$ and $V = V(q)$, as is the case in many situations, then $H = T + V$. However the above expression is the more general. The Hamiltonian equations are then, from (1.8)

$$\begin{aligned} \dot{q}_r &= \frac{\partial H}{\partial p_r} \\ \dot{p}_r &= - \frac{\partial H}{\partial q_r} \end{aligned} \tag{1.10}$$

This is a very similar to the situation in thermodynamics if we change from the energy, E, satisfying $dE = TdS - PdV$ where E is thought of as a function of S, V to the Free energy F which is thought of as a function of T, V and $dF = -SdT - PdV$. Recall that the relationship between E and F is $F = E - ST$. In fact, the correct way of thinking about this is to regard thermodynamics as a dynamical system whence the change from E to F is precisely a change such as from L to H. The Hamiltonian system is particularly useful when we consider quantum mechanics because q and p become non-commuting operators - something which makes sense if we use $H(p, q)$ but which requires more thought if we use $L(q, \dot{q})$. For our simple pendulum, Hamiltonian dynamics will produce a pair of first order equations.

Before leaving Hamiltonian mechanics, let is define the Poisson Bracket of any two functions of p and q. Let f and g be any functions of p, q then

$$\left\{f.g\right\} = \sum_r \left(\frac{\partial f}{\partial q_r}\frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial g}{\partial q_r}\right) \tag{1.11}$$

The Poisson bracket of the variables q_i and p_j are then

$$\begin{aligned} \left\{q_i, q_j\right\} &= 0 \\ \left\{q_i, p_j\right\} &= \delta_{ij} \\ \left\{p_i, p_j\right\} &= 0 \end{aligned} \tag{1.12}$$

A *Canonical* change of coordinates is a change from p, q to coordinates $Q(p, q)$ and $P(p, q)$ which maintain the above Poisson brackets. Hamiltonian dynamics is invariant under such canonical transformations. (As an extremely nasty technical point, Quantum mechanics is *not*. Thus there are many quantisations of the same classical system , in principle.)

The best known way of quantising a classical system uses the Hamiltonian formalisms, replaces q_r and p_r by operators and replacing the Poisson brackets by commutators

$$\left\{\cdots\right\} \rightarrow \left[\cdots\right]/i\hbar \tag{1.13}$$

2. Quantum Pictures

2.1 The Dirac or Interaction Picture

In the prerequisites, the are two equivalent pictures of Quantum mechanics: 1) the Schrödinger picture where the wavefunction is time dependent and the operators not and 2) the Heisenberg picture where the wavefunction is time-independent and the time-dependence is carried by the operators. I will introduce a third picture which is called the Dirac picture or, frequently, the interaction picture. First we set the scene. Take a typical situation where the Hamiltonian of a system is described as a "solvable piece" H_0 and a "small perturbation piece" H_I.

$$\hat{H} = \hat{H}_0 + \hat{H}_I \tag{2.1}$$

Actually the interaction picture doesn't care whether H_I is small or not but is really only useful when it is. One of the depressing/hopeful features of physics is how few problems have been solved exactly in quantum mechanics. There are actually only two. The first is the simple harmonic oscillator, the second is the hydrogen atom. (a third should or should not be added to this according to taste - it is the two dimensional Ising model.) All other cases which have been solved exactly are equivalent to these two cases. Free Field theory (non-interacting particles) is, as we will see, solvable because it can be related to a sum of independent harmonic oscillators. It is also amazing how far we have taken physics with just these few examples! Perhaps someday someone will solve a further model and physics will advance enormously.

Since there is so little we can solve exactly a great deal of effort has gone into developing approximate methods to calculate. The methods I will develop here are for calculating matrix elements and will be perturbative in the (assumed) small perturbation H_I. These have proved enormously successful (but don't answer all questions..) For a given operator $\hat{O}$, we can define the interaction picture operator $\hat{O}_I$ in terms of the Schrödinger operator by

$$\begin{aligned} \hat{O}_I =& e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t} \\ =& e^{i\hat{H}_0 t}e^{-i\hat{H}t}\hat{O}_H e^{i\hat{H}t}e^{-i\hat{H}_0 t} \\ =& \hat{U}(t)\hat{O}_H\hat{U}^{-1}(t) \end{aligned} \tag{2.2}$$

(We set $\hbar = 1$ unless explicitly stated otherwise - it is always a useful exercise to reinsert $\hbar$ in equations.) The operator

$$\hat{U}(t) \equiv e^{i\hat{H}_0 t}e^{-i\hat{H}t} \tag{2.3}$$

will be critical in what follows. In the case where $H_I = 0$ the interaction picture reduces to the Heisenberg picture and $U(t) = 1$. We must make a similar definition for the states in the Dirac picture

$$|a, t\rangle_I = e^{i\hat{H}_0 t}|a, t\rangle_S = \hat{U}(t)|a\rangle_H \tag{2.4}$$

Note that the Dirac picture states contain a time dependence. Since the operators are transformed as if in the Heisenberg picture for H_0 we have

$$i\frac{\partial}{\partial t}\hat{O}_I(t) = [\hat{O}_I(t), \hat{H}_0] \tag{2.5}$$

To calculate in the interaction picture we need to evaluate $\hat{U}(t)$. *It is this object which will be the focus of perturbation theory.* We have

$$
\begin{aligned}
i\frac{\partial}{\partial t}\hat{U}(t) =& -\hat{H}_0 e^{i\hat{H}_0 t}e^{-i\hat{H}t} + e^{i\hat{H}_0 t}e^{-i\hat{H}t}\hat{H} \\
=& e^{i\hat{H}_0 t}\hat{H}_I e^{-i\hat{H}t} \\
=& (\hat{H}_I)_I\hat{U}(t)
\end{aligned}
\tag{2.6}
$$

where the confusing notation $(\hat{H}_I)_I$ denotes that the *operator* H_I has been transformed into the interaction picture. Clearly if H_I is a function of operators, $H_I(\hat{O}^j)$, then $(\hat{H}_I)_I = H_I(\hat{O}^j_I)$.

We are now in a position to solve this equation perturbatively, always assuming that H_I forms a small perturbation. Expanding $U(t)$ as a series,

$$
U(t) = 1 + U_1 + U_2 + U_3 + \cdots \tag{2.7}
$$

We can then substitute this into the equation for $U(t)$ and solve order by order. We find for U_1,

$$
i\frac{\partial}{\partial t}U_1 = \hat{H}_I(t) \tag{2.8}
$$

which can be solved to give

$$
U_1 = -i\int_0^t \hat{H}_I(t_1)dt_1 \tag{2.9}
$$

and for U_2

$$
i\frac{\partial}{\partial t}U_2 = \hat{H}_I(t)U_1(t) \tag{2.10}
$$

giving

$$
U_2 = (-i)^2\int_0^t dt_2\int_0^{t_2} dt_1\hat{H}_I(t_2)\hat{H}_I(t_1) \tag{2.11}
$$

From this we can guess the rest (or prove recursively)

$$
U_n = (-i)^n\int_0^t dt_n\int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} dt_1\hat{H}_I(t_n)\hat{H}_I(t_{n-1})\cdots\hat{H}_I(t_2)\hat{H}_I(t_1) \tag{2.12}
$$

Notice that in the above $t_n > t_{n-1} > \ldots t_2 > t_1$. This can all be massaged into a more standard form. We define the *time ordered product* of any two operators by

$$
\begin{aligned}
T(\hat{A}(t_1),\hat{B}(t_2)) =& \hat{A}(t_1)\hat{B}(t_2);\;\; t_1 > t_2 \\
=& \hat{B}(t_2)\hat{A}(t_1);\;\; t_2 > t_1
\end{aligned}
\tag{2.13}
$$

Note that within a time ordered product we can commute two operators as we like. Now the expression for U_2 may be written

$$
(-i)^2\int_0^t dt_2\int_0^{t_2} dt_1\hat{H}_I(t_2)\hat{H}_I(t_1) = \frac{(-i)^2}{2}\int_0^t dt_2\int_0^t dt_1 T(\hat{H}_I(t_2),\hat{H}_I(t_1)) \tag{2.14}
$$

where the integrations now both run from 0 to t. The times ordered product ensures that the ordering of operators is as before and the factor of 1/2 comes because the integral now "overcounts". Similarly we obtain,

$$U_n = \frac{(-i)^n}{n!} \int_0^t \prod_i dt_i T(\hat{H}_I(t_n), \hat{H}_I(t_{n-1}), \cdots, \hat{H}_I(t_2), \hat{H}_I(t_1)) \tag{2.15}$$

We are now in a position to formally sum the contributions into an exponential,

$$U(t) = T(exp(-i \int_0^t \hat{H}_I(t) dt)) \tag{2.16}$$

This is in many senses a formal solution. As we will see later the perturbative evaluation typically involves finding U_1, U_2 themselves. We will spend a considerable effort in evaluating the U_i operators later.

2.2 Lagrangian Quantum Mechanics and the Path Integral

We now turn to the second distinct part of this section on Quantum mechanics. This will involve a formulation of quantum mechanics which involves the Lagrangian rather than the Hamiltonian. We will present this for a single coordinate q and momentum p. We will take two steps later: firstly to consider q as a vector of coordinates and secondly to take it as a field. We will initially work with a simplified Hamiltonian,

$$\hat{H}(\hat{p}, \hat{q}) = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{2.17}$$

Recall that we can consider eigenstates or *either* position $|q\rangle$ satisfying $\hat{q}|q\rangle = q|q\rangle$ or momentum $|p\rangle$ satisfying $\hat{p}|p\rangle = p|p\rangle$ but we cannot have simultaneous eigenstates. In fact the momentum and position eigenstates can be expressed in terms of each other via

$$|q\rangle = \int \frac{dp}{2\pi} e^{-ipq} |p\rangle \,, \quad |p\rangle = \int dq e^{ipq} |q\rangle \tag{2.18}$$

We consider the amplitude for a particle to start at initial point q_i at time $t = t_i$ and end up at point q_f at $t = t_f$. In the Schrödinger picture this is

$$A = \langle q_f | e^{-i\hat{H}t} | q_i \rangle \tag{2.19}$$

where $|q\rangle$ are the time independent eigenstates of $\hat{q}$ and we take $t_i = 0, t_f = t$. The following manipulation of this amplitude is due to Feynman originally. We split up the time interval t into a large number, n, of small steps of length $\Delta = (t_f - t_i)/n$. Then, trivially,

$$e^{-i\hat{H}t} = e^{-i\hat{H}\Delta} \cdot e^{-i\hat{H}\Delta} \cdot e^{-i\hat{H}\Delta} \cdots e^{-i\hat{H}\Delta} \tag{2.20}$$

and

$$A = \langle q_f | e^{-i\hat{H}\Delta} \cdot e^{-i\hat{H}\Delta} \cdot e^{-i\hat{H}\Delta} \cdots e^{-i\hat{H}\Delta} | q_i \rangle \tag{2.21}$$

In between the terms we now insert representations of one (quantum mechanically)

$$\begin{aligned}\int dq|q\rangle\langle q| &=1\\ \int \frac{dp}{2\pi}|p\rangle\langle p| &=1\end{aligned}\tag{2.22}$$

to obtain the following expression for A,

$$\begin{aligned}\int_{q_i,p_i} \langle q_f|p_n\rangle\langle p_n|e^{-i\hat{H}\Delta}|q_{n-1}\rangle\langle q_{n-1}|p_{n-1}\rangle\langle p_{n-1}|e^{-i\hat{H}\Delta}|q_{n-2}\rangle\\ \times\langle q_{n-2}|p_{n-2}\rangle\cdots|q_1\rangle\langle q_1|p_1\rangle\langle p_1|e^{-i\hat{H}\Delta}|q_i\rangle\end{aligned}\tag{2.23}$$

In the above we may make the replacement

$$\langle q_i|p_i\rangle = e^{iq_ip_i}\tag{2.24}$$

We may also evaluate approximately

$$\begin{aligned}\langle p_n|e^{-i\hat{H}\Delta}|q_{n-1}\rangle \sim&\langle p_n|(1-i\hat{H}(\hat{p},\hat{q})\Delta)|q_{n-1}\rangle\\ =&\langle p_n|(1-iH(p_n,q_{n-1})\Delta)|q_{n-1}\rangle\\ =&e^{-iH(p_n,q_{n-1})\Delta}e^{-ip_nq_{n-1}}\end{aligned}\tag{2.25}$$

where we are using the fact that Δ is small and the form of $\hat{H}$. Note that we have turned operators into numbers in the above. We can now rewrite the amplitude and take the limit $n\rightarrow\infty$,

$$\begin{aligned}A =& \lim_{n\rightarrow\infty}\int\prod_{i=1}^{n}dp_i\prod_{i=1}^{n-1}dq_i\left\{\prod_i e^{-iH(p_i,q_{i-1})\Delta}e^{-ip_iq_{i-1}}e^{ip_iq_i}\right\}\\ =& \lim_{n\rightarrow\infty}\int\prod_{i=1}^{n}dp_i\prod_{i=1}^{n-1}dq_i\left\{\exp\Big(i\sum_i\Delta(\frac{(q_i-q_{i-1})p_i}{\Delta}-H(p_i,q_{i-1})\Big)\right\}\\ \equiv& \int[dq][dp]e^{i\int dt[p\dot{q}-H]}\end{aligned}\tag{2.26}$$

The last line is the *Path Integral formulation.* It is an interesting question what the symbols mean in this equation!. In the integrations all intermediate values of p,q contribute. We can interpret this as an integral over all possible paths a particle may take between q_i and q_f. This expression is commonly used but is not quite the Lagrangian formalism. To obtain this we must evaluate the dp_j integrals at the penultimate step (before $n\rightarrow\infty$). The integral is assuming the simplified form for $H=p^2/2m+V(q)$,

$$\begin{aligned}\int dp_ie^{-i\frac{p_i^2}{2m}\Delta}e^{ip_i(q_i-q_{i-1})} =& e^{i\frac{(q_i-q_{i-1})^2m}{2\Delta}}\\ &\sim e^{i\Delta\frac{m\dot{q}^2}{2}}\end{aligned}\tag{2.27}$$

where we approximate $(q_i - q_{i-1})$ by $\dot{q}_i\Delta$. Using this we can again take $n \to \infty$ to obtain the expression

$$\int [dq] e^{i\int dt L(q,\dot{q})} \tag{2.28}$$

This Formulation of Quantum mechanics is one we will use extensively. A useful object is the *Action*, S, defined as

$$S = \int dt L \tag{2.29}$$

whence the path integral is

$$\int [dq] e^{iS/\hbar} \tag{2.30}$$

(just for fun I reinserted $\hbar$ in this equation.) The classical significance of S is that it may be used to obtain the equations of motion. Lagrange's equations arise by demanding the Action is at an extremal value. That is, at the classical path

$$\delta S = 0 \tag{2.31}$$

If we have path $q(t)$ and we vary by $\delta q(t)$ then

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}$$

since

$$\delta \dot{q} = \frac{d}{dt}\delta q \tag{2.32}$$

we may partial integrate to find

$$\delta S = \int dt \Big[-\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) + \frac{\partial L}{\partial q}\Big]\delta q(t) + [\delta q(t_f) - \delta q(t_i)] \tag{2.33}$$

I have included the boundary term for completeness. A correct statement of the principle is the the classical path is the one which extremises the action with the variation zero at initial and final state. Demanding $\delta S = 0$ for arbitrary such $\delta q(t)$ then forces Lagrange's equation.

A common way to express the path integral, is to say that all paths are summed over, weighted by $e^{i \times \text{action}}$. This has a certain appeal. Think about what happens as $\hbar \to 0$. This formulation has strong analogies with statistical mechanics where the partition function is the sum over all configurations weighted by the energy

$$Z \sim \sum_i e^{-E_i/kT} \tag{2.34}$$

however the factor of i should never be forgotten!

3. Field Theory: A Free Boson

3.1 The classical treatment

In this section we will examine our first Field Theory, look at it initially and then quantise and solve. This will only be possible because it is a non-interacting field theory. We will consider a field, $\phi(x)$. That is an object which has a value at every point in space. This is unlike the harmonic oscillator where, although wavefunctions depend on space these are merely the probability of observing a particle at that point. A field configuration is then described by a (continuous) infinity of real numbers as opposed to the single number describing a harmonic oscillator. This infinity will, of course, complicate the mathematics. We can regard this as the transition from a finite system described by q_r to the case where the r-index becomes the continuous x,

$$q_r \rightarrow \phi(x) \tag{3.1}$$

In this limit we we have to replace

$$\sum_r \rightarrow \int dx\ ,\ \delta_{rs} \rightarrow \delta(x-y) \tag{3.2}$$

We can easily postulate the Kinetic energy of such a field to be

$$T = \int d^3x \frac{1}{2}(\frac{\partial\phi(x,t)}{\partial t})^2 \tag{3.3}$$

This gives the field a Kinetic energy at each point. The potential term we take as

$$V = \frac{m^2}{2}\int d^3x\phi^2(x,t) + \int d^3x \frac{c^2}{2}\sum_{i=1}^{3}(\frac{\partial\phi(x,t)}{\partial x^i})^2 \tag{3.4}$$

The "mass term" $\phi^2(x,t)$ is easy to understand. The remaining kinetic term $(\frac{d\phi(x,t)}{dx^i})^2$ is necessary by Lorentz invariance. (Or one may consider the model of an electric sheet with potential energy, consider small perturbations and then evaluate the potential energy: a term such as this then appears.) The c should be the speed of light for Lorentz invariance.

From this we may construct the Lagrangian,

$$L = \int d^3x \left[\frac{1}{2}(\frac{\partial\phi(x,t)}{\partial t})^2 - \frac{c^2}{2}\sum_{i=1}^{3}(\frac{\partial\phi(x,t)}{\partial x^i})^2 - \frac{m^2}{2}\phi^2(x,t)\right] \tag{3.5}$$

which we may apply Lagrange's method to. For fields we often speak of the Lagrangian density $\mathcal{L}$ where $L = \int d^3x\mathcal{L}$. Before doing so we will rewrite this form in a more Lorentz covariant manner. Define a four-vector x^μ where $\mu = 0\cdots3$ and $x^0 = t$. We henceforth set $c = 1$ (otherwise $\hbar$ would be jealous). Then

$$\begin{aligned}\partial_\mu\phi =& \frac{\partial\phi}{\partial t} : \mu = 0\\ =& \frac{\partial\phi}{\partial x^i} : \mu = i\end{aligned} \tag{3.6}$$

It is a fundamental fact of relativity that x^μ and $\partial^\mu\phi$ are 4-vectors. I.e. they transform in a well behaved fashion under Lorentz transformations. Four vectors are similar to normal vectors if one remembers the important minus signs. From the vector x^μ one can define a "co-vector" x_μ by $x_0 = x^0, , x_i = -x^i, i = 1,2,3$. In more fancy language $x_\mu = \sum_\nu g_{\mu\nu}x^\nu$ where $g_{\mu\nu}$ are the elements of a 4×4 matrix g. In this case $g = \mathrm{diag}(+1,-1,-1,-1)$. I mention this to introduce ***the Einstein summation convention*** where we write $x_\mu = g_{\mu\nu}x^\nu$ and the summation is understood. With this convention, $x_\mu x^\mu = t^2 - x^2 - y^2 - z^2$.

The dot product of two four vectors,

$$A \cdot B \equiv A_\mu B^\mu = A_0 B_0 - \sum_{i=1}^{3} A_i B_i \tag{3.7}$$

is invariant under Lorentz transformations. The action S is

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \; - \frac{m^2}{2}\phi^2\right] \tag{3.8}$$

which since the measure $d^4x \equiv dtd^3x$ is invariant under Lorentz transformation. I am actually slipping in a very very important concept here. Namely that symmetries of the theory are *Manifest* in the action or Lagrangian. (By contrast the Hamiltonian formulation also gives Lorentz invariant behaviour but it is not manifestly Lorentz invariant.) Since symmetries are very important, the Lagrangian formalism is a good place to study them. We can define the momenta conjugate to the field ϕ

$$\Pi(\underline{x}, t) = \frac{\partial \mathcal{L}}{\partial[\partial_0\phi(\underline{x}, t)]} \; = \partial^0\phi \tag{3.9}$$

whence the Hamiltonian becomes

$$H = \int d^3x \left[\frac{\Pi^2(\underline{x}, t)}{2} + \frac{1}{2}\Big(\frac{\partial\phi(\underline{x}, t)}{\partial x^i}\Big)^2 + \frac{m^2}{2}\phi^2(\underline{x}, t)\right] \tag{3.10}$$

Notice that this is *not* invariant under Lorentz transformations. let us now solve this system classically now. First we must present Lagrange's equations for a field. Because of the space derivatives $\partial\phi/\partial x$ the equations become modified. (We could see this by returning to S and examining the conditions that S is extremised.)

$$\frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right] + \frac{\partial}{\partial x^i}\left[\frac{\partial \mathcal{L}}{\partial(\frac{\partial\phi}{\partial x^i})}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{3.11}$$

(where the sum over i is implied). For our Lagrangian this yields

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x_i^2} + m^2\phi = 0 \tag{3.12}$$

or

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{3.13}$$

We now find the general solution to this equation. Since the system is linear in ϕ the sum of any two solutions is also a solution. Try a plane wave solution,

$$\phi(\underline{x}, t) = Ae^{i(\underline{k}\cdot\underline{x}-\omega t)} \tag{3.14}$$

then substituting this into eq.(3.12) gives

$$A\left[-\omega^2 + \underline{k}^2 + m^2\right]e^{i(\underline{k}\cdot\underline{x}-\omega t)} = 0 \tag{3.15}$$

so that the trial form will be a solution provided

$$\omega(k) = \pm\sqrt{\underline{k}^2 + m^2} \tag{3.16}$$

Notice that there are two solutions. From now on take $\omega(k)$ to denote the positive one. The general solution will be

$$\phi(\underline{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)}\left[a(k)e^{i(\underline{k}\cdot\underline{x}-\omega t)} + a^*(-k)e^{i(\underline{k}\cdot\underline{x}+\omega t)}\right] \tag{3.17}$$

The $a(k)$ and $a^*(k)$ are constants. We have also imposed the condition $\phi^* = \phi$ which is necessary for a real field. For purely conventional reasons we have chosen the normalisations given. A classical problem would now just degenerate to finding the $a(k)$ and $a^*(k)$ by e.g., examining the boundary conditions. To finish this section on the classical properties note that

$$\Pi(\underline{x}, t) = \int \frac{d^3k}{(2\pi)^3 2}\left[-ia(k)e^{i(\underline{k}\cdot\underline{x}-\omega t)} + ia^*(-k)e^{i(\underline{k}\cdot\underline{x}+\omega t)}\right] \tag{3.18}$$

3.2 The Quantum theory

We will now quantise the theory. The field variables are $\phi(\underline{x}, t)$ and $\Pi(\underline{x}, t)$. we must decide upon the commutation relations for these objects. That is, we want the appropriate generalisations of (1.12) for the case where the q and p now are a continuous infinite set. These are

$$\begin{aligned}
\left[\hat{\phi}(\underline{x}, t),\ \hat{\phi}(\underline{y}, t)\right] &= 0 \\
\left[\hat{\Pi}(\underline{x}, t),\ \hat{\phi}(\underline{y}, t)\right] &= -i\delta^3(\underline{x} - \underline{y}) \\
\left[\hat{\Pi}(\underline{x}, t),\ \hat{\Pi}(\underline{y}, t)\right] &= 0
\end{aligned} \tag{3.19}$$

This looks reasonable except that the δ_{ij} present for a discrete number of coordinate is replaced by the Dirac-δ function. I'll try to elucidate this in an exercise.

Let us now, in the Heisenberg picture examine the equations of motion for $\hat{\phi}$ and $\hat{\Pi}$,

$$
\begin{aligned}
i\dot{\hat{\phi}}(\underline{x},t) &= \left[\hat{\phi}(\underline{x},t),\hat{H}\right] \\
&= \int d^3y\left[\hat{\phi}(\underline{x},t),\frac{\hat{\Pi}^2(y,t)}{2}\right] \\
&= \int d^3y\frac{1}{2}\left(\left[\hat{\phi}(\underline{x},t),\hat{\Pi}(y,t)\right]\hat{\Pi}(y,t)+\hat{\Pi}(y,t)\left[\hat{\phi}(\underline{x},t),\hat{\Pi}(y,t)\right]\right) \\
&= \int d^3y i\delta^3(\underline{x}-\underline{y})\hat{\Pi}(y,t) \\
&= i\hat{\Pi}(x,t)
\end{aligned}
\tag{3.20}
$$

and for $\hat{\Pi}$,

$$
\begin{aligned}
i\dot{\hat{\Pi}}(\underline{x},t) &= \left[\hat{\Pi}(\underline{x},t),\hat{H}\right] \\
&= \int d^3y\sum_i\left[\hat{\Pi}(\underline{x},t),\frac{\partial\hat{\phi}(\underline{y},t)}{\partial y_i}\right]\frac{\partial\hat{\phi}(\underline{y},t)}{\partial y_i}+\int d^3ym^2\left[\hat{\Pi}(\underline{x},t),\hat{\phi}(\underline{y},t)\right]\hat{\phi}(\underline{y},t) \\
&= \int d^3y\left\{-i\sum_i\frac{\partial}{\partial y_i}\delta^3(\underline{x}-\underline{y})\frac{\partial\hat{\phi}(\underline{y},t)}{\partial y_i}-im^2\hat{\phi}(\underline{y},t)\delta^3(\underline{x}-\underline{y})\right\} \\
&= i\frac{\partial^2\hat{\phi}(\underline{x},t)}{\partial x^2}-im^2\hat{\phi}(\underline{x},t)
\end{aligned}
\tag{3.21}
$$

We can combine and rewrite these two equations as

$$
\begin{aligned}
\frac{\partial^2\hat{\phi}(\underline{x},t)}{\partial t^2} &= \nabla^2\hat{\phi}-m^2\hat{\phi} \\
\hat{\Pi}(\underline{x},t) &= \dot{\hat{\phi}}(\underline{x},t)
\end{aligned}
\tag{3.22}
$$

which is just as before. However, now these are operator equations with the solution

$$
\hat{\phi}(\underline{x},t)=\int\frac{d^3k}{(2\pi)^3 2\omega(k)}\left[\hat{a}(k)e^{i(\underline{k}\cdot\underline{x}-\omega t)}+\hat{a}^\dagger(-k)e^{i(\underline{k}\cdot\underline{x}+\omega t)}\right]
\tag{3.23}
$$

Now the $\hat{a}$ and $\hat{a}^\dagger$ are operators. This can be rewritten using four vectors in the form

$$
\hat{\phi}(\underline{x},t)=\int\frac{d^3k}{(2\pi)^3 2\omega(k)}\left[\hat{a}(k)e^{-ik\cdot x}+\hat{a}^\dagger(k)e^{ik\cdot x}\right]
\tag{3.24}
$$

Where the four vector k^μ is formed from ω and $\underline{k}$. (It requires a little care and relabelling under the integral sign to show this.) We can deduce the commutation relationships for them from those for $\hat{\phi}$ and $\hat{\Pi}$,

$$
\begin{aligned}
\left[\hat{a}(\underline{k}),\hat{a}(\underline{k}')\right] &= 0 \\
\left[\hat{a}(\underline{k}),\hat{a}^\dagger(\underline{k}')\right] &= (2\pi)^3.2\omega.\delta^3(\underline{k}-\underline{k}') \\
\left[\hat{a}^\dagger(\underline{k}),\hat{a}^\dagger(\underline{k}')\right] &= 0
\end{aligned}
\tag{3.25}
$$

Thus as promised we find an infinite set of harmonic oscillators labeled by the momenta $\underline{k}$. If we substitute the forms for $\hat{\phi}$ into the Hamiltonian we find (tediously)

$$\hat{H} = \int \frac{d^3k}{2(2\pi)^3}\hat{a}^\dagger(\underline{k})\hat{a}(\underline{k}) + \text{const.} \tag{3.26}$$

So that the Hamiltonian is a sum of independent harmonic oscillators. We can thus apply our knowledge of such objects to this case. If we denote the ground state by $|0\rangle$ then we will form states by applying raising operators to the vacuum. $\hat{a}^\dagger(\underline{k})$ will create a particle of momentum $\underline{k}$ and energy $\hbar\omega(k)$. (try reinserting the $\hbar$s!) We can easily check

$$\hat{H}\hat{a}^\dagger(\underline{k})|0\rangle = \omega(k)a^\dagger(\underline{k})|0\rangle \tag{3.27}$$

Similarly we may create the two particle states

$$a^\dagger(\underline{k}_1)a^\dagger(\underline{k}_2)|0\rangle \tag{3.28}$$

etc, etc. Notice that because of the commutation relationships that the 2-particles states are even under exchange. That means our system is a system of non-interacting bosons.

We have taken ϕ to be a real field. In practise we wish to consider complex fields. Suppose we have two real fields of the same mass,

$$S = \int d^4x \sum_{r=1}^{2}\left(\frac{1}{2}\partial^\mu\phi_r\partial_\mu\phi_r - \frac{m^2}{2}\phi_r^2\right) \tag{3.29}$$

then we may define the complex field

$$\begin{aligned} \chi &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \chi^\dagger &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \end{aligned} \tag{3.30}$$

Then we may easily check

$$S = \int d^4x\left[\partial^\mu\chi^\dagger\partial_\mu\chi - m^2\chi^\dagger\chi\right] \tag{3.31}$$

Solving Heisenberg's equations as before we find

$$\hat{\chi}(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)}\left[\hat{b}(\underline{k})e^{-ik\cdot x} + \hat{d}^\dagger(\underline{k})e^{ik\cdot x}\right] \tag{3.32}$$

where $\hat{b}$ and $\hat{d}$ are now independent because χ is a complex field. these must have commutation relationships

$$\begin{aligned} \left[\hat{b}(\underline{k}), \hat{b}^\dagger(\underline{k}')\right] &= (2\pi)^3.2\omega.\delta^3(\underline{k} - \underline{k}') \\ \left[\hat{d}(\underline{k}), \hat{d}^\dagger(\underline{k}')\right] &= (2\pi)^3.2\omega.\delta^3(\underline{k} - \underline{k}') \end{aligned} \tag{3.33}$$

all others being zero, with the Hamiltonian

$$\hat{H} = \int \frac{d^3k}{2(2\pi)^3}\left(\hat{b}^\dagger(\underline{k})\hat{b}(\underline{k}) + \hat{d}^\dagger(\underline{k})\hat{d}(\underline{k}) + \text{const.}\right) \tag{3.34}$$

This is fairly important. So far no fundamental scalars have not been observed experimentally although the standard models as we know it contains a fundamental scalar - the Higgs boson. The Higgs boson is complex rather than real. (if it exists!).

4. An interacting Boson Theory: Canonical Quantisation and Feynman Diagrams

We are now in a position to consider an interacting theory. As an example consider a theory which contains a real scalar ϕ and a complex scalar χ. The Lagrangian density we take to be

$$\mathcal{L}_\phi + \mathcal{L}_\chi + \mathcal{L}_{int} \tag{4.1}$$

where $\mathcal{L}_\phi$ and $\mathcal{L}_\chi$ are the Lagrangian densities for a free real and complex scalar (see (3.8) and (3.31)). The interaction term we take

$$\mathcal{L}_{int} = -g\hat{\chi}^\dagger\hat{\chi}\hat{\phi} \tag{4.2}$$

We now work with this system. The Heisenberg equations (which we could solve in the non-interacting case) are

$$\begin{aligned}\left(\partial^2 + m_\phi^2\right)\hat{\phi} + g\hat{\chi}^\dagger\hat{\chi} &= 0\\ \left(\partial^2 + m_\chi^2\right)\hat{\chi} + g\hat{\phi}\chi &= 0\end{aligned} \tag{4.3}$$

where $\partial^2 = \partial_\mu\partial^\mu$. † These non-linear operator equations have no known solution. We must attack them approximately. As we can see our system *provided g is small* is suited for analysis in the interaction picture. We can split the Hamiltonian into the non-interacting piece H_0 plus the small additional $H_I = g\hat{\chi}^\dagger\hat{\chi}\hat{\phi}$. This will allow us to evaluate transitions and scattering perturbatively.

Recall that in the interaction picture, the crucial object is the operator $\hat{U}(t)$. In lowest order this is

$$\begin{aligned}\hat{U}(t_i, t_f) &= -i\int_{t_i}^{t_f} \hat{H}_I(t)dt\\ &= -ig\int_{t_i}^{t_f} d^4x\hat{\chi}^\dagger\hat{\chi}\hat{\phi}\end{aligned} \tag{4.4}$$

We shall use this to examine the transition probability from an initial state containing a single ϕ boson and a final state consisting of a $\chi\chi^\dagger$ pair. We will take the initial time t_i to be $-\infty$ and the final times $t_f = \infty$, we have then,

$$\begin{aligned}|t = -\infty\rangle &= \hat{a}^\dagger(\underline{k})|0\rangle\\ |t = \infty\rangle &= \hat{b}^\dagger(\underline{p})\hat{d}^\dagger(\underline{q})|0\rangle\\ \langle t = \infty| &= \langle 0|b(\underline{p})d(\underline{q})\end{aligned} \tag{4.5}$$

The initial ϕ boson has four momenta k and the final pair of $\chi - \chi^\dagger$ particles have momenta p and q. Recall that in the interaction picture the states evolve with time via the $U(t)$ operator, $|a,t\rangle_I = \hat{U}(t)|a\rangle_H$. Thus the initial state $\hat{a}^\dagger(\underline{k})|0\rangle$ at $t = -\infty$ will evolve into

$$\hat{U}(-\infty, \infty)\hat{a}^\dagger(\underline{k})|0\rangle \tag{4.6}$$

† I have slipped over the issue of how to deal with complex fields. The correct procedure turns out to be to treat χ and $\chi^\dagger$ as independent fields. This can be justified be rewriting χ in terms of it's real components.

(Note that if $H_I = 0$ then the state remains fixed.) The probability that this state at $t = \infty$ is a $\chi\chi^\dagger$ pair is the overlap of this with $\hat{b}^\dagger(\underline{p})\hat{d}^\dagger(\underline{q})|0\rangle$. This is the *matrix element*

$$\langle t = \infty|\hat{U}(-\infty, \infty)|t = -\infty\rangle = \langle 0|\hat{b}(\underline{p})d(\underline{q})\left(g\int d^4x\hat{\chi}^\dagger\hat{\chi}\hat{\phi}\right)\hat{a}^\dagger(\underline{k})|0\rangle \tag{4.7}$$

This probability we now evaluate. Using the expansions for ϕ and χ this is

$$\begin{aligned} &= -ig\int d^4x\int \bar{d}^3p'\int \bar{d}^3k'\int \bar{d}^3q'\langle 0|\hat{b}(\underline{p})d(\underline{q})\left(\hat{d}(\underline{p}')e^{ip'\cdot x} + \hat{b}^\dagger(\underline{p}')e^{-ip'\cdot x}\right) \\ &\quad\times\left(\hat{a}(\underline{k}')e^{ik'\cdot x} + \hat{a}^\dagger(\underline{k}')e^{-ik'\cdot x}\right)\left(\hat{b}(\underline{q}')e^{iq'\cdot x} + \hat{d}^\dagger(\underline{q}')e^{iq'\cdot x}\right)\hat{a}^\dagger(\underline{k})|0\rangle \end{aligned} \tag{4.8}$$

where $\bar{d}^3p = d^3p/2(2\pi)^3\omega$. We will evaluate this by commuting the annihilation operators to the right where they vanish when acting on the vacuum and the creation operators to the left where they vanish when multiplied by $\langle 0|$. Since, for example $\hat{b}$ commutes with $\hat{a}^\dagger$ we can throw away the $\hat{b}(\underline{q}')$ terms. Similarly the $\hat{a}^\dagger(\underline{k}')$ term disappears. (and also the $\hat{d}(\underline{p}')$ with a little more thought) leaving

$$-ig\int d^4x\int \bar{d}^3p'\int \bar{d}^3k'\int \bar{d}^3q'\langle 0|\hat{b}(\underline{p})d(\underline{q})\hat{b}^\dagger(\underline{p}')\hat{a}(\underline{k}')\hat{d}^\dagger(\underline{q}')e^{-ix\cdot(p'+q'-k')}\hat{a}^\dagger(\underline{k})|0\rangle \tag{4.9}$$

We can continue commuting each annihilation operator to the right, obtaining a variety of δ-functions on route. The final result is

$$-ig\int d^4x e^{-i(p+q-k)\cdot x}\langle 0|0\rangle = -ig(2\pi)^4\delta^4(p+q-k)\langle 0|0\rangle \tag{4.10}$$

The δ-function imposes conservation of four-momentum. This is in fact a real perturbative calculation. Notice that it doesn't make a lot of sense unless g is small.

In general, to evaluate to a given order, we need to calculate objects of the form

$$\int dt_1 dt_2\cdots dt_n T(\hat{H}_I(t_1)\hat{H}_I(t_2)\cdots\hat{H}_I(t_n)) \tag{4.11}$$

In principle we can carry out the same procedure as before. This is sandwiching between states and commuting annihilation operators to the right until we obtain some kind of result. There is a very well specified procedure for doing so in a systematic manner which is known as Wick's theorem. The diagrammatic representation of this is more or less the Feynman diagram approach. We will now think a little more generally in terms of operators. Since we wish to have operators with annihilation operators acting on the right we define the *normal ordered* operator to be precisely this. For example consider the composite operator $T(\hat{\phi}(x)\hat{\phi}(y))$ then

$$:\hat{\phi}(x)\hat{\phi}(y): \tag{4.12}$$

is the same operator but with the annihilation operators pushed to the right. $T(\phi(x)\phi(y))$ and $:\phi(x)\phi(y):$ differ by a term which we call the contraction

$$T(\hat{\phi}(x)\hat{\phi}(y)) =: \hat{\phi}(x)\hat{\phi}(y) : +\hat{\phi}(x)\hat{\phi}(y) \tag{4.13}$$

since ϕ is linear in operators and hence $T(\phi(x)\phi(y))$ quadratic the contraction term will be a pure number (that is no operator). We may evaluate this by sandwiching the above equation between $\langle 0|$ and $|0\rangle$ so that

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \hat{\phi}(x)\hat{\phi}(y) \tag{4.14}$$

We now present Wick's theorem which tells us how to evaluate large collection of operators into the normal ordered pieces and the contraction terms. Consider a large class of operators $A, B, C \cdots X, Y, Z$ which are linear in annihilation/creation operators. Then the time ordered product may be expanded,

$$\begin{aligned} T(ABC\cdots XYZ) = &: ABC\cdots XYZ : \\ &+ AB : CD\cdots XYZ : + AC : BD\cdots XYZ : +\text{perms.} \\ &+ AB\ CD\ : E\cdots XYZ : +\text{perms.} \\ &+ \cdots \\ &+ AB\ CD\ \cdots YZ + \text{perms.} \end{aligned} \tag{4.15}$$

(This needs a little modification for fermions.) Now we apply this to the case we are interested in. Namely the decay of a ϕ particle into a $\chi\chi^\dagger$ pair. We need to sandwich the time-ordered products of Hamiltonians

$$\int dt_1 dt_2 \cdots dt_n T(H_I(t_1)H_I(t_2)\cdots H_I(t_n)) \tag{4.16}$$

between the initial and final states to evaluate the matrix element. We have done this for $n = 1$. Let us examine the systematics of $n > 1$. First we define 'initial' and final state operators (also linear in creation operators),

$$|i>= O^i_\phi|0\rangle, \quad |f>= O^f_\chi O^f_{\chi^\dagger}|0\rangle \tag{4.17}$$

(The operator for creating a ϕ-state is in many ways a "sub-operator" of the $\hat{\phi}$ operator.) The first correction we can take as

$$g\langle 0|T\Big(O^f_\chi O^f_{\chi^\dagger}\hat{\phi}(x_1)\hat{\chi}(x_1)\hat{\chi}^\dagger(x_1)O^i_\phi\Big)|0\rangle \tag{4.18}$$

We can evaluate this using Wicks theorem and throwing away all the normal ordered terms since they vanish we sandwiched between $\langle 0|$ and $|0\rangle$. Fortunately a large number of the possible contractions are zero. For example the contraction between a ϕ and a χ field is zero since the operators in ϕ commute with those in χ. Thus we have

$$\phi(x)\chi(y) = \chi(x)\chi(y) = \chi^\dagger(x)\chi^\dagger(y) = 0 \tag{4.19}$$

and the only non-zero contractions will be between pairs of $\hat{\phi}$ operators and pairs of χ and $\chi^\dagger$ operators. It is a very useful exercise to repeat the previous calculation using Wick's theorem. Note that the contraction between a $\hat{\phi}(x)$ operator and an initial state operator is rather simple $\phi(x)\hat{O}^i_\phi = e^{ik\cdot x}$. If we consider the next case the correction is

$$g^2\langle 0|T\Big(O^f_\chi O^f_{\chi^\dagger}\hat{\phi}(x_1)\hat{\chi}(x_1)\hat{\chi}^\dagger(x_1)\hat{\phi}(x_2)\hat{\chi}(x_2)\hat{\chi}^\dagger(x_2)O^i_\phi\Big)|0\rangle \tag{4.20}$$

Since we have an odd number of ϕ terms the contractions must leave a single $\hat{\phi}$ operator which will vanish when sandwiched. Thus the second correction will be identically zero. The third is

$$g^3\langle 0|T\Big(O^f_\chi O^f_{\chi^\dagger}\hat{\phi}(x_1)\hat{\chi}(x_1)\hat{\chi}^\dagger(x_1)\hat{\phi}(x_2)\hat{\chi}(x_2)\hat{\chi}^\dagger(x_2)\hat{\phi}(x_3)\hat{\chi}(x_3)\hat{\chi}^\dagger(x_3)O^i_\phi\Big)|0\rangle \tag{4.21}$$

This will be non-zero and by Wick's theorem will produce a whole splurge of terms. Let us try to organise them. A term will be,

$$O^f_\chi\hat{\chi}^\dagger(x_1)\; O^f_{\chi^\dagger}\hat{\chi}(x_1)\; \hat{\phi}(x_1)\hat{\phi}(x_2)\; \hat{\chi}(x_2)\hat{\chi}^\dagger(x_3)\; \hat{\chi}^\dagger(x_2)\hat{\chi}(x_3)\; \hat{\phi}(x_3)O^i_\phi \tag{4.22}$$

If we draw a diagram with three points x_1, x_2 and x_3 then we can "join the dots" using the contraction terms as labelled lines and obtain a diagram

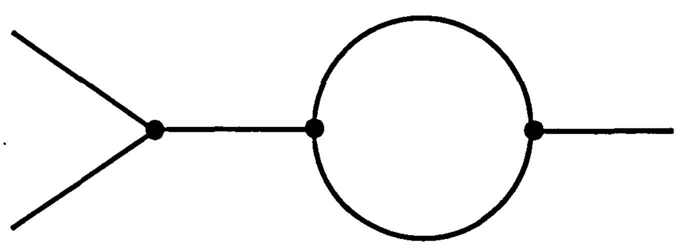

Figure 2. A Feynman Diagram.

Similarly for the other terms we can also draw diagrams. The real trick is, of course, not to do it this way but in reverse. It is much easier to draw diagrams to keep track of contributions than to look after terms. We draw diagrams with the "Feynman rules" which are rules for sewing together vertices with propagators. These may be written down directly from the Lagrangian. In our case we have Hamiltonian $\phi\chi\chi^\dagger$ and the rule for vertices is that we have a three point vertex with one ϕ line, one χ line and one $\chi^\dagger$ line. The general case is easy to see (and to understand in terms of what has gone before). For example if we had

$$H_I = \hat{\phi}^n \tag{4.23}$$

then we would have a n-point vertex. The vertices are joined together with lines to form all possibilities. We can then associate with each diagram the appropriate contribution. The contributions are given in terms of the contractions of pairs of fields. This contraction is known as the Feynman propagator. Let us now evaluate the Feynman propagator for the ϕ field

$$
\begin{aligned}
i\Delta_F(x,y) &\equiv \hat{\phi}(x)\hat{\phi}(y) = \langle 0|T(\phi(x)\phi(y))|0\rangle \\
&= \langle 0|\int \bar{d}k \int \bar{d}q \left(\hat{a}(\underline{k})e^{i(\underline{k}\cdot\underline{x}-\omega(k)t_1)}\right)\left(\hat{a}^\dagger(\underline{q})e^{-i(\underline{q}\cdot\underline{y}-\omega(q)t_2)}\right)|0\rangle
\end{aligned}
\tag{4.24}
$$

(we have dropped the terms giving zero trivially) The two operator terms can be commuted past each other to yield

$$
i\Delta_F(x,y) = \int \bar{d}k \int \bar{d}q (2\pi)^3 2\omega\delta^3(\underline{k}-\underline{q})e^{i(\underline{k}\cdot\underline{x}-\underline{q}\cdot\underline{y})-i(t_1\omega(k)-t_2\omega(q))} \tag{4.25}
$$

The δ-function can now be evaluated. In the above we assumed $t_1 > t_2$ when evaluating. The result in general is

$$
i\Delta_F(x,y) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left[\theta(t_1-t_2)e^{i\underline{k}\cdot(\underline{x}-\underline{y})-i(t_1-t_2)\omega}+\theta(t_2-t_1)e^{-i\underline{k}\cdot(\underline{x}-\underline{y})-i(t_2-t_1)\omega}\right] \tag{4.26}
$$

where $\theta(t) = 1, t > 0$ and $\theta(t) = 0, t < 0$. There is a more Lorentz invariant looking expression for the above which is

$$
i\Delta_F(x,y) = \int \frac{d^4k e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon} \tag{4.27}
$$

where we have slipped into relativistic four vector notation. The proof of the equivalence of these two forms relies upon Cauchy's theorem. For the more mathematically inclined we can prove this by examining the integration in ik_0 and continuing to a complex integration. The poles in the integral occur when

$$
(k^0)^2 - \underline{k}^2 - m^2 + i\epsilon = 0 \tag{4.28}
$$

which happens when $k_0 = \pm\omega(k) \mp i\epsilon$ The integral in the complex ik_0 plane now lies along the real axis with poles lying at $(-\omega(k), +i\epsilon)$ and $(\omega(k), -i\epsilon)$.

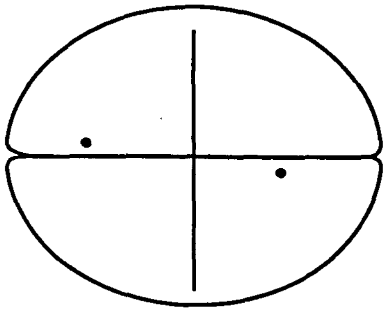

Figure 3. The contour integrations for the Feynman propagator.

We can close the contour with a semi-circle at infinity to obtain a curve which we then apply Cauchy's theorem to. Whether we use the upper or lower hemisphere depends upon whether $t_1 > t_2$ or not. If $t_1 < t_2$ then we close in the upper plane and have to evaluate the residue at $(-\omega(k), +i\epsilon$. The general case can be combined

$$\theta(t_1 - t_2) \int \frac{d^3k}{(2\pi)^3 2\omega} e^{i\underline{k}\cdot(\underline{x}-\underline{y})-i(t_1-t_2)\omega} + \theta(t_2 - t_1) \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-i\underline{k}\cdot(\underline{x}-\underline{y})-i(t_2-t_1)\omega} \quad (4.29)$$

which is as before. We now have a form of the propagator which integrates over d^4k rather than d^3k. We are thus integrating over particles which need not be on mass-shell.

5. Functional Methods

I will now rework some of the results of the previous section but using the path integral approach instead. This is in many ways much slicker. First for a set of discrete coordinates q_i define

$$W[J_i] = \int \prod_i [dq_i] \exp\Big(i \int L(q_i, \dot{q}_i)dt + \sum_j J_j q_j \Big) \tag{5.1}$$

The J_i are dummy variables which will allow us to calculate expectations of q_i etc by derivatives of $W[J]$. For example

$$\frac{\partial W[J_i]}{\partial J_k}\bigg|_{J_i=0} = \int [dq_i] q_k e^{iS} \tag{5.2}$$

We with to extend this concept to a field theory. This means extending $q_i \to \phi(x)$. This gives

$$W[J(x)] = \int [d\phi] \exp\Big(i \int d^4x \mathcal{L} + \int d^4x J(x)\phi(x) \Big) \tag{5.3}$$

Now $W[J(x)]$ is a *functional.* That is something which takes a function and produces a number. Before continuing we must define a functional derivative. Consider a functional $F[J(x)]$ then

$$\frac{\delta F}{\delta J(y)} \equiv \lim_{\epsilon \to 0} \frac{F[J(x) + \epsilon\delta(x-y)] - F[J(x)]}{\epsilon} \tag{5.4}$$

If we consider a simple example,

$$F[J(x)] = \int d^4x J(x)\phi(x) \tag{5.5}$$

then

$$\begin{aligned} \frac{\delta F}{\delta J(y)} &= \lim_{\epsilon \to 0} \int \delta(x-y)\phi(x) \\ &= \phi(y) \end{aligned} \tag{5.6}$$

We now will apply these methods to the theory with Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\frac{\partial\phi}{\partial t})^2 - \sum_{i=1}^{3} \frac{1}{2}(\frac{\partial\phi}{\partial x_i})^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda\phi^4}{4!} \tag{5.7}$$

This Lagrangian has the free part plus an interaction terms ϕ^4. We will consider the free part first. The path integral for the free theory is Gaussian and hence calculable by our favourite integrals. However we must carefully take the $q_i \to \phi(x)$ transition carefully. Recall that we can carry our Gaussian integrals where the exponential contains the term,

$$\sum_{i,j} q_i K_{ij} q_j \tag{5.8}$$

where K is a matrix. The correct generalisation will be to replace K by an operator. We thus wish to transform the exponent in the path integral into the form

$$\int d^4x \int d^4y \phi(x) \cdot \text{Operator} \cdot \phi(y) \tag{5.9}$$

By integrating by parts (and neglecting surface terms) the Lagrangian density may be written,

$$\phi(x)\Big[-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\Big]\phi(x) \tag{5.10}$$

whence we may rewrite $W[J]$ as

$$W_0[J] = \int [d\phi] \exp\Big(-\frac{1}{2}\int d^4x \int d^4y \phi(x) K(x,y)\phi(y) - \int d^4x J(x)\phi(x)\Big) \tag{5.11}$$

where

$$K(x,y) = \delta^4(x-y)\Big[-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\Big] \tag{5.12}$$

We may now evaluate $W_0[J]$ in terms of the inverse operator of K. This is the operator satisfying

$$\int d^4y K(x,y)\Delta(y,z) = \delta(x-z) \tag{5.13}$$

and we find

$$W_0[J] = \exp\Big(-\frac{1}{2}\int d^4x d^4y J(x)\Delta(x,y)J(y)\Big) \tag{5.14}$$

whence

$$\frac{\delta^2 W_0[J]}{\delta J(x)\delta J(y)}\bigg|_{J=0} = \Delta(x,y) \qquad (5.15)515$$

Now, the inverse operator Δ is in fact precisely the Feynman propagator encountered in canonical methods (up to the odd normalisation factor of i or -1). To see this

$$\begin{aligned}\int d^4z K(x,z) i\Delta_F(z,y) &= \int d^4z \delta^4(x-z)\Big(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\Big)\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(z-y)}}{k^2-m^2} \\ &= \int d^4z \delta^4(x-z)\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(z-y)} \\ &= \int d^4z \delta^4(x-z)\delta^4(z-y) \\ &= \delta^4(x-y)\end{aligned} \tag{5.16}$$

Now if we wish to evaluate, using functional methods, objects such as

$$\int dt_1 dt_2 \langle 0|T(\phi(x)\phi(y)H_I(t_1)H_I(t_2))|0\rangle \tag{5.17}$$

then we can obtain these by acting upon $W_0[J]$ with

$$\frac{\delta}{J(x)} \frac{\delta}{J(y)} \frac{\delta^4}{J(u)^4} \frac{\delta^4}{J(v)^4} \tag{5.18}$$

and then setting $J = 0$. (together with integrating d^4u and d^4v.) Since, the exponential is quadratic, and we set $J = 0$ finally, every time a propagator is brought down a further functional derivative must act. The end result is that the object is a sum of products of propagators.

As in the canonical case the simplest way to keep track of the terms is by drawing Feynman diagrams. This functional approach provides an alternate derivation. In the cases considered up till now we have seen simple vertices (corresponding to just polynomial terms in H_I) this will now be the case for gauge theories but the methods still apply. †

5.2 Momentum space Feynman diagrams

The Feynman diagrams I have drawn are not really the conventional ones. These are normally drawn in momentum space rather then x space. The very good reason for this is that the external states are normally momentum eigenstates. The momentum space is really just a Fourier transform of the configuration space rules -and it may be regarded as an exercise to transform these. Just a few points, the rules then require that we draw all diagrams, the momenta now flowing through the legs is now integrated over and each vertex has a δ-function in momenta. Tree level diagrams in momentum space are then merely the product of the propagators $1/(k^2 - m^2)$ however loop diagrams have more integrations over momenta than there are δ- functions and we obtain (the infamously difficult to evaluate) loop momentum integrations. We always obtain (look at our example) a δ-function in our results which imposes total conservation of energy and momentum. From the examples we can easily (?!) see what the general rule for vertices will be - whatever is in $\mathcal{L}_I$ will be reflected in terms of the rules for the vertex: A $\phi\chi\chi^\dagger$ vertex leads to a vertex with a ϕ a χ and a $\chi^\dagger$ outgoing state: A : $\phi^n(x)$: Lagrangian will yield a vertex with n outgoing ϕ states. Constants multiplying the vertex (such as g) get reflected in the rules.

† I have cut more corners in this section than I care to think about in an attempt to convey some understanding of the path integral approach. Some of these corners came back to haunt me in tutorials.

6. Gauge Theories 1: Electro-Magnetism

The great success in particle physics has been the ability to use gauge theories to describe the fundamental forces. As far as we know, both the strong and electro-weak forces are described by gauge theories. The strong force is believed to be described by a $SU(3)$ gauge theory known as QCD and the Electro-weak by $SU(2) \times U(1)$. Hopefully these terms will become clearer. I'll take two "bites" at this very important type of field theory. (Graham will also spend a lot of time on gauge theories as will Jonathon). The first bite will be simply electro-magnetism or a U(1) gauge theory - although it might not seem so simple and on the second pass I'll extend to $SU(3)$ and $SU(2)$ (or in fact any gauge group).

The theory of electromagnetism as described by Maxwell's equations is our proto-gauge theory. Maxwell's equations are

$$\begin{aligned}\nabla \cdot \underline{B} &= 0 \\ \nabla \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\ \nabla \cdot \underline{E} &= \rho \\ \nabla \times \underline{B} &= \underline{j} + \frac{\partial \underline{E}}{\partial t}\end{aligned} \tag{6.1}$$

As might be familiar to you, it is common to express $\underline{E}$ and $\underline{B}$ in terms of the vector and scalar potentials

$$\underline{E} = -\nabla\phi - \frac{\partial \underline{A}}{\partial t} \qquad \underline{B} = \nabla \times \underline{A} \tag{6.2}$$

whence the two equations $\nabla \cdot \underline{B} = 0$ and $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ become automatic. Our first task will be to write these equations in manifestly Lorentz covariant form. Firstly we form a 4-vector potential $A_\mu = (\phi, -\underline{A})$ and $j_\mu = (\rho, -\underline{j})$ and define a *field strength* $F_{\mu\nu}$ such that

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{6.3}$$

This definition is in fact equivalent to

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.4}$$

With this definition it is fairly easy to see that the last two of Maxwell's equations (four equations really) can be written (don't forget the Einstein summation convention!)

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{6.5}$$

We now wish to provide a Lagrangian formalism for these equations. It turns out that the appropriate Lagrangian density is given by

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad + j_\mu A^\mu \tag{6.6}$$

whose Lagrange equations are just those of (6.5) . To see this, for example, take the Lagrange equation for A_0,

$$\begin{aligned}
&\frac{\partial}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{A}_0}\right]+\frac{\partial}{dx}\left[\frac{\partial \mathcal{L}}{\partial(\partial_x A_0)}\right]+(\text{y and z terms})-\frac{\partial \mathcal{L}}{\partial A_0}=0\\
&0+\frac{\partial}{dx}\left[-F_{01}\right]+(\text{y and z terms})+\rho=0\\
&\nabla\cdot\underline{E}=\rho
\end{aligned}\tag{6.7}$$

There is a difficulty in carrying out a Hamiltonian approach to electro-magnetism. This is because the momentum which is conjugate to A_0 is identically zero,

$$\Pi_{A_0}=\frac{\partial \mathcal{L}}{\partial \dot{A}_0}\equiv 0\tag{6.8}$$

since the Lagrangian density does not depend upon $\dot{A}_0$.

Although not so obvious a problem in the Lagrangian formalism, this will rear it's ugly head fairly soon. The reason that there is a problem is because, in some ways, we have too many variables A_μ describing the fields. This will lead us into gauge symmetry. Notice that the field strength $F_{\mu\nu}$ is invariant under a transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda(x)\tag{6.9}$$

where $\Lambda(x)$ is an arbitrary function of x. Now, classically, the two choices of A_μ give the same values of $\underline{E}$ and $\underline{B}$ thus since everything can be written in terms of $\underline{E}$ and $\underline{B}$ this symmetry in merely a curiosity. †

Before discussing the quantisation of Electro-magnetism I will consider the theory coupled to Dirac fermion (or scalar) If we consider a Dirac fermion ψ then the Lagrangian

$$\mathcal{L}_\psi = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi\tag{6.10}$$

will be invariant under the transformation,

$$\psi \to \psi' = e^{-ig\alpha}\psi\tag{6.11}$$

where here α is a constant and not a function of x. (We could also consider coupling to the scalar Lagrangian $\partial_\mu\chi^\dagger\partial^\mu\chi$.) Suppose we would like to extend our transformation so that $\alpha(x)$. Then the Lagrangian is not invariant but an extra term

$$-ig\bar{\psi}\gamma_\mu\psi\partial^\mu\alpha\tag{6.12}$$

† An analogy of the problems we are encountering is the simple pendulum. Suppose I was silly enough to *over* specify my system by describing it by x, y, and θ. I might be tempted (obviously not but..) because the kinetic term is simple in x and y whereas the potential is simple in terms of θ. If I then chose $L=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\theta^2)$ we would obtain the momentum $p_\theta=0$. This constraint on (p,q) space is similar to the electromagnetism case.)

arises. Now we could make the Lagrangian invariant if we add an interaction term

$$\mathcal{L}_{int} = -gA^{\mu}\bar{\psi}\gamma_{\mu}\psi \tag{6.13}$$

and the combination

$$\mathcal{L}_A + \mathcal{L}_{\psi} + \mathcal{L}_{int} \tag{6.14}$$

will be invariant under the combined gauge transformation.

$$\psi \to e^{-ig\alpha(x)}\psi \qquad A_{\mu} \to A_{\mu} + \partial_{\mu}\alpha(x) \tag{6.15}$$

In terms of the fermions the transformation act via multiplication by a phase $e^{i\alpha}$. Such phases form a group. A very simple group which is known as $U(1)$- the group of 1×1 unitary matrices. ($U(n)$ will be the group of $n \times n$ unitary matrices). We can include the interaction term with the kinetic term for ψ by defining the ***covariant derivative***

$$D_{\mu}\psi \equiv (\partial_{\mu} + igA_{\mu})\psi \tag{6.16}$$

This is known as the covariant derivative because it transforms in the same way as ψ, namely with just a phase.

$$D_{\mu}\psi \to e^{-ig\alpha(x)}D_{\mu}\psi \tag{6.17}$$

This general trick of gauging symmetries has been enormously useful. It allows us to build models which have proved enormously useful in describing physics.

There are several conventions for phases in this area. Later I will use a different convention which can be obtained by replacing α by $-\alpha/g$. Whence the fields transforms as

$$\psi \to e^{i\alpha}\psi, \;\; A_{\mu} \to A_{\mu} - \frac{1}{g}\partial_{\mu}\alpha \tag{6.18}$$

whence

$$D_{\mu} = \partial_{\mu} - igA_{\mu} \tag{6.19}$$

6.2 Quantum Gauge Theories

Our naive attempts to quantise electrodynamics will prove to be sick because we are missing an important point. however, let us see how the sickness develops in the path integral formulation. We attempt to find the propagator. To do so, we must write the quadratic part of the Lagrangian as FIELD.OPERATOR.FIELD. The action may be rewritten

$$\begin{aligned}
&\int d^4x(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) \\
&= \int d^4x A_{\nu}\Big(-2\eta^{\nu\nu'}\partial_{\mu}\partial^{\mu} + 2\partial^{\nu}\partial^{\nu'}\Big)A_{\nu'} \\
&= \int d^4x \int d^4x' A_{\nu}(x')\Big(\delta^4(x - x')(-2\eta^{\nu\nu'}\partial_{\mu}\partial^{\mu} + 2\partial^{\nu}\partial^{\nu'})\Big)A_{\nu'}(x)
\end{aligned} \tag{6.20}$$

we thus have the inverse-propagator organised in position space. When we Fourier transform the above we obtain the momentum space inverse propagator,

$$P_{\mu\nu} = (k_\mu k_\nu - k^2 \eta_{\mu\nu}) \tag{6.21}$$

This "inverse propagator" has the unfortunate property that it does not have an inverse (so it is not the inverse of anything!). To observe this note that

$$\begin{aligned} P_{\mu\nu} P_{\nu\rho} &= - k^2 (k_\mu k_\rho - k^2 \eta_{\mu\rho}) \\ &= - k^2 P_{\mu\rho} \end{aligned} \tag{6.22}$$

Now any matrix satisfying $M^2 = \lambda.M$ cannot be invertible (unless $M = \lambda.I$ which P clearly is not.) so P is not an invertible operator.

Now we have reached a problem in the path integral formalism (just as we would have in canonical methods.) What is the reason for this? The interpretation of the "sickness" is that we are actually counting too many states in our path integral. If we have field configurations A_μ and $A_{\mu'}$ related by a gauge transformation, they only represent a single equivalent states so we should only count them once rather than twice. In fact an infinite over-counting occurs in the path integral. Consider the following diagram, where I have "squeezed" the integration of the path integral onto two dimensions. Configurations related to a field configuration lie in the *orbit* of the configuration.

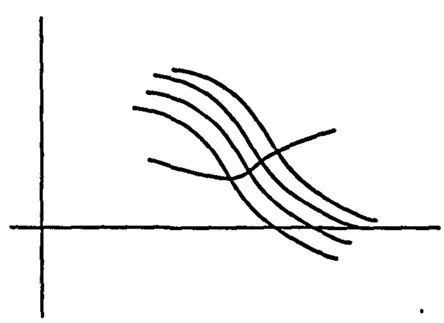

Figure 4. Orbits in gauge configuration space.

In this figure the orbits are shown and a curve which cuts each orbit is shown. Such a curve is given generically by

$$g\left[A_\mu\right] = 0 \tag{6.23}$$

We can think of implementing the gauge fixing by inserting a δ-function into the path integral. (However they are important coefficients!). Such a condition is called a gauge fixing condition. A good function $g[A]$ is clearly one which cuts each orbit once and once only. The implementation of gauge-fixing is important technically in quantising a gauge theory. I will demonstrate (rather than prove) how to implement this. I will try to switch back and forth between a two-dimensional analogy and the real situation.

Consider a two dimensional integral

$$I = \int dx dy f(x,y) \tag{6.24}$$

in analogy with the gauge theories the function f is invariant under rotations thus

$$f(x,y) = F(r,\theta) = F(r) \tag{6.25}$$

by analogy with gauge symmetries let us assume that the different values of θ should not be counted. Thus we wish to evaluate

$$I' = \int dr r F(r) \tag{6.26}$$

rather than (6.24) (which differs by a factor of $\int_0^{2\pi} d\theta = 2\pi$. Now we can just implement this by inserting a δ-function within the integral. We define

$$I_\phi = \int d^2 r f(x,y)\delta(\theta - \phi) = \int r dr d\theta F(r,\theta)\delta(\theta - \phi) = \int r dr F(r) = I' \tag{6.27}$$

We can define this for any function and by definition

$$I = \int d\phi I_\phi \tag{6.28}$$

however only for rotationally invariant functions will I_ϕ be independent of ϕ. Since I_ϕ is independent of ϕ,

$$I = \int d\phi I_\phi = 2\pi I_{\phi_0} \tag{6.29}$$

where ϕ_0 is any value of ϕ. In many ways I have just cheated! - I "knew" that the curve $\theta = const.$ cut each orbit one and one only (and also smoothly!). In general we want to consider a general curve $g(x,y) = 0$. (analogous to (6.23)). Again I want to insert $\delta(g(x,y))$ into the integral but now we need factors. We can see these from the identity,

$$\left|\frac{\partial g}{\partial \theta}\right|_{g=0} \int d\phi \delta(g(\underline{r}_\phi)) = 1 \tag{6.30}$$

(For intuition on this equation look, for example, at the prerequisites where $\delta(ax) = \delta(x)/|a|$.) It is important that

$$\Delta_g(\underline{r}_\phi) \equiv \left|\frac{\partial g}{\partial \theta}\right|_{g=0} \tag{6.31}$$

is rotation invariant. To see this note

$$\Delta_g^{-1}(\underline{r}_\phi) = \int d\theta \delta(g(r_{\theta+\phi})) = \int d\theta' \delta(g(r_{\theta'})) = \Delta_g^{-1}(\underline{r}) \tag{6.32}$$

We may now insert the factor of one in (6.30) into the integral I

$$I = \int d\theta r dr f(x,y) \int d\phi \Delta_g(\underline{r}) \delta(g(\underline{r}_\phi)) = \left(\int d\phi\right) \int d^2 r f(x,y) \Delta_g(\underline{r}) \delta(g(\underline{r}_\phi)) \quad (6.33)$$

So we can obtain

$$I' = \int d^2 r f(x,y) \Delta_g(\underline{r}) \delta(g(\underline{r}_\phi)) \quad (6.34)$$

As expected we have introduced a δ-function but we have a correcting factor Δ_g. In a quite considerable generalisation to gauge theories there is an identity,

$$1 = \Delta_g(A^\mu) \int \prod_x dU(x) \prod_x \delta(g(A^{\mu U}) \quad (6.35)$$

where

$$\Delta_g(A^\mu) = \det\left(\frac{\delta g}{\delta U}\right) \quad (6.36)$$

and $U(x) = e^{i\alpha(x)}$ -we are integrating over elements of the $U(1)$ group. Inserting this into the functional integral we obtain,

$$\begin{aligned} &\int d[A^\mu] e^{-Action} \int [dU] \Delta_g(A^g_\mu) \prod \delta(g(A^{\mu U})) \\ &\quad = \int [dU] \int d[A^\mu] e^{-Action} \Delta_g(A^g_\mu) \prod \delta(g(A^{\mu U}) \end{aligned} \quad (6.37)$$

The formal method of quantising is now rather simple - we just throw away the integration of the group variables $\int [dU]$. (analogously to $\int d\phi$) leaving us with a "gauge fixed" path integral which only counts each orbit once.

Great. We however have one more step before this is any use!. (How do we implement a general gauge fixing δ-function?) Obviously, the gauge fixed path integral is independent of g. (It's not easy to show this...) So using the gauge fixing functional

$$g' = g - B \quad (6.38)$$

where B is just a function of x (just a constant really in functional space!) will give just the same result. Inserting a factor

$$\int [dB] \prod \delta(g(A^{\mu U}) - B) e^{-\frac{1}{2\xi} \int d^4 x B^2(x)} \quad (6.39)$$

instead of $\prod \delta(g(A^{\mu U}))$ merely changes the path integral by a constant. This is really just averaging (or smearing) over the gauge functions $g - B$ with a factor e^{B^2}. This trivial trick allows us to get rid of the δ-functions and the gauge fixed path integral is

$$\begin{aligned} &\int d[A^\mu] \int [dB] e^{-Action} \Delta_g(A^g_\mu) \prod \delta(g(A^{\mu U} - B) e^{-\frac{1}{2\xi} \int d^4 x B^2(x)} \\ &\quad = \int d[A^\mu] e^{-Action - \frac{1}{2\xi} \int d^4 x g[A]^2} \Delta_g(A^g_\mu) \end{aligned} \quad (6.40)$$

So we have promoted the δ-function to an extra term in the action - the "gauge-fixing" term plus a determinant in the action (maybe more later). Many choices of "gauge-fixing" exist (and thus much effort to find good gauges - in some sense). I'll try to illustrate one approach via the so-called covariant gauges.

6.3 The Covariant gauges

This gauge choice uses the gauge fixing term,

$$g\left[A^\mu\right] = \partial_\mu A^\mu \tag{6.41}$$

With this gauge choice we find that the gauge fixing term in the action becomes

$$\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{6.42}$$

This will affect the quadratic terms in the action (thankfully!) to be

$$A_\mu\left(k_\mu k_\nu(1-\frac{1}{\xi}) - k^2\eta_{\mu\nu}\right)A_\nu \tag{6.43}$$

Now, we *can* invert this operator and obtain a propagator in momentum space

$$\frac{\left(\eta_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2}\right)}{k^2 + i\epsilon} \tag{6.44}$$

Amongst this class of gauge choices two special ones are when $\xi = 0, 1$ These are

$$\begin{aligned} \text{Feynman Gauge,} \quad \xi = 1, \quad P_{\mu\nu} &= \frac{\eta_{\mu\nu}}{k^2} \\ \text{Landau Gauge,} \quad \xi = 0, \quad P_{\mu\nu} &= \frac{\eta_{\mu\nu} - k_\mu k_\nu/k^2}{k^2} \end{aligned} \tag{6.45}$$

So gauge fixing has resolved this (and in fact all other) problems with quantisation of the gauge theory.

In the absence of either scalars or fermions, the quantised theory is a free theory and we may solve as for free scalar theory. (The Lagrangian contains only quadratic terms and, in the Feynman gauge, the propagator is just $\delta_{\mu\nu}/k^2$ which means the A_μ act just like multiple scalar fields.) In the presence of scalar or fermion fields the theory becomes a real live interacting quantum theory - QED for fermions or scalar-QED for scalars. For a fermion the covariant derivative contains an interaction term

$$ig\bar{\psi}\gamma_\mu A^\mu\psi = ig\sum_{a,b}\bar{\psi}_a(\gamma_\mu)_{ab}\psi_b A^\mu \tag{6.46}$$

implying a Feynman vertex

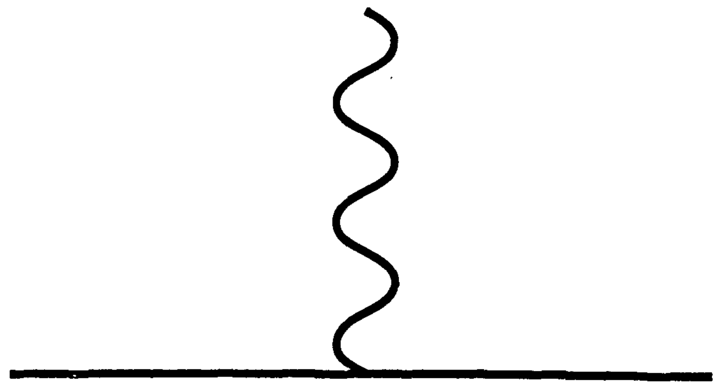

Figure 5. Feynman Diagram for QED.

7. Gauge Theories 2: Non-Abelian gauge theories

In this section we will generalise the concept of a gauge theory to that of a non-Abelian gauge theory. Both the strong and weak interactions appear to be described by such theories. Recall that the action of a gauge transformations for electromagnetism act as

$$e^{i\alpha(x)} \tag{7.1}$$

Now complex phases could, if one were perverse, be described as 1×1 unitary matrices. The $U(1)$ such matrices form a group. The basic definition of group's I quickly review here

7.1 basic group theory

A group G, is a set of objects with an action, or multiplication, defined such that the following axioms are satisfied,

$$\begin{aligned}
&1 :\text{if } a, b \in G, \text{then } a.b \in G \ \ \text{(closure)}\\
&2 :\text{there exists an identity }, e, \text{s.t.} a.e = e.a = a, \forall a \in G\\
&3 :\text{for all } a \in G, \text{there exists an inverse } a^{-1}, a.a^{-1} = e, a^{-1}.a = e\\
&4 :a.(b.c) = (a.b).c \ \ \forall a, b, c
\end{aligned} \tag{7.2}$$

There are many examples of groups. For example,

a) the numbers $\{1, -1\}$ under multiplication

b) the real numbers under addition (but *not* multiplication since zero has no inverse.)

c) the set of $n \times n$ matrices which are unitary ($A^{-1} = A^\dagger$) and which have determinant one. This group is known as $SU(N)$.

d) the set of orthogonal matrices ($A^{-1} = A^T$) of determinant one. This is known as $SO(N)$.

Examples c) and d) are examples of *Lie Groups*. Lie groups are groups which depend smoothly (in a well defines mathematical sense) on parameters. For example, a general $SO(2)$ matrix can be written in the form,

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{7.3}$$

which we can parametrise by θ. Clearly group multiplication (and inverses etc) depend smoothly upon θ, for example

$$M(\theta)M(\phi) = M(\theta + \phi), \ \ M^{-1}(\theta) = M(-\theta) \tag{7.4}$$

(If you are particularly observant you might notice that there is a lot of similarity between these matrices and $U(1)$. In fact $SO(2)$ and $U(1)$ are essentially the some algebraic structure.) If all elements of a group commute,

$$a.b = b.a \ \ \forall a, b \tag{7.5}$$

then we call the group Abelian.

7.2 Lie Algebras

An important object of interest in a Lie group is it's *algebra.* This is defined in terms of the behaviour of the group elements near the identity. For example consider the group $SU(2)$, ($A^\dagger A = 1$, $\det(A) = 1$). If we have an arbitrary element near the identity, $A = I + iT$ (where T is small) then T must satisfy,

$$T^\dagger = T, \ \ \mathrm{tr}(T) = 0 \tag{7.6}$$

thus T can be parametrised as

$$T = \sum_{a=1}^{3} \alpha^a T^a \tag{7.7}$$

where

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.8}$$

The matrices T_i generate an algebra under commutation. That is the commutator of any two T matrices is a sum of T matrices. For example

$$\left[T^1, T^2\right] = iT^3 \tag{7.9}$$

In general for $SU(N)$, if we consider the algebra, then it is generated by hermition traceless matrices of which there are $N^2 - 1$. This is the dimension of the Lie algebra. For $SU(3)$ there are thus eight matrices. A standard basis is

$$\lambda^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \lambda^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \lambda^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \lambda^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{7.10}$$

which are closed under commutation. Elements of the Lie algebra are linear combinations of these. There is a very important relationship between the elements of the algebra and the group itself. Essentially the group elements can be obtained by exponentiating the algebra,

$$U(\alpha) = \exp \sum_a \alpha^a T^a \tag{7.11}$$

where the α are no longer infinitesimal. Similar to the case of $SU(2)$, the T_a obey commutation relations,

$$\left[T^a, T^b\right] = if^{abc}T^c \tag{7.12}$$

where f^{abc} are known as the structure constants of the algebra. For $SU(2)$, $f^{abc} = \epsilon^{abc}$. (We normally normalise the T^a such that $\mathrm{tr}(T^a T^b) = \delta^{ab}/2$.) Although I won't really justify this, the structure constants really contain all the information in the group.

7.3 Representations

The structure of a group is defined abstractly in terms of the multiplication. A concrete realisation of a group is called A representation. A representation has two objects. Firstly, there must be a specific object for each element of the group. Normally we will be interested in matrix representations of a group. So we will have a mapping between the group and our set of matrices,

$$f : f(G) \to M \tag{7.13}$$

which preserves the multiplication structure i.e. $f(G.H) = f(G).f(H)$. For our $SU(2)$ and $SU(3)$ groups we have actually been looking at a representation of the formal mathematical structure. However, it has been a very special representation - the *fundamental*. For a given group there are many representations. For example the is always the trivial representation where every matrix gets mapped to the number 1. Also very importantly, the matrices must have a vector space to act upon. Normally we view this as column vectors. A cultural gap between mathematicians and physicists is that mathematicians focus upon the matrices whereas physicists focus upon the vector space.

7.4 Non Abelian Gauge symmetries

Let us generalise our gauge transformation acting upon a fermion

$$\underline{\psi} \to U(x)\underline{\psi} \tag{7.14}$$

where U is an element of a group G such as $SU(2)$ and $\underline{\psi}$ lies in a representation of G. For example for $SU(2)$ we could take $\underline{\psi}$ to be a doublet of fermions

$$\underline{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{7.15}$$

If U did not vary with x then the Lagrangian

$$\bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1 + \bar{\psi}_2 \gamma^\mu \partial_\mu \psi_2 = \bar{\underline{\psi}}^T \gamma^\mu \partial_\mu \underline{\psi} \tag{7.16}$$

is invariant, however for a gauge symmetry we wish the gauge transformation to vary with x. The technique will be to construct a covariant derivative D_μ such that

$$D_\mu \underline{\psi} \to U(x) D_\mu \underline{\psi} \tag{7.17}$$

which will require

$$U(x) D_\mu U^{-1}(x) = D'_\mu \tag{7.18}$$

We will postulate a form for D^μ analogously to the $U(1)$ case,

$$D_\mu = \partial_\mu + ig \sum_a T^a W^a_\mu$$

where T^a are the generators of the algebra and W^a_μ and the gauge fields - which now carry a group label a. (I have also introduced a coupling constant g.) This implies that the W^a_μ transform as

$$W'_\mu = UW_\mu U^{-1} - \frac{i}{g}U(x)\partial_\mu U^{-1}(x) \tag{7.19}$$

where we define $W_\mu = \sum_a W^a_\mu T^a$. Given this strange transformation the covariant derivative will transform appropriately. We can also define the field strength $F_{\mu\nu} = \sum_a F^a_{\mu\nu}T^a$ by

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig\Big[W^\mu, W^\nu\Big] \tag{7.20}$$

In terms of $F^a_{\mu\nu}$ this is

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\sum_{b,c} f^{abc}W^b_\mu W^c_\nu \tag{7.21}$$

From this matrix, due to this simple transformation property, it is possible to construct gauge invariant Lagrangians. The appropriate one is

$$\mathcal{L} = +\frac{1}{2}trace(F^{\mu\nu}F_{\mu\nu}) = \frac{1}{4}\sum_a F^a_{\mu\nu}F^{a\mu\nu} \tag{7.22}$$

which is invariant under Lorentz and gauge transformations. (there are other possibilities such as using *det* but these have problems.)

The gauge fixing we applied to the $U(1)$ case will also work here if we chose a gauge fixing term

$$trace(\partial \cdot A)^2 \tag{7.23}$$

7.5 Feynman Rules

We now look at the Lagrangian and determine the Feynman rules and comment on the consequences. Firstly the propagator. The propagator will only be determined by the quadratic terms in $\hat{H}$. These will just look like

$$W^\mu_a P^{-1}_{\mu\nu}\delta_{ab}W^\nu_b \tag{7.24}$$

where P is the propagators for the $U(1)$ case. Thus the (unsurprising) result is that

$$P^{a,b}_{\mu\nu} = P_{\mu\nu}\delta_{ab} \tag{7.25}$$

However when we examine the Lagrangian we find there are terms which are both cubic and quartic in the W-fields. In particular, the cubic terms are

$$f^{abc}\partial_\mu W_{\nu a}W^\mu_b W^\nu_c \tag{7.26}$$

What does this imply for our Feynman rules?. We will still have a 3-point vertex but now there is considerably more structure in the vertex. When evaluating we will have derivatives of the propagator, which in momentum space will lead to k_μ terms. The precise answer for the three point momentum space Feynman vertex, in the Feynman gauge, is

$$V^{abc}_{\mu\nu\rho}(p,q,r) = gf^{abc}\Big(\delta_{\nu\rho}(q_\mu - r_\mu) + \delta_{\rho\mu}(r_\nu - p_\nu) + \delta_{\mu\nu}(p_\rho - q_\rho)\Big) \tag{7.27}$$

as we show diagrammatically,

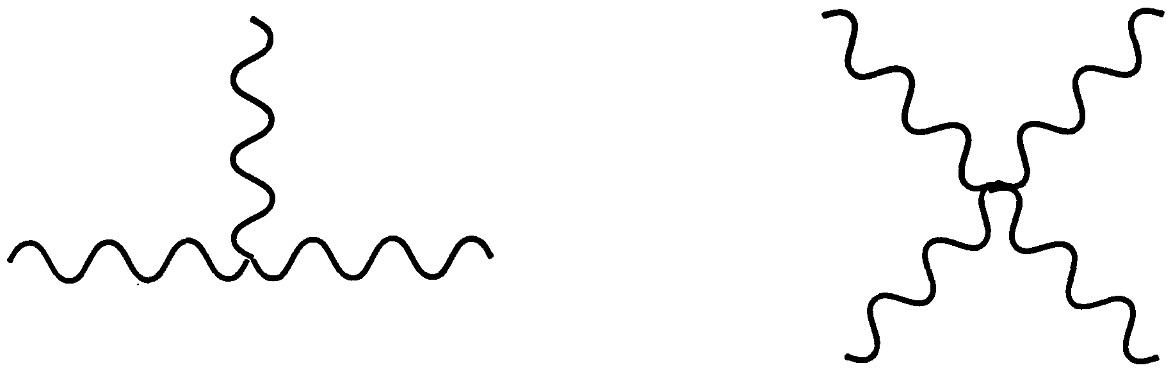

Figure 6. Feynman Diagrams for Non-Abelian Gauge Theory.

Note that is has crossing symmetry under interchange of legs and has one power of momentum in the vertex. The general situation is probably fairly clear from now on. There will also be a 4-point vertex. This contains no momentum (but a factor of g^2 rather than g.)

8. Critique of Perturbation theory

Perturbation theory has been enormously successful but it does have limitations. First I'll try to illustrate the "light" and then the "shade"

The Light

Perhaps the most impressive demonstration of perturbative field theory is the evaluation of $g-2$ of the electron in QED. The magnetic moment of a fermion is related to it's spin via

$$\mu = -g\frac{e}{2m}S \tag{8.1}$$

The *classical* Dirac Lagrangian gives a prediction for g to be exactly 2. However, as a purely Quantum mechanical effect, g may not exactly equal 2 but may be anomalous. This is calculable, using Feynman diagrams, perturbatively.

The great success is

$$\begin{aligned}(\frac{g-2}{2}) =&1159657.7 \pm 3.5 \times 10^{-9} : \text{Experiment} \\ =&1159655.4 \pm 3.3 \times 10^{-9} : \text{From Theory}\end{aligned} \tag{8.2}$$

The theoretical, prediction includes Feynman diagrams up to three loops. The only sensible conclusion is that

PERTURBATION THEORY WORKS

The Shade

Consider the function

$$\begin{aligned} f(x) &= 0 : x = 0 \\ f(x) &= e^{-\frac{1}{x^2}} \end{aligned} \tag{8.3}$$

This little function has a lot to teach us. It is not a particularly badly behaved function or very exciting to look at. It is continuous differentiable and it isn't very difficult to show that

$$f'(0) = 0 \tag{8.4}$$

If fact, with a little more work we can show that

$$f^{(n)}(0) = 0 \tag{8.5}$$

Thus the Taylor series of $f(x)$ around $x = 0$ is

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n} = 0 \neq f(x) \tag{8.6}$$

Thus it is a fairly simple example where the Taylor series does not equal the function. Now a typical decay amplitude is a function of the coupling constant g

$$R(g)$$

We attempt to evaluate $R(g)$ by perturbation theory - this is essentially just it's Taylor series. So any component of R which takes the form

$$\sim \epsilon^{1/g^2} f(g) \tag{8.7}$$

will *never* show up in a perturbative expansion. One might argue that such functions are pathological. I.e. that they are really just mathematical and don't effect real problems however I'll try to argue the reverse. Consider $SU(2)$ pure gauge theory. Rescale the potential field

$$W_\mu \to \frac{1}{g} W'_\mu \tag{8.8}$$

whence

$$F_{\mu\nu} \to \frac{1}{g} F'_{\mu\nu}$$

where F' has no explicit dependence on g. Then the Path integral looks a bit like

$$\sim \int [dW] e^{\frac{1}{g^2} \int (F'_{\mu\nu})^2} \tag{8.9}$$

Which definitely looks dangerous! Thus we can easily see how contributions not accessible by perturbative results can creep in. This is especially true in any form of classical background

$$A_\mu = A^B_\mu + A'_\mu \tag{8.10}$$

(I.e. looking at transitions in the presence of a non-zero background.)

I present this example (another good example is $1/(1+g^2$) not to try to destroy Feynman diagram techniques but to point out that they are not *everything*. We must consider the realm of validity. Unfortunately, we have few alternate techniques. One technique is to take the path integral and just evaluate it numerically. To do so we must discretise space-time , the configuration etc etc. It takes a lot of computing effort and still has yet to be enormously fruitful but , at present, we have nothing else other than Feynman diagrams (and variations thereof). Despite these concerns, field theory does "produce the goods".

9. Some Things to look out for

In an informal session we tred to peek beyond the standard model and give some theorists prejudices. I have summarised somr of the areas we covered here. These very much reflect my personal prejudes and imperfect recollections. *you should under no circumstances take me too seriously!.* Firstly, I will try to explain some of the issues inspiring theoretical interest.

One of the biggest areas of theoretical work is in supersymmetry and many of you will no doubt be involved in searches for "supersymmetric" partners of the known particles. Supersymmetry is a symmetry which relates particles of different spins. This is very different from any other symmetry we have presented to you - and so far has no experimental realisation. The main prediction of supersymmetry is that every particle should have a "superpartner" of the same quantum numbers but differing by 1/2 spin. That is the "photino" would be a fermion of spin 1/2. Examining the standard model no such pairing has yet been observed!. Thus if supersymmetry were true, there must be a whole set of partners of the existing known particles waiting to be found. Supersymmetry predicts that the superpartners have the same mass - It is obviously a broken symmetry!. Theorists love supersymmetry because supersymmetric theories have much better quantum properties than non-supersymmetric theories. Theorists in the USA and Europe have had, in general, rather different "cultures" regarding supersymmetry. In Europe it come close to religious fervour amongst some whereas in certain, influential, circles in the USA there is strong disbelief. This unfortunately has led the US experimentalism to be less aggressive in many cases in supersymmetric partner searches than in Europe. The discovery of supersymmetric partners to the existing particles would be a great vindication for the believers. (and extremely exciting for everyone!) The running of the coupling constants to "unification" scales tends to support supersymmetry rather than not but, *despite the hype*, most "neutrals" take rather a cynical view.

An area of great theoretical interest has been in formulations of quantum gravity. Although of no experimental significance very many theorists think this is a valid area of research both in it's own right and also for the implications if has for the other forces. The basic problem is that if one takes the classical Lagrangian known to describe general relativity (the Einstein-Hilbert action) and applies the perturbative techniques described in these lectures is just gives nonsense. The nonsense takes the form of infinities which one encounters in scattering amplitudes and we have no, sensible, way to eliminate these infinities. One can take two approaches to this problem 1) modify Quantum Mechanics or 2) modify Gravity. Personally I don't like the approaches which involve modifying quantum mechanics but a surprising number of smart people do. One of the most interesting modifications to gravity has been "String Theory". Instead of having point particles the fundamental objects in one's theory are one-dimensional strings. Obviously a string has an infinite number of degrees of freedom compared to a point particle - easily thought of a the modes- but one can still just apply Quantum mechanics *in principle* to the theory. Although the mathematics is shockingly difficult the Quantum behaviour of the theory is very good. It appears that the theory is quantum consistent and includes gravity. So far string theory provides an honest answer to a real question. whether it is the only solution and whether it is the solution chosen by nature who knows. The mathematics of string theory have proven so rich that many theorists are difficult to recognise as physicists now.

To finish with, I'll mention something which doesn't see a lot of theoretical output these days but which many people if you ask them in a quite corner, hand on heart might agree with. This is that quarks and leptons and especially the Higgs might be composite. Again many of you might get involved in compositness searches. Theories where the Higgs is composite can be very attractive. These are often called Technicolour theories. However it's rather difficult to get them to work convincingly. One major problem with any such calculations is that, basically, we can only handle weakly interacting theories with much success. As soon as a theory is strongly coupled life becomes very hard. And any theory which binds composite objects into quarks/leptons/Higgs must be strongly interacting. The difficulty in calculating makes it very hard to speculate on the type of theory. Take the case of QCD: the lattice gauge theory community has poured huge amounts of effort and time and computer power into evaluating QCD quantities numerically. So far this has had limited success (this isn't to criticise - lattice techniques are the only techniques available). If I visited the office of a local lattice theorist and asked him what are the masses of the bound states of $SU(4)$ or some other weird theory (so I might compare to the quark mass spectrum for example) I wont b get a lot of sympathy. Despite the fact that humans can't calculate strongly interacting theories doesn't mean that the universe can't and compositness is a very real possibility.

10. Exercises (selected)

1.1 Consider the double pendulum.

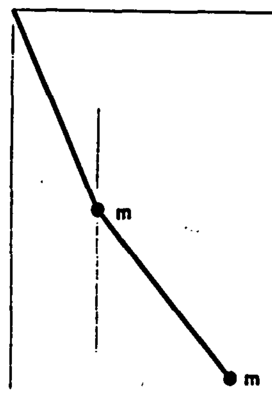

Figure E1. The Double pendulum.

Assuming small oscillations find $L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)$. Find Lagranges equations for this system. Find H and evaluate Hamilton's equations. *Optional-* Solve.

1.2 Calculate the Poisson brackets,

$$\left\{q^2, p\right\}, \quad \left\{q^2, p^2\right\}$$

How do these compare with

$$[\hat{q}^2, \hat{p}], \quad [\hat{q}^2, \hat{p}^2]$$

1.3 Suppose

$$L = \frac{1}{2} v(q)^{-1} \dot{q}^2$$

then what is H?.

1.4 Show that the time dependence of any function $F(p_r, q_r)$ is given by

$$\dot{F} = \left\{F, H\right\} \tag{10.1}$$

2.1 In the low temperature limit of the partition function in statistical mechanics it is the low-energy states whose contributions dominate. In the small-$\hbar$ limit which paths will dominate in the path integral?

3.1 Suppose

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \lambda \phi^n$$

what are Lagranges equations for this?.

4.1 Suppose we have a single real scalar field $\phi(\underline{x}, t)$ and

$$\mathcal{L}_{int} = -\lambda \phi^n$$

so that $H_{int} = \int d^3x\lambda : \hat{\varphi}^n(\underline{x}, t) :$. For an initial state with two particles of momenta $\underline{k}_1$ and $\underline{k}_2$

$$|t = -\infty\rangle = a^\dagger(\underline{k}_1)a^\dagger(\underline{k}_2)|0\rangle(*)$$

Suppose the final state has M-particles

$$|t = +\infty\rangle = a^\dagger(\underline{p}_1)a^\dagger(\underline{p}_2)\cdots a^\dagger(\underline{p}_M)|0\rangle$$

To leading order, (U_1) what is the value of M so that the transition is non-zero?

4.2 Suppose we have $\phi(\underline{x}, t)$ and a complex $\chi(\underline{x}, t)$ with $\mathcal{L}_{int} = -g : \chi^\dagger\chi\phi$ the case in the notes). If we have initial state (*) and final state

$$|t = +\infty\rangle = b^\dagger(\underline{p}_1)\cdots b^\dagger(\underline{p}_M)d^\dagger(\underline{q}_1)\cdots d^\dagger(\underline{q}_N)|0\rangle$$

then

(a) Show to lowest order (U_1) that the matrix element is zero

(b) What is the first order where the matrix element is non-zero and for this order what are the values of M and N?

5.1 Compute $\frac{\delta}{\delta J(y)}$‘ and $\frac{\delta^2}{\delta J(y)\delta J(z)}$ of

$$a)\ \int dx\phi(x)J(x)$$

$$b)\ \left[\int dx\phi(x)J(x)\right]^2$$

$$c)\ \int dx\phi(x)J(x)^2$$

6.1 Express both $F_{\mu\nu}F^{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F^{\rho\sigma}$ in terms of E and B.

7.1 An alternate Definition of $F_{\mu\nu}$ is

$$F_{\mu\nu} = \left[D_\mu, D_\nu\right]$$

7.2 Find a set of 3×3 matrices which form a representation of $SU(2)$. i.e. matrices satisfying (7.9)

INTRODUCTION TO QUANTUM ELECTRODYNAMICS AND QUANTUM CHROMODYNAMICS

By Dr S King
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Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1996

Introduction to QED and QCD

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Contents

1 Introduction

The traditional aim of this course is to teach you how to calculate amplitudes, cross-sections and decay rates, particularly for quantum electrodynamics, QED, but in principle also for quantum chromodynamics, QCD. By the end of the course you should be able to go from a Feynman diagram such as the one for $e^+e^- \to \mu^+\mu^-$ in Figure 1.1(a), to a number for the cross-section, for example.

We will restrict ourselves to calculations at *tree level* but will also look qualitatively at higher order *loop* effects which amongst other things are responsible for the running of the QCD coupling constant, where the coupling appears weaker when you measure it at higher enegy scales. This running underlies the useful application of perturbative QCD calculations to high-energy processes. As you can guess, the sort of diagrams which are important here have closed loops of particle lines in them: in Figure 1.1(b) is one example contributing to the running of the strong coupling (the curly lines denote gluons).

In order to do our calculations we will need a certain amount of technology. In particular, we will need to describe particles with spin, especially the spin-1/2 leptons and quarks. We will therefore spend some time looking at the Dirac equation and its free particle solutions. After this will come revision of Fermi's golden rule to find probability amplitudes for transitions, followed by some general results on normalisation, flux factors and phase space, which will allow us to obtain formulas for cross sections and decay rates.

With these tools in hand, we will look at some examples of tree level QED processes. Here you will get hands-on experience of calculating transition amplitudes and getting from them to cross sections. We then move on to QCD. This will entail a brief introduction to renormalisation in both QED and QCD. We will introduce the idea of the running coupling constant and look at asymptotic freedom in QCD.

In reference [1] you will find a list of textbooks which may be useful.

1.1 Units and Conventions

I will use natural units, $c = 1$, $\hbar = 1$, so mass, energy, inverse length and inverse time all have the same dimensions.

$$\begin{array}{lll} \text{4-vector} & a^\mu \quad \mu = 0,1,2,3 & a = (a^0, \mathbf{a}) \\ \text{scalar product} & a{\cdot}b = a^0b^0 - \mathbf{a}{\cdot}\mathbf{b} = g_{\mu\nu}a^\mu b^\nu & \end{array} \tag{1.1}$$

From the scalar product you see that the metric is:

$$g = \mathrm{diag}(1,-1,-1,-1), \qquad g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{1.2}$$

For $c = 1$, $g^{\mu\nu}$ and $g_{\mu\nu}$ are numerically the same.

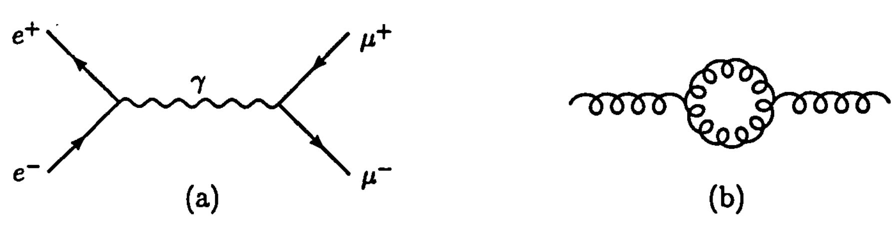


Figure 1.1 Examples of Feynman diagrams contributing to (a) $e^+e^- \to \mu^+\mu^-$ and (b) the running of the strong coupling constant.

From the above, you would think it natural to write the space components of a 4-vector as a^i for $i = 1, 2, 3$. However, for 3-vectors I will normally write the components as a_i. This is confusing only when you convert between ordinary vector equations and their covariant forms, when you have to remember the sign difference between a^i and a_i.

Note that ∂_μ is a covector,

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial_\mu x^\nu = \delta^\nu_\mu, \tag{1.3}$$

so $\nabla^i = -\partial^i$ and $\partial^\mu = (\partial^0, -\nabla)$

My convention for the totally antisymmetric Levi-Civita tensor is:

$$\epsilon^{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{if } \{\mu, \nu, \lambda, \sigma\} \text{ an even permutation of } \{0, 1, 2, 3\} \\ -1 & \text{if an odd permutation} \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

Note that $\epsilon^{\mu\nu\lambda\sigma} = -\epsilon_{\mu\nu\lambda\sigma}$, and $\epsilon^{\mu\nu\lambda\sigma} p_\mu q_\nu r_\lambda s_\sigma$ changes sign under a parity transformation (which is obvious because it contains an odd number of spatial components).

1.2 Relativistic Wave Equations

The starting point for this course is the good old Schrodinger equation which can be written quite generally as:

$$H\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{1.5}$$

where H is the Hamiltonian (i.e. the energy operator). In this equation $\psi(t)$ is the wavefunction describing the single particle probability amplitude. In this course we shall reserve the Greek symbol ψ for spin 1/2 fermions and ϕ for spin 0 bosons. So for pions and the like we shall write:

$$H\phi(t) = i\frac{\partial\phi(t)}{\partial t} \tag{1.6}$$

Now in this course we want to extend non-relativistic quantum mechanics into the relativistic domain. The good news is that the Schrodinger equation as written above applies equally well in relativistic quantum mechanics. However care must be taken with the Hamiltonian to ensure that it is relativistically invariant. For example, in non-relativitic quantum mechanics you are used to writing

$$H = T + V \tag{1.7}$$

where T is the kinetic energy and $V(\mathbf{r})$ is the potential energy. A particle of mass m and momentum $\mathbf{p}$ has non-relativistic kinetic energy,

$$T = \frac{\mathbf{P}^2}{2m} \tag{1.8}$$

where capital $\mathbf{P}$ is the operator corresponding to momentum $\mathbf{p}$. For a slow moving particle $v \ll c$ (e.g. an electron in a Hydrogen atom) this is adequate, but for relativistic systems $v \sim c$ the Hamiltonian above breaks down. For a free relativistic particle the total energy E is given by the Einstein equation

$$E^2 = \mathbf{p}^2 + m^2 \tag{1.9}$$

Thus the square of the relativistic Hamiltonian H^2 is simply given by promoting the momentum to operator status,

$$H^2 = \mathbf{P}^2 + m^2 \tag{1.10}$$

So far so good, but now the question arises of how to implement the Schrodinger equation which is expressed in terms of H rather than H^2. Naively the relativistic Schrodinger equation looks like

$$\sqrt{\mathbf{P}^2 + m^2}\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{1.11}$$

but this is difficult to interpret because of the square root. There are two ways forward:

(1) Work with H^2. By iterating the Schrodinger equation we have

$$H^2\phi(t) = -\frac{\partial^2\phi(t)}{\partial t^2} \tag{1.12}$$

which is known as the Klein-Gordon (KG) equation. In this case the wavefunction describes spinless bosons.

(2) Invent a new Hamiltonian H_D which is linear in momentum, and whose square is equal to H^2 given above, $H_D^2 = \mathbf{P}^2 + m^2$. In this case we have

$$H_D\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{1.13}$$

which is known as the Dirac equation, with H_D being the Dirac Hamiltonian. In this case the wavefunction describes spin one half fermions, as we shall see.

1.3 Wavefunctions vs. Fields

You may be wondering why I am talking about wavefunctions while in your *field theory* course Dave Dunbar is telling you about fields. Some of you may even be wondering what is the difference between a wavefunction and a field. Well, you all know that wavefunctions are just probablility amplitudes for finding the particle. This is fine and dandy - so why can't we stick with wavefunctions rather than go to the trouble of inventing fields? The answer has to do with some problems faced by relativistic quantum mechanics. As we know from the *non-relativistic* Schrodinger equation one can define a probability density

$$\rho = \psi^*\psi \tag{1.14}$$

and a current density

$$\mathbf{J} = \frac{-i}{2m}(\phi^*\nabla\phi - \phi\nabla\phi^*) \tag{1.15}$$

which satisfy the so called continuity equation

$$\frac{\partial\rho}{\partial t} = -\nabla.\mathbf{J} \tag{1.16}$$

which just expresses conservation of probability. The existence of this equation enables one to interpret $\psi^*\psi$ as a probability distribution. (This is why probability is indentified with $|\psi|^2$ rather than say $|\psi|$ for example.) OK - now what about relativity?

In the relativistic domain for the KG and Dirac equations there are analagous continuity equations for probability but here there are some problems of interpretation. To begin with the KG equation turns out to have a probability density which can be negative! This is the tip of a conceptual iceberg because even if it were positive all the time, we have no right to expect that probability is conserved for bosons which can be created and destroyed in arbitrary numbers (e.g. any number of pions can be produced when a high energy proton beam hits a target). There is clearly a conceptual problem with the single particle interpretation of the wavefunction ψ in this case, and quantum field theory is the solution! The KG equation also suffers from the problem of having negative energy solutions, and here again one finds the solution in quantum field theory.

Quantum field theory draws much of its inspiration from electromagnetic fields. We are used to thinking of an electromagnetic field as a real physical quantity which can occupy space and which can contain energy. When you wiggle around a bar magnet as a child it is quite natural to think of the magnetic field as a real quantity which exists in the space around the magnet, and gets carried around with the magnet. When this field encounters some iron filings it interacts with them, and so on. We also know that photons are packets of energy and that they must be regarded as the result of quantising the electromagnetic field. Like the pions, photons can be created and destroyed in arbitrary numbers (e.g. an excited atom can emit one or more photons). This presents no problem if photons are regarded as quanta of the electromagnetic field, since a state with n photons just corresponds to a higher level of exitation of the electromagnetic field than a state with no photons (the vacuum or ground state of the field). Given our experience with electromagnetism it seems perfectly natural to try to play the same game with spinless bosons such as pions, and invent a new field analagous to the electromagnetic field, whose quantum excitations can be interpreted as spinless bosons.

Such considerations led inevitably to the development of quantum field theory as the solution to the problem with probability faced by the KG equation. However the same problem also led Dirac to invent his equation, for which the probability is always positive, and his now famous prediction of spin and antiparticles. However the existence of antiparticles implies that particle-antiparticle pairs can be created and destroyed in arbitrary numbers, just as in the case of bosons, so again there is the problem with the single particle wavefunction interpretation, and again one is led to quantum field theory. Indeed here the case is even more compelling since one desires to treat electrons and photons on the same basis in order to understand their interactions properly, and given that photons are quanta of the e.m. field one is led to the Dirac field immediately.

It is important to emphasise that a field is a very different beast from a wavefunction. A *wavefunction* $\phi(\mathbf{r}, t)$ is just a mathematical object, a complex number from which we can extract information about the whereabouts of the particle. A *field* on the other hand is a physical object which exists in space and which can have energy. In order to be able to carry energy, the field is described by a function $\phi(\mathbf{r}, t)$ which is regarded as a dynamical variable or generalised coordinate. I find it useful to think of the value of the field at a point in space as a coordinate describing the motion of some (ficticious) infinitesimal harmonic oscillator associated with that point. The total field describes the collection of all such little (coupled) harmonic oscillators corresponding to all the points in space. Each little pretend oscillator is described by its own coordinate, and carries an infinitesimal energy. The field is in fact analagous to a set of oscillating coupled atoms in a crystal lattice. However the field variable $\phi(\mathbf{r}, t)$ is not to be thought of literally as the displacement of some oscillator, but rather something akin to an electric or magnetic field. Nevertheless the field coordinate at a particular space point can oscillate, does couple to neighbouring field points, and does carry an infinitesimal energy - just like a vibrating atom in a crystal - so it is a useful picture to have in your mind.

When Dave Dunbar considers quantum mechanics of the field (quantum field theory) all he has to do is quantise each of the little oscillators for each space point, in just the same way as we would quantise the coordinates of atoms in a crystal. Since coordinates become operators in quantum mechanics this implies that the field variables $\phi(\mathbf{r}, t)$ get promoted to the status of operator. Of course there are some slight technicalities involved with this proceedure, and so it will take Dave a whole course to explain how to do it!

You have probably noticed that I have used the same symbol ϕ for both the wavefunction and the field, even though one is a complex number and the other is a dynamical variable which must be regarded as a quantum operator. The reason I am able to get away with such sloppiness is that it turns out (although this is not obvious) that the wavefunction and the field obey the same equation of motion, even though they are very different beasts. So when I talk about the KG equation, I can equally well be talking about the equation for the wavefunction or the field – they are the same. So you must have your wits about you at all times to decide if I am talking about the wavefunction or the field!

1.4 The Klein-Gordon Equation

We now write the KG equation 1.12 in position space, using a rather fancy notation with which you can impress your friends. In position space we write the momentum operator as

$$\mathbf{p} \rightarrow -i\nabla, \tag{1.17}$$

so that the KG equation 1.12 becomes,

$$\left(\Box + m^2\right)\phi(x) = 0 \tag{1.18}$$

where we have introduced the impressive box notation,

$$\Box = \partial_\mu \partial^\mu = \partial^2/\partial t^2 - \nabla^2 \tag{1.19}$$

and x is the 4-vector $(t, \mathbf{x})$.

The operator $\Box$ is Lorentz invariant, so the Klein-Gordon equation is relativistically covariant (that is, transforms into an equation of the same form) if ϕ is a scalar function. That is to say, under a Lorentz transformation $(t, \mathbf{x}) \rightarrow (t', \mathbf{x}')$,

$$\phi(t, \mathbf{x}) \rightarrow \phi'(t', \mathbf{x}') = \phi(t, \mathbf{x})$$

so ϕ is invariant. In particular ϕ is then invariant under spatial rotations so it represents a spin-zero particle (more on spin when we come to the Dirac equation), there being no preferred direction which could carry information on a spin orientation.

The Klein-Gordon equation has plane wave solutions

$$\phi(x) = Ne^{-i(Et - \mathbf{p}\cdot\mathbf{x})} \tag{1.20}$$

where N is a normalisation constant and $E = \pm\sqrt{\mathbf{p}^2 + m^2}$. Thus, there are both positive and negative energy solutions. In the quantum field ϕ, these are just associated with operators which create or destroy particles. However, they are a severe problem if you try to interpret ϕ as a wavefunction. The spectrum is no longer bounded below, and you can extract arbitrarily large amounts of energy from the system by driving it into ever more negative energy states. Any external perturbation capable of pushing a particle across the energy gap of $2m$ between the positive and negative energy continuum of states can uncover this difficulty.

A second problem with the wavefunction interpretation arises when you try to find a probability density. Since ϕ is Lorentz invariant, $|\phi|^2$ doesn't transform like a density. To search for a candidate we derive a continuity equation, rather as you did for the Schrödinger equation in the pre-school problems. Defining ρ and $\mathbf{J}$ by

$$\begin{aligned} \rho &\equiv i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) \\ \mathbf{J} &\equiv -i\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right) \end{aligned} \tag{1.21}$$

you obtain (see problem) a covariant conservation equation

$$\partial_\mu J^\mu = 0 \tag{1.22}$$

where J is the 4-vector $(\rho, \mathbf{J})$. It is natural to interpret ρ as a probability density and $\mathbf{J}$ as a probability current. However, for a plane wave solution (1.20), $\rho = 2|N|^2E$, so ρ is not positive definite since we've already found E can be negative.

▷**Exercise 1.1**

Derive the continuity equation (1.22). Start with the Klein-Gordon equation multiplied by ϕ^* and subtract the complex conjugate of the K-G equation multiplied by ϕ.

Thus, ρ may well be considered as the density of a conserved quantity (such as electric charge), but we cannot use it for a probability density. To Dirac, this and the existence of negative energy solutions seemed so overwhelming that he was led to introduce another equation, first order in time derivatives but still Lorentz covariant, hoping that the similarity to Schrödinger's equation would allow a probability interpretation. In fact, with the interpretation of ϕ as a quantum field, these problems are not problems at all: the negative energy solutions will find an explanation in terms of antiparticles and ρ will indeed be a charge density as hinted above. Moreover, Dirac's hopes were unfounded because his new equation also turns out to admit negative energy solutions. Fortunately it is just what we need to describe particles with half a unit of spin angular momentum, so we will now turn to it.

2 The Dirac Equation

Dirac wanted an equation first order in time derivatives and Lorentz covariant, so it had to be first order in spatial derivatives too. His starting point was to assume a Hamiltonian of the form,

$$H_D = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \beta m \tag{2.1}$$

where P_i are the three components of the momentum operator $\mathbf{P}$, and α_i and β are some "unknown quantities", which as will be seen below cannot simply be commuting numbers. When the requirement that the $H_D^2 = \mathbf{P}^2 + m^2$ is imposed, this implies that α_i and β must be interpreted as 4×4 matrices, as we shall discuss. The first step is to write the momentum operators explicitly in terms of their differential operators, using Eq.1.17, then the Dirac equation 1.13 becomes, using the Dirac Hamiltonian in Eq.2.1,

$$i\frac{\partial\psi}{\partial t} = (-i\,\alpha\cdot\nabla + \beta\mathrm{m})\psi \tag{2.2}$$

which is the position space Dirac equation. Remember that in field theory, the Dirac equation is the equation of motion for the field operator describing spin 1/2 fermions. In order for this equation to be Lorentz covariant, it will turn out that ψ cannot be a scalar under Lorentz transformations. In fact this will be precisely how the equation turns out to describe spin 1/2 particles. We will return to this below.

If ψ is to describe a free particle it is natural that it should satisfy the Klein-Gordon equation so that it has the correct energy-momentum relation. This requirement imposes relationships among the α and β. To see these, apply the operator on each side of equation (2.2) twice, i.e. iterate the equation,

$$-\frac{\partial^2\psi}{\partial t^2} = [-\alpha^i\alpha^j\nabla^i\nabla^j - i\,(\beta\alpha^i + \alpha^i\beta)m\nabla^i + \beta^2 m^2]\psi$$

with an implicit sum over i and j from 1 to 3. The Klein-Gordon equation by comparison is

$$-\frac{\partial^2\psi}{\partial t^2} = [-\nabla^i\nabla^i + m^2]\psi \tag{2.3}$$

If we do not assume that the α^i and β commute then the KG will clearly be satisfied if

$$\begin{aligned} \alpha_i\alpha_j + \alpha_j\alpha_i &= 2\delta_{ij} \\ \beta\alpha_i + \alpha_i\beta &= 0 \\ \beta^2 &= 1 \end{aligned} \tag{2.4}$$

for $i, j = 1, 2, 3$. It is clear that the α_i and β cannot be ordinary numbers, but it is natural to give them a realisation as matrices. In this case, ψ must be a multi-component *spinor* on which these matrices act.

▷**Exercise 2.1**
Prove that any matrices α and β satisfying equation (2.4) are traceless with eigenvalues ± 1. Hence argue that they must be even dimensional.

In two dimensions a natural set of matrices for the α would be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

However, there is no other independent 2×2 matrix with the right properties for β, so the smallest dimension for which the Dirac matrices can be realised is four. One choice is the *Dirac representation*

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.6}$$

Note that each entry above denotes a two-by-two block and that the 1 denotes the 2×2 identity matrix.

There is a theorem due to Pauli which states that all sets of matrices obeying the relations in (2.4) are equivalent. Since the Hermitian conjugates $\alpha^\dagger$ and $\beta^\dagger$ clearly obey the relations, you can, by a change of basis if necessary, assume that α and β are Hermitian. All the common choices of basis have this property. Furthermore, we would like α_i and β to be Hermitian so that the Dirac Hamiltonian (2.18) is Hermitian.

▷**Exercise 2.2**
Derive the continuity equation $\partial_\mu J^\mu = 0$ for the Dirac equation with

$$\rho = J^0 = \psi^\dagger(x)\psi(x), \qquad \mathbf{J} = \psi^\dagger(x)\alpha\psi(\mathbf{x}). \tag{2.7}$$

We will see in section 2.6 that $(\rho, \mathbf{J})$ does indeed transform as a four-vector.

2.1 Free Particle Solutions I: Interpretation

We look for plane wave solutions of the form

$$\psi = \begin{pmatrix} \chi(\mathbf{p}) \\ \phi(\mathbf{p}) \end{pmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})} \tag{2.8}$$

where $\phi(\mathbf{p})$ and $\chi(\mathbf{p})$ are two-component spinors which depend on momentum $\mathbf{p}$ but are independent of $\mathbf{x}$. Using the Dirac representation of the matrices, and inserting the trial solution into the Dirac equation gives the pair of simultaneous equations

$$E\begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -m \end{pmatrix}\begin{pmatrix} \chi \\ \phi \end{pmatrix}, \tag{2.9}$$

There are two simple cases for which Eq.2.9 can readily be solved, namely

(1) $\mathbf{p} = 0$, $m \neq 0$ corresponding physically to an electron in its rest frame.

(2) $m = 0$, $\mathbf{p} \neq 0$ corresponding physically to a massless neutrino.

For case (1), an electron in its rest frame, the equations 2.9 decouple and become simply,

$$E\chi = m\chi, \qquad E\phi = -m\phi \tag{2.10}$$

so that in this case we see that χ corresponds to solutions with $E = m$, while ϕ corresponds to solutions with $E = -m$: negative energy solutions!

These negative energy solutions persist for an electron with $\mathbf{p} \neq 0$ for which the solutions to Eq.2.9 are readily seen to be

$$\phi = \frac{\sigma\cdot\mathbf{p}}{\mathrm{E+m}}\chi, \qquad \chi = \frac{\sigma\cdot\mathbf{p}}{\mathrm{E-m}}\phi. \tag{2.11}$$

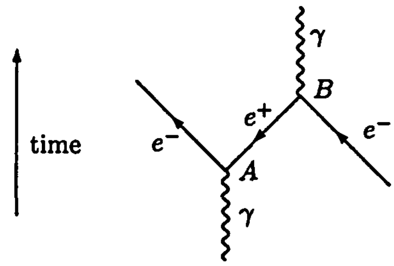


Figure 2.1 Feynman interpretation of a process in which a negative energy electron is absorbed. Time increases moving upwards.

Thus the general positive energy solutions with $E = +|\sqrt{\mathbf{p}^2 + m^2}|$ are:

$$\psi(x) = \begin{pmatrix} \chi \\ \frac{\sigma \cdot \mathbf{p}}{\mathrm{E+m}}\chi \end{pmatrix} e^{-i(Et - \mathbf{p}\cdot\mathbf{x})}, \tag{2.12}$$

while the general negative energy solutions with $E = -|\sqrt{\mathbf{p}^2 + m^2}|$ are:

$$\psi(x) = \begin{pmatrix} \frac{\sigma \cdot \mathbf{p}}{\mathrm{E-m}}\phi \\ \phi \end{pmatrix} e^{-i(Et - \mathbf{p}\cdot\mathbf{x})}, \tag{2.13}$$

for arbitrary constant ϕ and χ. Clearly when $\mathbf{p} = 0$ these solutions reduce to the positive and negative energy solutions discussed previously. Now, since $E^2 = \mathbf{p}^2 + m^2$ by construction, we find, just as we did for the Klein-Gordon equation (1.18), that there exist positive and negative energy solutions given by equations (2.12) and (2.13) respectively. Once again, the existence of negative energy solutions vitiates the interpretation of ψ as a wavefunction.

Dirac interpreted the negative energy solutions by postulating the existence of a "sea" of negative energy states. The vacuum or ground state has all the negative energy states full. An additional electron must now occupy a positive energy state since the Pauli exclusion principle forbids it from falling into one of the filled negative energy states. By promoting one of these negative energy states to a positive energy one, by supplying energy, you create a pair: a positive energy electron and a hole in the negative energy sea corresponding to a positive energy positron. This was a radical new idea, and brought pair creation and antiparticles into physics. Positrons were discovered in cosmic rays by Carl Anderson in 1932.

The problem with Dirac's hole theory is that it doesn't work for bosons, such as particles governed by the Klein Gordon equation, for example. Such particles have no exclusion principle to stop them falling into the negative energy states, releasing their energy. We need a new interpretation and turn to Feynman for our answer.

According to Feynman and quantum field theory, we should interpret the emission (absorption) of a negative energy particle with momentum p^μ as the absorption (emission) of a positive energy antiparticle with momentum $-p^\mu$. So, in Figure 2.1, for example, an electron–positron pair is created at point A. The positron propagates to point B where it is annihilated by another electron.

Thus Feynman tells us to keep both types of free particle solution. One is to be used for particles and the other for the accompanying antiparticles. Let's return to our spinor solutions and write them in a conventional form. Take the positive energy solution of equation (2.12) and write,

$$\sqrt{E+m}\begin{pmatrix}\chi_r \\ \frac{\sigma\cdot\mathbf{p}}{\mathrm{E+m}}\chi_r\end{pmatrix} e^{-ip\cdot x} \equiv u_p^r e^{-ip\cdot x}. \tag{2.14}$$

For the former negative energy solution of equation (2.13), change the sign of the energy, $E \to -E$, and the three-momentum, $\mathbf{p} \to -\mathbf{p}$, to obtain,

$$\sqrt{E+m}\begin{pmatrix}\frac{\sigma\cdot\mathbf{p}}{\mathrm{E+m}}\chi_r \\ \chi_r\end{pmatrix} e^{ip\cdot x} \equiv v_p^r e^{ip\cdot x}. \tag{2.15}$$

In these two solutions E is now (and for the rest of the course) always positive and given by $E = (\mathrm{p}^2 + m^2)^{1/2}$. The subscript r takes the values 1, 2, with

$$\chi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi_2 = \begin{pmatrix}0\\1\end{pmatrix}. \tag{2.16}$$

For the simple case $\mathbf{p} = 0$ we may interpret χ_1 as the spin-up state and χ_2 as the spin-down state. Thus for $\mathbf{p} = 0$ the 4-component wavefunction has a very simple interpretation: the first two components describe electrons with spin-up and spin-down, while the second two components describe positrons with spin-up and spin-down. Thus we understand on physical grounds why the wavefunction had to have four components. The general case $\mathbf{p} \neq 0$ is slightly more involved and is considered in the next section.

At this point I would like to introduce another notation, and define

$$\omega_p \equiv \sqrt{\mathrm{p}^2 + m^2}, \tag{2.17}$$

so that, ω_p is the energy (positive) of a particle or anti-particle with three-momentum $\mathbf{p}$ (I write the subscript p instead of $\mathbf{p}$, but you should remember it really means the three-momentum). I will tend to use E or ω_p interchangeably.

The u-spinor solutions will correspond to particles and the v-spinor solutions to antiparticles. The role of the two χ's will become clear in the following section, where it will be shown that the two choices of r are spin labels. Note that each spinor solution depends on the three-momentum $\mathbf{p}$, so it is implicit that $p^0 = \omega_p$.

2.2 Free Particle Solutions II: Spin

Now it's time to justify the statements we have been making that the Dirac equation describes spin-1/2 particles. The Dirac Hamiltonian in momentum space is given in Eq.2.1 as

$$H_D = \alpha\cdot\mathbf{P} + \beta\mathrm{m} \tag{2.18}$$

and the orbital angular momentum operator is

$$\mathbf{L} = \mathbf{R} \times \mathbf{P}.$$

Normally you have to worry about operator ordering ambiguities when going from classical objects to quantum mechanical ones. For the components of $\mathbf{L}$ the problem does not arise — why not?

Evaluating the commutator of $\mathbf{L}$ with H_D,

$$\begin{aligned} [\mathbf{L}, H_D] &= [\mathbf{R}\times\mathbf{P}, \alpha\cdot\mathbf{P}] \\ &= [\mathbf{R}, \alpha\cdot\mathbf{P}]\times\mathbf{P} \\ &= i\alpha\times\mathbf{P}, \end{aligned} \tag{2.19}$$

we see that the orbital angular momentum is not conserved (otherwise the commutator would be zero). We'd like to find a *total* angular momentum $\mathbf{J}$ which *is* conserved, by adding an additional operator $\mathbf{S}$ to $\mathbf{L}$,

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \quad [\mathbf{J}, H_D] = 0 \tag{2.20}$$

To this end, consider the three matrices,

$$\Sigma \equiv \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\alpha. \tag{2.21}$$

where the first equivalence is merely a definition of Σ and the last equality can readily be verified. The $\Sigma/2$ have the correct commutation relations to represent angular momentum, since the Pauli matrices do, and their commutators with α and β are,

$$[\Sigma, \beta] = 0, \qquad [\Sigma_i, \alpha_j] = 2i\epsilon_{ijk}\alpha_k. \tag{2.22}$$

▷ **Exercise 2.3**

Verify the commutation relations in equation (2.22).

From the relations in (2.22) we find that

$$[\Sigma, H_D] = -2i\alpha\times\mathbf{P}.$$

Comparing this with the commutator of $\mathbf{L}$ with H_D in equation (2.19), you readily see that

$$\left[\mathbf{L} + \tfrac{1}{2}\Sigma, H_D\right] = 0,$$

and we can identify

$$\mathrm{S} = \frac{1}{2}\Sigma.$$

as the additional quantity which when added to $\mathbf{L}$ in Eq.2.20 yields a conserved total angular momentum J. We interpret S as an angular momentum *intrinsic* to the particle. Now

$$\mathrm{S}^2 = \frac{1}{4}\begin{pmatrix} \sigma\cdot\sigma & 0 \\ 0 & \sigma\cdot\sigma \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and recalling that the eigenvalue of $\mathbf{J}^2$ for spin j is $j(j+1)$, we conclude that $\mathbf{S}$ represents spin-1/2 and the solutions of the Dirac equation have spin-1/2 as promised.

We worked in the Dirac representation of the matrices for convenience, but the result is of course independent of the representation.

Now consider the u-spinor solutions u_p^r of equation (2.14). Choose $\mathbf{p} = (0, 0, p_z)$ and write

$$u_\uparrow = u_{p_z}^1 = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix}, \qquad u_\downarrow = u_{p_z}^2 = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix}. \tag{2.23}$$

It is easy to see that,

$$S_z u_\uparrow = \frac{1}{2} u_\uparrow, \qquad S_z u_\downarrow = -\frac{1}{2} u_\downarrow.$$

So, these two spinors represent spin up and spin down along the z-axis respectively. For the v-spinors, with the same choice for $\mathbf{p}$, write,

$$v_\downarrow = v^1_{p_z} = \begin{pmatrix} \sqrt{E-m} \\ 0 \\ \sqrt{E+m} \\ 0 \end{pmatrix}, \qquad v_\uparrow = v^2_{p_z} = \begin{pmatrix} 0 \\ -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} \end{pmatrix}, \tag{2.24}$$

where now,

$$S_z v_\downarrow = \frac{1}{2} v_\downarrow, \qquad S_z v_\uparrow = -\frac{1}{2} v_\uparrow.$$

This apparently perverse choice of up and down for the v's is because, as you see later for the quantum Dirac field, $u_\uparrow$ multiplies an annihilation operator which *destroys* a particle with momentum p_z and spin up, whereas $v_\downarrow$ multiplies an operator which *creates* an antiparticle with momentum p_z and spin up.

2.3 Normalisation, Gamma Matrices

We have included a normalisation factor $\sqrt{E+m}$ in our spinors. With this factor,

$$u^{r\dagger}_p u^s_p = v^{r\dagger}_p v^s_p = 2\omega_p \delta^{rs}. \tag{2.25}$$

This corresponds to the standard relativistic normalisation of $2\omega_p$ particles per unit volume. It also means that $u^\dagger u$ transforms like the time component of a 4-vector under Lorentz transformations as we will see in section 2.6.

▷Exercise 2.4
Check the normalisation condition for the spinors in equation (2.25).

I will now introduce (yet) more standard notation. Define the *gamma matrices*,

$$\gamma^0 = \beta, \qquad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha}. \tag{2.26}$$

In the Dirac representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{2.27}$$

In terms of these, the relations between the α and β in equation (2.4) can be written compactly as,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{2.28}$$

Combinations like $a_\mu \gamma^\mu$ occur frequently and are conventionally written as,

$$\not{a} = a_\mu \gamma^\mu = a^\mu \gamma_\mu,$$

pronounced "a slash." Note that γ^μ is not, despite appearances, a 4-vector — it just denotes a set of four matrices. However, the notation is deliberately suggestive, for when combined with Dirac fields you can construct quantities which transform like vectors and other Lorentz tensors (see the next section).

Let's close this section by observing that using the gamma matrices the Dirac equation (2.2) becomes

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{2.29}$$

or in momentum space,

$$(p\!\!\!/ - m)\psi = 0. \tag{2.30}$$

The spinors u and v satisfy

$$\begin{aligned} (p\!\!\!/ - m)u_p^r &= 0 \\ (p\!\!\!/ + m)v_p^r &= 0 \end{aligned} \tag{2.31}$$

▷**Exercise 2.5**

Derive the momentum space equations satisfied by u_p^r and v_p^r.

2.4 Lorentz Covariance

We want the Dirac equation (2.29) to preserve its form under Lorentz transformations (LT's). Let $\Lambda^\mu{}_\nu$ represent an LT,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.32}$$

A familiar example of a LT is a boost along the z-axis, for which

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix},$$

with as usual $\beta = v$ (in units of c) and $\gamma = (1-\beta^2)^{-1/2}$. LT's can be thought of as generalised rotations.

The requirement is,

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \quad \longrightarrow \quad (i\gamma^\mu\partial'_\mu - m)\psi'(x') = 0,$$

where $\partial_\mu = \Lambda^\sigma{}_\mu\partial'_\sigma$. This last equality follows because

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\sigma}{\partial x^\mu}\frac{\partial}{\partial x'^\sigma} = \Lambda^\sigma{}_\mu\frac{\partial}{\partial x'^\sigma}$$

where Eq.2.32 has been used in the last step. We know that 4-vectors get their components mixed up by LT's, so we expect that the components of ψ might get mixed up also,

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x') \tag{2.33}$$

where $S(\Lambda)$ is a 4×4 matrix acting on the spinor index of ψ. Note that the argument $\Lambda^{-1}x'$ is just a fancy way of writing x, so each component of $\psi(x)$ is transformed into a linear combination of components of $\psi(x)$.

It is helpful to recall that for a vector field, the corresponding transformation is

$$A^{\mu}(x) \to A'^{\mu}(x')$$

where $x' = \Lambda x$. This makes sense physically if one thinks of space rotations of a vector field. For example the wind arrows on a weather map of England are an example of a vector field: at each point on the map there is associated an arrow. Consider the wind direction at a particular point on the map, say Abingdon. If the map of England is rotated, then one would expect on physical grounds that the wind vector at Abingdon always point in the same physical direction and have the same length. In order to achieve this, both the vector itself must rotate, and the point to which it is attached (Abingdon) must be correctly identified after the rotation. Thus the vector at the point x' (corresponding to Abingdon in the rotated frame) is equal to the vector at the point x (corresponding to Abingdon in the unrotated frame), but rotated so as to keep the physical sense of the vector the same in the rotated frame (so that the wind always blows towards Oxford, say, in the two frames). Thus having correctly identified the same point in the two frames all we need to do is rotate the vector:

$$A'^{\mu}(x') = \Lambda^{\mu}_{\nu} A^{\nu}(x).$$

A similar thing also happens in the case of the 4-component spinor field above, except that we do not (yet) know how the components of the wavefunction themselves must transform, i.e. we do not know S.

To determine S we rewrite the Dirac equation in terms of the primed variables (just a mathematical substitution),

$$(i\gamma^{\mu}\Lambda^{\sigma}{}_{\mu}\partial'_{\sigma} - m)\psi(\Lambda^{-1}x') = 0. \tag{2.34}$$

Some new matrices can be defined, $\gamma'^{\sigma} \equiv \gamma^{\mu}\Lambda^{\sigma}{}_{\mu}$ which satisfy the same anticommutation relations as the γ^{μ}'s in equation (2.28),

$$\{\gamma'^{\mu}, \gamma'^{\nu}\} = 2g^{\mu\nu}. \tag{2.35}$$

▷ **Exercise 2.6**
Check relation (2.35).

Now we invoke the theorem (Pauli's theorem) which states that any two representations of the gamma matrices are equivalent. This means that there is a matrix $S(\Lambda)$ such that

$$\gamma'^{\mu} = S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda). \tag{2.36}$$

This allows us to rewrite equation (2.34) as

$$(i\gamma^{\mu}\partial'_{\mu} - m)S(\Lambda)\psi(\Lambda^{-1}x') = 0,$$

or using Eq.2.33,

$$(i\gamma^{\mu}\partial'_{\mu} - m)\psi'(x') = 0, \tag{2.37}$$

so that the Dirac equation does indeed preserve its form in the primed frame.

To construct S explicitly we must solve Eq.2.36, which may be written as,

$$\gamma^{\sigma}\Lambda^{\mu}{}_{\sigma} = S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda). \tag{2.38}$$

For an infinitesimal LT, it can be verified that,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \epsilon(g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu) \tag{2.39}$$

where ϵ is an infinitesimal parameter and ρ and σ are fixed. Since this expression is antisymmetric in ρ and σ there are six choices for the pair (ρ, σ) corresponding to three rotations and three boosts.

For example a boost along the z-axis corresponds to $\rho = 0$, $\sigma = 3$, since in this case,

$$\begin{aligned}\Lambda^\mu{}_\nu &= \delta^\mu{}_\nu - \epsilon(g^{0\mu}\delta^3{}_\nu - g^{3\mu}\delta^0{}_\nu)\\ &= \begin{pmatrix} 1 & 0 & 0 & -\epsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon & 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

which can be identified with the previous example with $\beta = -\epsilon$ and $\gamma = 1$ in the low velocity limit.

Writing,

$$S(\Lambda) = 1 + i\epsilon s^{\rho\sigma} \tag{2.40}$$

where $s^{\rho\sigma}$ is a matrix to be determined for each choice of ρ and σ, we find that equation (2.36) for γ' is satisfied by,

$$s^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma] \equiv \frac{1}{2}\sigma^{\rho\sigma}. \tag{2.41}$$

Here, I have taken the opportunity to define the matrix $\sigma^{\rho\sigma}$. Thus S is given explicitly in terms of gamma matrices, for any LT specified by ρ, σ and ϵ.

▷Exercise 2.7

Verify that equation (2.36) relating γ' and γ is satisfied by $s^{\rho\sigma}$ defined through equations (2.40) and (2.41).

We have thus determined how ψ transforms under LT's. To find quantities which are Lorentz invariant, or transform as vectors or tensors, we need to introduce the Pauli and Dirac adjoints. The Pauli adjoint $\overline{\psi}$ of a *spinor* ψ is defined by

$$\overline{\psi} \equiv \psi^\dagger\gamma^0 = \psi^\dagger\beta. \tag{2.42}$$

The Dirac adjoint of a *matrix* A is defined by

$$(\overline{\psi}A\phi)^* = \overline{\phi}\,\overline{A}\psi. \tag{2.43}$$

For Hermitian γ^0 it is easy to show that

$$\overline{A} = \gamma^0 A^\dagger \gamma^0. \tag{2.44}$$

Some properties of the Pauli and Dirac adjoints are:

$$\begin{aligned}\overline{(\lambda A + \mu B)} &= \lambda^*\overline{A} + \mu^*\overline{B},\\ \overline{AB} &= \overline{B}\,\overline{A},\\ \overline{A\psi} &= \overline{\psi}\,\overline{A}.\end{aligned}$$

With these definitions, $\overline{\psi}$ transforms as follows under LT's:

$$\overline{\psi} \to \overline{\psi}' = \overline{\psi}S^{-1}(\Lambda) \tag{2.45}$$

▷Exercise 2.8

(1) Verify that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. This says that $\overline{\gamma}^\mu = \gamma^\mu$.

(2) Using (2.40) and (2.41) verify that $\gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda)$, i.e. $\overline{S} = S^{-1}$. So S is not unitary in general, although it *is* unitary for rotations (when ρ and σ are spatial indices). This is because the rotations are in the unitary $O(3)$ subgroup of the nonunitary Lorentz group. Here you show the result for an infinitesimal LT, but it is true for finite LT's.

(3) Show that $\overline{\psi}$ satisfies the equation

$$\overline{\psi}(-i\overleftarrow{\not\partial} - m) = 0$$

where the arrow over $\not\partial$ implies the derivative acts on $\overline{\psi}$.

(4) Hence prove that $\overline{\psi}$ transforms as in equation (2.45).

Note that result (2) of the problem above can be rewritten as $\overline{S}(\Lambda) = S^{-1}(\Lambda)$, and equation (2.36) for the similarity transformation of γ^μ to γ'^μ takes the form,

$$\overline{S}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu. \tag{2.46}$$

Combining the transformation properties of ψ and $\overline{\psi}$ in equations (2.33) and (2.45) we see that the bilinear $\overline{\psi}\psi$ is Lorentz invariant. In section 2.6 we'll consider the transformation properties of general bilinears.

Let me close this section by recasting the spinor normalisation equations (2.25) in terms of "Dirac inner products." The conditions become,

$$\begin{aligned} \overline{u}_p^r u_p^s &= 2m\delta^{rs} \\ \overline{u}_p^r v_p^s &= 0 \qquad = \overline{v}_p^r u_p^s \\ \overline{v}_p^r v_p^s &= -2m\delta^{rs} \end{aligned} \tag{2.47}$$

▷ **Exercise 2.9**

Verify the normalisation properties in the above equations (2.47).

2.5 Parity

In the next section we are going to construct quantities bilinear in ψ and $\overline{\psi}$, and classify them according to their transformation properties under LT's. We normally use LT's which are in the connected Lorentz Group, $SO(3,1)$, meaning they can be obtained by a continuous deformation of the identity transformation. Indeed in the last section we considered LT's very close to the identity in equation (2.39). The full Lorentz group has four components generated by combining the $SO(3,1)$ transformations with the discrete operations of parity or space inversion, P, and time reversal, T,

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

LT's satisfy $\Lambda^T g\Lambda = g$ (see the preschool problems), so taking determinants shows that $\det\Lambda = \pm 1$. LT's in $SO(3,1)$ have determinant 1, since the identity does, but the P and T operations have determinant -1.

Let's now find the action of parity on the Dirac wavefunction and determine the wavefunction ψ_P in the parity-reversed system. According to the discussion of the previous section, and using the result of equation (2.46), we need to find a matrix S satisfying

$$\overline{S}\gamma^0 S = \gamma^0, \qquad \overline{S}\gamma^i S = -\gamma^i.$$

It's not hard to see that $S = \overline{S} = \gamma^0$ is an acceptable solution, from which it follows that the wavefunction ψ_P is

$$\psi_P(t, \mathbf{x}) = \gamma^0 \psi(t, -\mathbf{x}). \tag{2.48}$$

In fact you could multiply γ^0 by a phase and still have an acceptable definition for the parity transformation.

In the nonrelativistic limit, the wavefunction ψ approaches an eigenstate of parity. Since

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the u-spinors and v-spinors at rest have opposite eigenvalues, corresponding to particle and antiparticle having opposite *intrinsic* parities.

2.6 Bilinear Covariants

Now, as promised, we will construct and classify the bilinears. To begin, observe that by forming products of the gamma matrices it is possible to construct 16 linearly independent quantities. In equation (2.41) we have defined

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu],$$

and now it is convenient to define

$$\gamma_5 \equiv \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.49}$$

with the properties,

$$\gamma_5^\dagger = \gamma_5, \qquad \{\gamma_5, \gamma^\mu\} = 0.$$

Then the set of 16 matrices

$$\Gamma : \{1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$$

form a basis for gamma matrix products.

Using the transformations of ψ and $\overline{\psi}$ from equations (2.33) and (2.45), together with the similarity transformation of γ^μ in equation (2.46), construct the 16 fermion bilinears and their transformation properties as follows:

$$\begin{array}{rcll} \overline{\psi}\psi & \to & \overline{\psi}\psi & \text{S scalar} \\ \overline{\psi}\gamma_5\psi & \to & \det(\Lambda)\,\overline{\psi}\gamma_5\psi & \text{P pseudoscalar} \\ \overline{\psi}\gamma^\mu\psi & \to & \Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\psi & \text{V vector} \\ \overline{\psi}\gamma^\mu\gamma_5\psi & \to & \det(\Lambda)\,\Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\gamma_5\psi & \text{A axial vector} \\ \overline{\psi}\sigma^{\mu\nu}\psi & \to & \Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma\overline{\psi}\sigma^{\lambda\sigma}\psi & \text{T tensor} \end{array} \tag{2.50}$$

▷ **Exercise 2.10**
Verify the transformation properties of the bilinears in equation (2.50).

Observe that $\overline{\psi}\gamma^{\mu}\psi = (\rho, \mathbf{J})$ is just the current we found earlier in equation (2.7). Classically ρ is positive definite, but for the quantum Dirac field you find that the space integral of ρ is the charge operator, which counts the number of electrons minus the number of positrons,

$$Q \sim \int d^3x\, \psi^\dagger \psi \sim \int d^3p\, [b^\dagger b - d^\dagger d].$$

The continuity equation $\partial_\mu J^\mu = 0$ expresses conservation of electric charge.

2.7 Charge Conjugation

There is one more discrete invariance of the Dirac equation in addition to parity. It is charge conjugation, which takes you from particle to antiparticle and vice versa. For scalar fields the symmetry is just complex conjugation, but in order for the charge conjugate Dirac field to remain a solution of the Dirac equation, you have to mix its components as well:

$$\psi \to \psi_C = C\overline{\psi}^T.$$

Here $\overline{\psi}^T = \gamma^{0T}\psi^*$ and C is a matrix satisfying the condition

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu.$$

In the Dirac representation,

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}.$$

I refer you to textbooks such as [1] for details.

When Dirac wrote down his equation everybody thought parity and charge conjugation were exact symmetries of nature, so invariance under these transformations was essential. Now we know that neither of them, nor the combination *CP*, are respected by the standard electroweak model.

2.8 Neutrinos

In the particle data book [2] you will find only upper limits for the masses of the three neutrinos, and in the standard model they are massless. Let's look therefore at solutions of the Dirac equation with $m = 0$. From Eq.2.9 we have in this case

$$E\phi = \sigma\cdot\mathbf{p}\,\chi, \qquad E\chi = \sigma\cdot\mathbf{p}\,\phi. \tag{2.51}$$

These equations can easily be decoupled by taking the linear combinations and defining in a suggestive way the two component spinors ν_L and ν_R,

$$\nu_R \equiv \chi + \phi, \quad \nu_L \equiv \chi - \phi \tag{2.52}$$

which leads to

$$E\nu_R = \sigma\cdot\mathbf{p}\,\nu_R, \qquad E\nu_L = -\sigma\cdot\mathbf{p}\,\nu_L. \tag{2.53}$$

Since $E = |\mathbf{p}|$ for massless particles, these equations may be written,

$$\frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}\nu_L = -\nu_L, \qquad \frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}\nu_R = \nu_R \tag{2.54}$$

Since $\frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|}$ is identified as the helicity operator (i.e. the spin operator projected in the direction of motion of the momentum of the particle) we see that the ν_L corresponds to solutions with negative helicity, while ν_R corresponds to solutions with positive helicity. In other words ν_L descibes a left-handed neutrino while ν_R describes a right-handed neutrino – and each type of neutrino is descibed by a two-component spinor.

The two-component spinors describing neutrinos transform very simply under LT's,

$$\nu_L \to e^{\frac{i}{2}\sigma\cdot(\theta - i\phi)}\nu_L \tag{2.55}$$

$$\nu_R \to e^{\frac{i}{2}\sigma\cdot(\theta + i\phi)}\nu_R \tag{2.56}$$

where $\theta = \mathbf{n}\theta$ corresponding to space rotations through an angle θ about the unit $\mathbf{n}$ axis, and $\phi = \mathbf{v}\phi$ corresponding to Lorentz boosts along the unit vector $\mathbf{v}$ with a speed $v = \tanh\phi$. Under parity transformations they become transformed into each other,

$$\nu_L \leftrightarrow \nu_R \tag{2.57}$$

so a theory which involves only ν_L without ν_R (such as the standard model) manifestly violates parity.

Although massless neutrinos can be described very simply using two component spinors as above, they may also be incorporated into the four-component formalism as follows. From equation (2.2) we have, in momentum space,

$$|\mathbf{p}|\psi = \alpha\cdot\mathbf{p}\,\psi.$$

For such a solution,

$$\gamma_5\psi = \gamma_5\frac{\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\psi = 2\frac{\mathbf{S}\cdot\mathbf{p}}{|\mathbf{p}|}\psi,$$

using the spin operator $\mathbf{S} = \frac{1}{2}\Sigma = \frac{1}{2}\gamma_5\alpha$, with Σ defined in equation (2.21). But $\mathbf{S}\cdot\mathbf{p}/|\mathbf{p}|$ is the projection of spin onto the direction of motion, known as the *helicity*, and is equal to $\pm 1/2$. Thus $(1+\gamma_5)/2$ projects out the neutrino with helicity $1/2$ (right handed) and $(1-\gamma_5)/2$ projects out the neutrino with helicity $-1/2$ (left handed),

$$\frac{(1+\gamma_5)}{2}\psi \equiv \psi_R, \quad \frac{(1-\gamma_5)}{2}\psi \equiv \psi_L, \tag{2.58}$$

which defines the four-component spinors ψ_R and ψ_L.

To date, only left handed neutrinos have been observed, and only left handed neutrinos appear in the standard model. Since

$$\gamma^0\frac{1}{2}(1-\gamma_5)\psi = \frac{1}{2}(1+\gamma_5)\gamma^0\psi,$$

any theory involving only left handed neutrinos necessarily violates parity - as we saw before in the two-component formalism.

Finally note that in the Dirac representation which we have been using,

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.59}$$

and the relation between the two-component and four-component formalisms is via the change of variables in Eq.2.52. However there exists a representation in which this change of variables is done automatically and the (massless) Dirac equation falls apart into the two two-component equations discussed above. In this chiral representation,

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.60}$$

and hence,

$$\frac{(1+\gamma_5)}{2}\psi = \begin{pmatrix} 0 \\ \nu_R \end{pmatrix}, \quad \frac{(1-\gamma_5)}{2}\psi = \begin{pmatrix} \nu_L \\ 0 \end{pmatrix}. \tag{2.61}$$

where we have identified ν_R and ν_L as the two-component spinors discussed previously. These results are also applicable to the electron in the approximation that its mass is neglected, by the simple transcription $\nu_R \to e_R$, $\nu_L \to e_L$. In fact in the standard model the electrons start out massless, so these results will be of use to Tim Morris in his course.

The standard model (and the minimal supersymmetric standard model) contains only left handed massless neutrinos, and neutrino mass terms are forbidden by gauge symmetry, at least given the limited number of fields present in the standard model. If extra fields (e.g. right handed neutrinos) are added then neutrino masses become possible. If neutrino oscillations are confirmed as the solution to the solar neutrino problem, or are discovered in laboratory experiments, then such a modification would become a necessity.

3 Cross Sections and Decay Rates

In section 4 we will learn how to calculate quantum mechanical amplitudes for electromagnetic scattering and decay processes. These amplitudes are obtained from the Lagrangian of QED, and contain information about the dynamics underlying the scattering or decay process. This section is a brief review of how to get from the quantum mechanical amplitude to a cross section or decay rate which can be measured. We will commence by recalling Fermi's golden rule for transition probabilities.

3.1 Fermi's Golden Rule

Consider a system with Hamiltonian H which can be written

$$H = H_0 + V \tag{3.1}$$

We assume that the eigenstates and eigenvalues of H_0 are known and that V is a small, possibly time-dependent, perturbation. The equation of motion of the system is,

$$i\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = (H_0 + V)\left|\psi(t)\right\rangle. \tag{3.2}$$

If V vanished, we could calculate the time evolution of $|\psi(t)\rangle$ by expanding it as a linear combination of energy eigenstates. When V does not vanish, the eigenstates of H_0 are no longer eigenstates of the full Hamiltonian so when we expand in terms of H_0 eigenstates, the coefficients of the expansion become time dependent. To develop a perturbation theory in V we will change our basis of states from the Schrödinger picture to the *interaction* or Dirac picture, where we hide the time evolution due to H_0 and concentrate on that due to V. Thus we define the interaction picture states and operators by,

$$\left|\psi_I(t)\right\rangle \equiv e^{iH_0t}\left|\psi(t)\right\rangle, \qquad \mathcal{O}_I(t) \equiv e^{iH_0t}\mathcal{O}(t)e^{-iH_0t}, \tag{3.3}$$

so that the interaction picture and Schrödinger picture states agree at time $t = 0$, $|\psi_I(0)\rangle = |\psi(0)\rangle$, with a similar relation for the operators. In the new basis, the equation of motion becomes,

$$i\frac{\partial}{\partial t}\left|\psi_I(t)\right\rangle = V_I(t)\left|\psi_I(t)\right\rangle, \tag{3.4}$$

which can be iterated to yield an infinite series in V,

$$\left|\psi_I(t)\right\rangle = \left[1 + \sum_{n=1}^{\infty}\frac{1}{i^n}\int_{-T/2}^{t}dt_1\int_{-T/2}^{t_1}dt_2\cdots\int_{-T/2}^{t_{n-1}}dt_n\, V_I(t_1)V_I(t_2)\cdots V_I(t_n)\right]\left|\psi_I(-T/2)\right\rangle. \tag{3.5}$$

The interation involves formally integrating Eq.3.4 by writing

$$\int_{-T/2}^{t} d\left|\psi_I(t_1)\right\rangle = \int_{-T/2}^{t}\frac{1}{i}V_I(t_1)\left|\psi_I(t_1)\right\rangle dt_1$$

To first order we insert $|\psi_I(t_1)\rangle \approx |\psi_I(-T/2)\rangle$ into the rhs so that we have,

$$\left|\psi_I(t)\right\rangle \approx \left|\psi_I(-T/2)\right\rangle + \int_{-T/2}^{t}\frac{1}{i}V_I(t_1)dt_1\left|\psi_I(-T/2)\right\rangle \tag{3.6}$$

Then this solution is used in the rhs of the original equation to improve the approximation, and so on. This process of iteration is useful only if the perturbation V_I is small and so a small number of terms in the series (i.e. a small number of iterations) may be taken.

Here, we have chosen to start with some (known) state $|\psi_I(-T/2)\rangle$, at time $-T/2$, and have evolved it to $|\psi_I(t)\rangle$ at time t. The evolution is done by the operator, U, that you've seen in the field theory course:

$$|\psi_I(t)\rangle = U(t, -T/2)\,|\psi_I(-T/2)\rangle\,.$$

For an infinitesimal time interval the operator U is given by

$$U(t+\delta t, t) = I - iV_I(t)\delta t \tag{3.7}$$

which is the formal solution to Eq.3.4 over an infinitesimal time.

Now consider the calculation of the probability of a transition to an eigenstate $|b\rangle$ at time t. The amplitude is,

$$\begin{aligned}\langle b|\psi(t)\rangle &= \langle b_I|\psi_I(t)\rangle \\ &= \langle b|\, e^{-iH_0 t}\,|\psi_I(t)\rangle \\ &= e^{-iE_b t}\langle b|\psi_I(t)\rangle,\end{aligned}$$

so $\left|\langle b|\psi(t)\rangle\right|^2 = \left|\langle b|\psi_I(t)\rangle\right|^2$. We let V be time independent and consider the amplitude for a transition from an eigenstate $|a\rangle$ of H_0 at $t = -T/2$ to an orthogonal eigenstate $|b\rangle$ at $t = T/2$. The idea is that at very early or very late times H_0 describes some set of free particles. We allow some of these particles to approach each other and scatter under the influence of V, then look again a long time later when the outgoing particles are propagating freely under H_0 again. To first order in V, using Eq.3.6 we find

$$\langle b\,|\psi_I(T/2)\rangle = -i\int_{-T/2}^{T/2}\langle b|\,V_I(t)\,|a\rangle\,dt = -i\langle b|V|a\rangle\int_{-T/2}^{T/2} e^{i\omega_{ba}t}dt,$$

$$\langle b\,|\psi_I(T/2)\rangle = -i\langle b|V|a\rangle\frac{2}{\omega_{ba}}\sin(\omega_{ba}T/2) \tag{3.8}$$

where $\omega_{ba} = E_b - E_a$.

▷**Exercise 3.1**

Show that for $T\to\infty$ the first order transition amplitude for general V can be written in the covariant form

$$\langle b\,|\psi_I(\infty)\rangle = -i\int d^4x\,\phi_b^*(x)V\phi_a(x),$$

where $\phi_i(x) \equiv \phi_i(\mathrm{x})e^{-E_i t}$ and $\phi_i(\mathrm{x})$ is the usual Schrödinger wavefunction for a stationary state of H_0, with energy E_i.

The transition rate W_{ba} for time independent V is just given by the probability of the scattering taking place divided by the time T taken,

$$W_{ba} = \frac{\left|\langle b\,|\psi_I(T/2)\rangle\,\right|^2}{T} = \left|\langle b|V|a\rangle\right|^2\frac{4\sin^2(\omega_{ba}T/2)}{\omega_{ba}^2 T}.$$

If $E_b \neq E_a$, this probability tends to zero as $T \to \infty$. However, for $E_b = E_a$ we use the result,

$$\frac{1}{2\pi T}\frac{\sin^2(\omega_{ba}T/2)}{(\omega_{ba}/2)^2} \xrightarrow{T\to\infty} \delta(\omega_{ba}). \tag{3.9}$$

For long times the transition rate becomes,

$$W_{ba} = 2\pi\left|\langle b|V|a\rangle\right|^2 \delta(E_b - E_a). \tag{3.10}$$

We need V small for the first order result to be useful and T large so that the delta-function approximation is good. However, T cannot be too large since the transition probability grows with time and we don't want probabilities larger than one.

The above result assumes a well-spaced discrete set of states. Typically scattering takes place from some initial state $|a\rangle$ into one of a continuous number dn of final states closely spaced around $|b\rangle$. In this case since there are dn states to scatter into rather than just one, we multiply by dn to give a differential transition rate,

$$dW_{ba} = 2\pi dn\left|\langle b|V|a\rangle\right|^2 \delta(E_b - E_a). \tag{3.11}$$

If we define a density of final states $\rho(E_b) = \frac{dn}{dE_b}$ around $|b\rangle$ with energy E_b, the differential transition rate may be integrated over the final state energy,

$$W_{ba} = \int 2\pi\left|\langle b|V|a\rangle\right|^2 \delta(E_b - E_a)\rho(E_b)dE_b = 2\pi\rho(E_a)\left|\langle b|V|a\rangle\right|^2. \tag{3.12}$$

This is *Fermi's golden rule.* In words it says simply:

$$transition\ rate = 2\pi \times density\ of\ final\ states \times |amplitude|^2.$$

▷**Exercise 3.2**

Justify the result of equation (3.9) and hence verify Fermi's golden rule in equation (3.12).

I'll stop at first order in V. The answer you get from the formal solution in equation (3.5) depends on the form of V and the initial conditions. Your field theory course gives you a systematic way to perform perturbative calculations of transition amplitudes in field theories by the use of Feynman diagrams. In particular, you've seen the operator method of generating these diagrams, which I've mirrored in deriving the Golden Rule. Let's now move on to see how to get from these amplitudes to cross-sections and decay rates. This corresponds to finding the density of states factor in the Golden Rule.

3.2 Transition Rates in Quantum Field Theory

We now apply these ideas to quantum field theory. We first discuss the quantum field theory prediction for the amplitude, then discuss the number of final states, hopefully getting all the normalisation factors straight. We then define the famous Lorentz Invariant Phase Space.

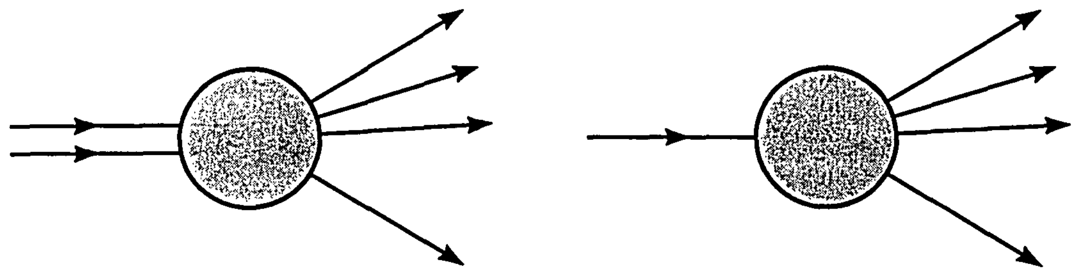


Figure 3.1 Scattering (a) and decay (b) processes.

3.2.1 The Amplitude

We saw in the previous section that $\langle b|\psi_I(\infty)\rangle$ gives the probability amplitude to go from state $|a\rangle$ in the far past to state $|b\rangle$ in the far future. In quantum field theory you calculate the amplitude to go from state $|i\rangle$ to state $|f\rangle$ to be,

$$i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f - P_i), \tag{3.13}$$

where $i\mathcal{M}_{fi}$ is the result obtained from a Feynman diagram calculation, and the overall energy-momentum delta function has been factored out (so when you draw your Feynman diagrams you conserve energy-momentum at every vertex). We have in mind processes where two particles scatter, or one particle decays, as shown in Figure 3.1.

Attempting to take the squared modulus of this amplitude produces a meaningless square of a delta function. This is a technical problem because our amplitude is expressed between non-normalisable plane wave states. These states extend throughout space-time so the scattering process occurs everywhere all the time. To deal with this properly you can construct normalised wavepacket states which do become well separated in the far past and the far future. We will be low-budget and put our system in a box of volume $V = L^3$ [1]. We also imagine that the interaction is restricted to act only over a time of order T. The final answers come out independent of V and T, reproducing the luxury wavepacket ones. We are in good company here: Nobel Laureate Steven Weinberg says in his recent book, when discussing cross sections and decay rates, "...(as far as I know) no interesting open problems in physics hinge on getting the fine points right regarding these matters."

In infinite spacetime with plane wave states the transition amplitude from i to f is given by (3.13). However in our box of finite size L and for our finite time T the amplitude is given by Eq.3.13 but with the Dirac delta functions replaced by well behaved functions:

$$(2\pi)^4\delta^4(P_f - P_i) \to I(E_f - E_i, T)I^3(\mathbf{P}_f - \mathbf{P}_i, L) \tag{3.14}$$

where for example,

$$I(E_f - E_i, T) = \frac{1}{\left(\frac{E_f - E_i}{2}\right)} \sin\left(\frac{(E_f - E_i)T}{2}\right) \tag{3.15}$$

[1]Please do not confuse the volume of the cube $V = L^3$ with the potential V introduced earlier

which is familiar from Eq.3.8. This function has the property that, as $T \to \infty$,

$$I(E_f - E_i, T) \to 2\pi\delta(E_f - E_i) \tag{3.16}$$

and also

$$I^2(E_f - E_i, T) \to 2\pi T\delta(E_f - E_i) \tag{3.17}$$

with analagous results for $I(P_f - P_i, L)$. Thus in our spacetime box we have the approximate result,

$$\left|(2\pi)^4\delta^4(P_f - P_i)\right|^2 \simeq VT\,(2\pi)^4\delta^4(P_f - P_i). \tag{3.18}$$

The second ingredient in the amplitude is a factor of $1/(2E_iV)^{1/2}$ for every particle in the initial or final state (here I am using E_i synonymously with ω_{k_i}). This comes from converting between relativistic and box normalisations for the states.

The box states are normalised to one particle in volume V and the relativistic states have $2\omega_k$ particles per unit volume, thus the states which occur in the amplitude are related by

$$|k\rangle_{\text{rel}} \longleftrightarrow \sqrt{2\omega_k}\sqrt{V}|\mathbf{k}\rangle_{\text{box}}.$$

We shall henceforth use box normalisation for the final states which we simply label by $|f\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle$ and similarly for the initial states which we write as

$$|i\rangle = \begin{cases} |\mathbf{p}\rangle & \text{one particle} \\ |\mathbf{p}_1, \mathbf{p}_2\rangle & \text{two particles} \end{cases} \tag{3.19}$$

Allowing for one or two particles in the initial state and N in the final state,

$$\text{box amp} = i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f - P_i)\prod_{f=1}^{N}\left[\frac{1}{\sqrt{2E_fV}}\right]\prod_{\text{in}}\left[\frac{1}{\sqrt{2E_iV}}\right],$$

The squared matrix element is thus:

$$|\text{box amp}|^2 = |\mathcal{M}_{fi}|^2VT(2\pi)^4\delta^4(P_f - P_i)\prod_{f=1}^{N}\left[\frac{1}{2E_fV}\right]\prod_{\text{in}}\left[\frac{1}{2E_iV}\right], \tag{3.20}$$

where we have used Eq.3.18.

3.2.2 The Number of Final States

For a single particle final state, the number of available states dn in some momentum range $\mathbf{k}$ to $\mathbf{k} + d\mathbf{k}$ is, in the box normalisation,

$$dn = \frac{d^3\mathbf{k}}{(2\pi)^3}V. \tag{3.21}$$

This result is proved by recalling that the allowed momenta in the box have components which can only take on discrete values such as $k_x = 2\pi n_x/L$ where n_x is an integer. Thus $dn = dn_x dn_y dn_z$ and the result follows.

For a two particle final state we have

$$dn = dn_1 dn_2$$

where

$$dn_1 = \frac{d^3\mathbf{k}_1}{(2\pi)^3}V, \quad dn_2 = \frac{d^3\mathbf{k}_2}{(2\pi)^3}V,$$

where dn is the number of final states in some momentum range $\mathbf{k}_1$ to $\mathbf{k}_1 + d\mathbf{k}_1$ for particle 1 and $\mathbf{k}_2$ to $\mathbf{k}_2 + d\mathbf{k}_2$ for particle 2. There is an obvious generalisation to an N particle final state,

$$dn = \prod_{f=1}^{N} \frac{d^3\mathbf{k}_f V}{(2\pi)^3}. \tag{3.22}$$

3.2.3 Lorentz Invariant Phase Space (LIPS)

Our experience with Fermi's Golden Rule tells us that the differential transition rate is given by

$$dW = \frac{|\text{box amp}|^2 dn}{T} \tag{3.23}$$

Note that the energy delta function in Fermi's Golden Rule has already been taken into account by the presence of the energy delta function multiplying the original amplitude in Eq.3.13.

Using Eqs.3.22 and Eq.3.20 we find,

$$dW = |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^4(P_f - P_i) V \prod_{f=1}^{N} \left[\frac{1}{2E_f V}\right] \prod_{\text{in}} \left[\frac{1}{2E_i V}\right] \prod_{f=1}^{N} \frac{d^3\mathbf{k}_f V}{(2\pi)^3}. \tag{3.24}$$

$$dW = |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^4(P_f - P_i) V \prod_{\text{in}} \left[\frac{1}{2E_i V}\right] \prod_{f=1}^{N} \frac{d^3\mathbf{k}_f}{(2\pi)^3 2E_f}. \tag{3.25}$$

This can be written as

$$dW = S\, |\mathcal{M}_{fi}|^2 V \prod_{\text{in}} \left[\frac{1}{2E_i V}\right] \times (LIPS)\ , \tag{3.26}$$

where the LIPS is,

$$LIPS = (2\pi)^4 \delta^4(P_f - P_i) \prod_{f=1}^{N} \frac{d^3\mathbf{k}_f}{(2\pi)^3\, 2E_f}. \tag{3.27}$$

Observe that everything in the transition rate is Lorentz invariant save for the initial energy factor and the factors of V (using $d^3k/2E = d^4k\, \delta^4(k^2 - m^2)\theta(k^0)$, which is manifestly Lorentz invariant, where $E = (\mathbf{k}^2 + m^2)^{1/2}$). For a one particle initial state the factor of V cancels, and we can breath a sigh of relief (after all we would not expect physical quantities to depend on the size of our artificial box). For a two initial particle scattering situation the factors of V will also cancel in the physical cross-section as we will show in the next section. I have smuggled in one extra factor, S, in equation (3.23) for the transition probability. If there are some identical particles in the final state, we will overcount them when integrating over all momentum configurations. The symmetry factor S takes care of this. If there n_i identical particles of type i in the final state, then

$$S = \prod_i \frac{1}{n_i!}. \tag{3.28}$$

▷ **Exercise 3.3**
Show that the expression for two-body phase space in the centre of mass frame is given by

$$\frac{d^3k_1}{(2\pi)^3\,2\omega_{k_1}}\frac{d^3k_2}{(2\pi)^3\,2\omega_{k_2}}(2\pi)^4\delta^4(P-k_1-k_2)=\frac{1}{32\pi^2 s}\lambda^{1/2}(s,m_1^2,m_2^2)d\Omega^*, \tag{3.29}$$

where $s=P^2$ is the centre of mass energy squared, $d\Omega^*$ is the solid angle element for the angle of one of the outgoing particles with respect to some fixed direction, and

$$\lambda(a,b,c)=a^2+b^2+c^2-2ab-2bc-2ca. \tag{3.30}$$

3.3 Cross Sections

The total cross-section for a static target and a beam of incoming particles is defined as the total transition rate for a single target particle and a unit beam flux. The differential cross-section is similarly related to the differential transition rate. We have calculated the differential transition rate with a choice of normalisation corresponding to a single "target" particle in the box, and a "beam" corresponding also to one particle in the box. A beam consisting of one particle per volume V with a velocity v has a flux N_0 given by

$$N_0=\frac{v}{V}$$

particles per unit area per unit time. Thus the differential cross-section σ is related to the differential transition rate in Eq.3.26 by

$$d\sigma=\frac{dW}{N_0}=dW\times\frac{V}{v} \tag{3.31}$$

where as promised the factors of V cancel in the cross-section.

Now let us generalise to the case where in the frame where you make your measurements the "beam" has a velocity v_1 but the "target" particles are also moving with a velocity v_2. In a colliding beam experiment for example v_1 and v_2 will point in opposite directions in the laboratory. In this case the definition of the cross-section is retained as above, but now the beam flux of particles N_0 is effectively increased by the fact that the target particles are moving towards it. The effective flux in the laboratory in this case is given by

$$N_0=\frac{|\vec{v}_1-\vec{v}_2|}{V}$$

which is just the total of particles per unit area which run past each other per unit time. I denote the velocities with arrows to remind you that they are vector velocities which must be added using the vector law of velocity addition not the relativistic law. In the general case, then, the differential cross-section is given by

$$d\sigma=\frac{dW}{N_0}=\frac{1}{|\vec{v}_1-\vec{v}_2|}\frac{1}{4E_1E_2}S|\mathcal{M}_{fi}|^2\times LIPS \tag{3.32}$$

where we have used Eq.3.26 for the transition rate, and the box volume V has again cancelled (phew!). [2] We re-emphasise that the velocities in the flux factor, $1/|\vec{v}_1-$

[2] Because the result is independent of the dimensions of the box, you can think of making the box as large as you like – say as large as CERN or perhaps as large as the Earth, or the Universe! This means that there is no reason to worry about the box.

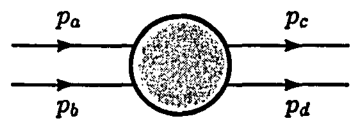


Figure 3.2 $2 \to 2$ scattering.

$\vec{v}_2|$, are subtracted *nonrelativistically.* The amplitude-squared and phase space factors are manifestly Lorentz invariant. What about the initial velocity and energy factors? Observe that

$$E_1E_2(\vec{v}_1 - \vec{v}_2) = E_2\mathbf{p}_1 - E_1\mathbf{p}_2.$$

In a frame where $\mathbf{p}_1$ and $\mathbf{p}_2$ are collinear,

$$|E_2\mathbf{p}_1 - E_1\mathbf{p}_2|^2 = (p_1 \cdot p_2)^2 - m_1^2 m_2^2,$$

and the last expression is manifestly Lorentz invariant. Hence the differential cross section is Lorentz invariant, as is the total cross section,

$$\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2|}\frac{1}{4E_1E_2} S \sum \int_{\text{final states}} |\mathcal{M}_{fi}|^2 \times LIPS. \tag{3.33}$$

3.3.1 Two-body Scattering

An important special case is $2 \to 2$ scattering (see Figure 3.2),

$$a(p_a) + b(p_b) \to c(p_c) + d(p_d).$$

▷**Exercise 3.4**
Show that in the centre of mass frame the differential cross section is,

$$\frac{d\sigma}{d\Omega^*} = \frac{S\,\lambda^{1/2}(s, m_c^2, m_d^2)}{64\pi^2 s\,\lambda^{1/2}(s, m_a^2, m_b^2)}|\mathcal{M}_{fi}|^2. \tag{3.34}$$

The result of equation (3.34) is valid for any $|\mathcal{M}_{fi}|^2$, but if $|\mathcal{M}_{fi}|^2$ is a constant you can trivially get the total cross section.

Invariant $2 \to 2$ scattering amplitudes are frequently expressed in terms of the *Mandelstam variables*, defined by,

$$\begin{aligned} s &\equiv (p_a + p_b)^2 &= (p_c + p_d)^2 \\ t &\equiv (p_a - p_c)^2 &= (p_b - p_d)^2 \\ u &\equiv (p_a - p_d)^2 &= (p_b - p_c)^2 \end{aligned} \tag{3.35}$$

In fact there are only two independent Lorentz invariant combinations of the available momenta in this case, so there must be some relation between s, t and u.

▷**Exercise 3.5**
Show that

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2.$$

▷**Exercise 3.6**
Show that for two body scattering of particles of equal mass m,

$$s \geq 4m^2, \qquad t \leq 0, \qquad u \leq 0.$$

3.4 Decay Rates

With one particle in the initial state the total transition rate is

$$W = \frac{1}{2E} S \sum \int_{\text{final states}} |\mathcal{M}_{fi}|^2 \times LIPS$$

Only the factor $1/2E$ is not manifestly Lorentz invariant. In the rest frame, for a particle of mass m, we have

$$\Gamma \equiv \frac{1}{2m} \sum \int_{\text{final states}} |\mathcal{M}_{fi}|^2 \times LIPS. \tag{3.36}$$

This is the "decay rate." In an arbitrary frame we find, $W = (m/E)\Gamma$, which has the expected Lorentz dilatation factor. In the master formula (equation 3.26) this is what the product of $1/2E_i$ factors for the initial particles does.

3.5 Optical Theorem

When discussing the Golden Rule, we encountered the evolution operator $U(t', t)$, which you also met in the field theory course. This takes a state at time t and evolves it to time t'. The scattering amplitudes we calculate in field theory are between states in the far past and the far future: hence they are matrix elements of $U(\infty, -\infty)$, which is known as the scattering operator or S-operator,

$$S \equiv U(\infty, -\infty)$$

Since the S-operator is unitary, we can write,

$$(S - I)(S^\dagger - I) = -\big((S - I) + (S - I)^\dagger\big). \tag{3.37}$$

Note that $S - I$ is the quantity of interest, since we generally ignore cases where there is no interaction (the "I" piece of S). In terms of the invariant amplitude,

$$\begin{aligned} \langle f|\, S - I \,|i\rangle &= i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f - P_i) \\ \langle f|\, (S - I)^\dagger \,|i\rangle &= -i\mathcal{M}^*_{if}(2\pi)^4\delta^4(P_f - P_i) \end{aligned}$$

Sandwiching the above unitarity relation (equation 3.37) between states $|i\rangle$ and $|f\rangle$, and inserting a complete set of states between the factors on the left hand side,

$$\begin{aligned} &\sum_m \langle f|\, S - I \,|m\rangle \langle m|\, S^\dagger - I \,|i\rangle \\ &= \sum_m \mathcal{M}_{fm}\mathcal{M}^*_{im}(2\pi)^8\delta^4(P_f - P_m)\delta^4(P_i - P_m) \prod_{j=1}^{r_m} \frac{d^3\mathbf{k}_j}{(2\pi)^3\, 2E_j} \\ &= \sum_m \int \mathcal{M}_{fm}\mathcal{M}^*_{im}(2\pi)^4\delta^4(P_f - P_i) D_m \end{aligned}$$

where D_m is the LIPS for the state labelled by m, containing r_m particles, $D_m \equiv D_{r_m}(P_i; k_1, \ldots, k_{r_m})$. Hence,

$$\sum_m \int \mathcal{M}_{fm}\mathcal{M}^*_{im} D_m = i(\mathcal{M}^*_{if} - \mathcal{M}_{fi}).$$

If the intermediate state m contains n_i identical particles of type i, there is an extra symmetry factor S, with,

$$S = \prod_i \frac{1}{n_i!}$$

on the left hand side of the above equation to avoid overcounting. The same factor (see equation 3.28) appears in the cross section formula (equation 3.32) when some of the final state particles are identical.

If $|i\rangle$ and $|f\rangle$ are the same two particle state, corresponding to two particles scattering elastically in the forward direction, then

$$2\,\mathrm{Im}\,\mathcal{M}_{ii} = 4E_T p_i\,\sigma. \tag{3.38}$$

This is the *optical theorem* which relates the forward part of the scattering amplitude to the total cross-section. If particles of masses m_1 and m_2 scatter, then $E_T = s^{1/2}$ and $4sp_i^2 = \lambda(s, m_1^2, m_2^2)$, where λ is the function defined in equation (3.30). Then the optical theorem reads, $\mathrm{Im}\,\mathcal{M}_{ii} = \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)\,\sigma$.

4 Quantum Electrodynamics

4.1 The Free Dirac Field

Dirac Field Theory is defined to be the theory whose field equations correspond to the Dirac equation. We regard the two Dirac fields $\psi(x)$ and $\overline{\psi}(x)$ as being dynamically independent fields and postulate the Dirac Lagrangian density:

$$\mathcal{L} = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{4.1}$$

The Euler-Lagrange equation

$$\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} - \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = 0 \tag{4.2}$$

leads to the Dirac equation.

The canonical momentum is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}(x)} = i\psi^\dagger(x) \tag{4.3}$$

The Hamiltonian density

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger i\frac{\partial\psi}{\partial t} \tag{4.4}$$

which is not positive definite. The general solution to the Dirac equation may be expanded in terms of plane waves

$$\psi(\mathrm{x}, t) = \int \frac{d^3\mathrm{k}}{(2\pi)^3}\frac{m}{k_0}\sum_{\alpha=1,2}[b_\alpha(\mathrm{k})u^\alpha(\mathrm{k})e^{-ik.x} + d^\dagger_\alpha(\mathrm{k})v^\alpha(\mathrm{k})e^{ik.x}] \tag{4.5}$$

$$\overline{\psi}(\mathrm{x}, t) = \int \frac{d^3\mathrm{k}}{(2\pi)^3}\frac{m}{k_0}\sum_{\alpha=1,2}[b^\dagger_\alpha(\mathrm{k})\overline{u}^\alpha(\mathrm{k})e^{ik.x} + d_\alpha(\mathrm{k})\overline{v}^\alpha(\mathrm{k})e^{-ik.x}] \tag{4.6}$$

The total Hamiltonian is

$$H = \int d^3x\mathcal{H} \tag{4.7}$$

After some algebra we find

$$H = \int \frac{d^3\mathrm{k}}{(2\pi)^3}\frac{m}{k_0}k_0\sum_{\alpha=1,2}[b^\dagger_\alpha(\mathrm{k})b_\alpha(\mathrm{k}) - d_\alpha(\mathrm{k})d^\dagger_\alpha(\mathrm{k})] \tag{4.8}$$

So far no commutation relations have been assumed, and H could quite easily be negative, unlike the Hamiltonian in the case of the charged scalars for example which was positive definite. In order to give a positive definite Hamiltonian we require the creation and annihilation operators to satisfy *anticommutation* relations, first proposed by Wigner:

$$\left\{b_\alpha(\mathrm{k}), b^\dagger_{\alpha'}(\mathrm{k}')\right\} = (2\pi)^3\frac{k_0}{m}\delta^3(\mathrm{k} - \mathrm{k}')\delta_{\alpha\alpha'} \tag{4.9}$$

$$\left\{d_\alpha(\mathrm{k}), d^\dagger_{\alpha'}(\mathrm{k}')\right\} = (2\pi)^3\frac{k_0}{m}\delta^3(\mathrm{k} - \mathrm{k}')\delta_{\alpha\alpha'} \tag{4.10}$$

$$\{b_\alpha(\mathbf{k}), b_{\alpha'}(\mathbf{k}')\} = 0 \tag{4.11}$$

$$\left\{b^\dagger_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}')\right\} = 0 \tag{4.12}$$

$$\{d_\alpha(\mathbf{k}), d_{\alpha'}(\mathbf{k}')\} = 0 \tag{4.13}$$

$$\left\{d^\dagger_\alpha(\mathbf{k}), d^\dagger_{\alpha'}(\mathbf{k}')\right\} = 0 \tag{4.14}$$

The Hamiltonian is then defined as the normal ordered version of Eq.4.8 *but with a change of sign for each interchange of operator*

$$H = \int d^3x : \psi^\dagger i \frac{\partial \psi}{\partial t} : \tag{4.15}$$

which results in

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{k_0} k_0 \sum_{\alpha=1,2} [b^\dagger_\alpha(\mathbf{k}) b_\alpha(\mathbf{k}) + d_\alpha(\mathbf{k}) d^\dagger_\alpha(\mathbf{k})] \tag{4.16}$$

which is now postive definite.

Anticommutation implies Fermi statistics for example:

$$\left\{b^\dagger_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}')\right\} = 0$$

$$\Rightarrow b^\dagger_\alpha(\mathbf{k}) b^\dagger_\alpha(\mathbf{k}) = 0$$

$$\Rightarrow b^\dagger_\alpha(\mathbf{k}) b^\dagger_\alpha(\mathbf{k}) |0> = 0$$

so that two quanta in the same state are not allowed (Pauli exclusion principle).

The charge operator is

$$Q = \int d^3\mathbf{x} : j_0(x) := \int d^3\mathbf{x} : \psi^\dagger i \partial \psi :$$

$$Q = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{k_0} k_0 \sum_{\alpha=1,2} [b^\dagger_\alpha(\mathbf{k}) b_\alpha(\mathbf{k}) - d^\dagger_\alpha(\mathbf{k}) d_\alpha(\mathbf{k})] \tag{4.17}$$

which shows that $b^\dagger$ creates fermions while $d^\dagger$ creates antifermions of opposite charge.

Finally the equal time commutation relations are (after some algebra):

$$\left\{\psi_i(\mathbf{x}, t), \psi^\dagger_j(\mathbf{x}', t)\right\} = \delta^3(\mathbf{x} - \mathbf{x}')\delta_{ij} \tag{4.18}$$

$$\{\psi_i(\mathbf{x}, t), \psi_j(\mathbf{x}', t)\} = 0 \tag{4.19}$$

$$\left\{\psi^\dagger_i(\mathbf{x}, t), \psi^\dagger_j(\mathbf{x}', t)\right\} = 0 \tag{4.20}$$

In fact at all times we have:

$$\{\psi(x), \psi(y)\} = 0 \tag{4.21}$$

4.2 The Free Electromagnetic Field

The four Maxwell equations are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

It is straightforward to show that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

In covariant form,

$$\partial_\mu j^\mu = 0$$

where $j^\mu = (c\rho, \mathbf{j})$.

It is convenient (and even essential) to introduce scalar and vector potentials ϕ and $\mathbf{A}$ by defining

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t.$$

whence two of the Maxwell equations become automatic.

Recall the gauge invariance of electrodynamics which says that $\mathbf{E}$ and $\mathbf{B}$ are unchanged when

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda \quad \text{and} \quad \phi \to \phi - \frac{\partial \Lambda}{\partial t}$$

for any scalar function Λ. Gauge invariance corresponds to a lack of uniqueness of the scalar and vector potentials. This lack of uniqueness can be reduced by imposing a further condition on the scalar and vector potentials, for example

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

Assuming that ϕ and $\mathbf{A}$ can be combined into a four vector

$$A^\mu = (\phi/c, \mathbf{A})$$

this can be written as

$$\partial_\mu A^\mu = 0 \tag{4.22}$$

which is known as the *Lorentz gauge* condition. Gauge invariance in four-vector notation is just:

$$A^\mu \to A^\mu + \partial_\mu \Lambda \tag{4.23}$$

Note that even the imposition of the Lorentz gauge condition does not completely fix the vector potential; it merely restricts the function Λ to satisfy

$$\partial^2 \Lambda = 0 \tag{4.24}$$

With the Lorentz gauge condition Maxwell's equations are equivalent to

$$\partial^2 A^\mu = \mu_0 j^\mu$$

The tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F_{\mu\nu}$ clearly has six independent components, and can be written:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

It is straightforward to show that,

$$F_{\mu\nu}F^{\mu\nu} = -2\left(\frac{\mathbf{E}^2}{c^2} - \mathbf{B}^2\right)$$

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -\frac{8}{c}\mathbf{E}\cdot\mathbf{B}$$

where

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

This gives the relativistic invariants which can be constructed from $\mathbf{E}$ and $\mathbf{B}$.

It is easy to see that in any gauge the Maxwell equations can be written,

$$\partial_\mu F^{\mu\nu} = j^\nu$$

The Maxwell equations, in this compact form, can be reproduced by the following Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \tag{4.25}$$

via the Euler-Lagrange equations for each of the four A_μ fields separately.

In Lorentz gauge the Lagrangian density has the more general form:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{4.26}$$

where ξ is a free parameter. The EL equations then imply

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi}\partial^\nu(\partial_\mu A^\mu) = j^\nu \tag{4.27}$$

which reduce to Maxwell's equations in Lorentz gauge. The extra term in the Lagrangian density $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ thus has no effect on physics in Lorentz gauge. In fact it is possible to turn the argument around and use this term to fix the gauge to be Lorentz gauge by imposing current conservation instead of obtaining it as a consequence of Maxwell's equations. If one adds the extra term to the Lagrangian and imposes current conservation then Eq.4.27 implies immediately the Lorentz gauge condition by the antisymmetry of $F^{\mu\nu}$. For this reason the extra term is referred to as a *gauge fixing term* and ξ is a Lagrange multiplier. The choice $\xi = 1$ is known as Feynman gauge although it is within the framework of the Lorentz gauge.

As usual we can expand the field $A_\mu(x)$ in its Fourier components

$$A_\mu(x)(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a_\mu(k)e^{-ik.x} + a^*_\mu(k)e^{ik.x}] \tag{4.28}$$

where $\omega = k_0 = |\mathbf{k}|$. The Lorentz gauge condition implies

$$k.a(k) = 0 \tag{4.29}$$

This implies that

$$a_0(k) = \hat{\mathbf{k}}.\mathbf{a}(k)$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. Thus the time component of a_μ equals the longitudinal component a.k. Of course only the transverse components are physical (since the $\mathbf{E}$ and $\mathbf{B}$ fields are always orthogonal to the three momentum) and it can be shown that the contribution to the Hamiltonian from the time component and longitudinal component cancel against each other. In fact it is possible to completely specify the gauge by requiring that

$$a_0(k) = \hat{\mathbf{k}}.\mathbf{a}(k) = 0$$

which is called Coulomb gauge. In Coulomb gauge we can write

$$a_\mu(k) = \sum_{\lambda=1,2} a^\lambda(k)\epsilon^\lambda_\mu(k)$$

where $\epsilon^\lambda_\mu(k)$ are two orthonormal spacelike vectors in the plane transverse to $\mathbf{k}$.

In a general Lorentz gauge we can write:

$$a_\mu(k) = \sum_{\lambda=0,1,2,3} a^\lambda(k)\epsilon^\lambda_\mu(k)$$

where now $\epsilon^\lambda_\mu(k)$ are arbitrary unit four-vectors. Suppose that $\mathbf{k}$ is along the third axis, $k = (\omega, 0, 0, \omega)$ then we can define the basis vectors as:

$$\epsilon^0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \epsilon^1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \epsilon^2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \epsilon^3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{4.30}$$

so that we call $\lambda = 1, 2$ the physical transverse polarisations, $\lambda = 0$ the unphysical timelike polarisation and $\lambda = 3$ the unphysical longitudinal polarisation. Clearly,

$$k.\epsilon^{1,2} = 0$$

and

$$\epsilon^\lambda.\epsilon^{\lambda'} = g^{\lambda\lambda'}$$

which is in fact a basis independent result, although we shall always work in this basis.

We shall now quantise the free e.m. theory ($j^\mu = 0$). To quantise the theory canonically we introduce the canonical momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \tag{4.31}$$

and impose the equal time covariant canonical commutation relations

$$[A_\mu(\mathbf{x},t),\pi_\nu(\mathbf{x}',t)] = ig_{\mu\nu}\delta^3(\mathbf{x}-\mathbf{x}') \tag{4.32}$$

$$[A_\mu(\mathbf{x},t),A_\nu(\mathbf{x}',t)] = 0 \tag{4.33}$$

$$[\pi_\mu(\mathbf{x},t),\pi_\nu(\mathbf{x}',t)] = 0 \tag{4.34}$$

Now if the Lagrangian were simply

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.35}$$

then we would find that

$$\pi^0 = \frac{\partial\mathcal{L}}{\partial\dot{A}_0} = 0$$

which would imply that π^0 always commutes with A^0, which loses us both covariance and quantum mechanics at a stroke!

We clearly need a π^0 that does not vanish. In order to do this we need to change the Lagrangian without changing the physics. But we have learned how to do this in Lorentz gauge which corresponds to the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{4.36}$$

and the field equations:

$$\partial^2 A_\mu - (1-\frac{1}{\xi})\partial_\mu(\partial_\nu A^\nu) = 0 \tag{4.37}$$

Henceforth for simplicity we shall take $\xi = 1$ which is called Feynman gauge (a sub-class of Lorentz gauge).

At first sight this doesn't help us because we find

$$\pi^0 = \frac{\partial\mathcal{L}}{\partial\dot{A}_0} = -\partial_\mu A^\mu$$

which apparently vanishes in Lorentz gauge. However we shall only assume that matrix elements of $\partial_\mu A^\mu$ vanish rather than imposing the operator condition that it vanish.

In Feynman gauge we have the field equations:

$$\partial^2 A_\mu = 0 \tag{4.38}$$

and we can once again expand the A_μ field in plane wave solutions similar to the previous section:

$$A_\mu(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}\sum_{\lambda=0}^{3}[\epsilon_\mu^\lambda(k)a^\lambda(k)e^{-ik.x} + \epsilon_\mu^{\lambda*}(k)a^\lambda(k)^\dagger e^{ik.x}] \tag{4.39}$$

Here $\epsilon_\mu^\lambda(k)$ are the set of four linearly independent vectors defined in Eq.4.30, but now we regard a and its hermitian conjugate as operators whose commutation relations readily follow from Eq.4.32

$$[a^\lambda(k), a^{\lambda'}(k')^\dagger] = -g^{\lambda\lambda'}2k_0(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}') \tag{4.40}$$

For longitudinal and transverse photons quantisation proceeds in the usual way. But for timelike photons with $\lambda = \lambda' = 0$ we have a negative quantity on the rhs which gives problems. This leads to timelike photons with negative norm. However it is possible to overcome these problems using the Gupta-Bleuler formalism. However at this point we prefer to abandon the canonical approach and move on to the path integral approach which has its own problems.

We have seen that the freedom to make gauge transformations means that the A^μ fields are not uniquely specified, and this causes problems with the theory in the canonical formalism. It should be no surprise that these problems persist in the path integral approach.

The generating functional in this case is

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x(\mathcal{L}+J^\mu A_\mu\right)} \tag{4.41}$$

where $\mathcal{L}$ is the Lagrangian for the free photon field which we might naively take to be

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.42}$$

(since we have already found problems with this form in the canonical formalism it really is naive to expect it to work here). The field equations in this case are as in Eq.4.27

$$\partial_\mu F^{\mu\nu} = 0 \tag{4.43}$$

which can be written as

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\mu = 0 \tag{4.44}$$

After partial integration and discarding surface terms we can write the generating functional as

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x\frac{1}{2}A^\mu[g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu]A^\nu+J^\mu A_\mu\right)} \tag{4.45}$$

By now we know that the photon propagator $D_{\mu\nu}$ is going to be the inverse of the operator in square brackets, and it will satisfy the equation:

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)D^{\nu\lambda}(x-y) = \delta^\lambda_\mu\delta^4(x-y) \tag{4.46}$$

If we multiply this equation by ∂^μ we get zero multiplying $D^{\nu\lambda}(x-y)$ on the lhs and something non-zero on thhe rhs, which would seem to imply that $D^{\nu\lambda}(x-y)$ is infinite. In fact the problem is that the operator in square brackets does not have an inverse! To show this all we need to do is show that it has a zero eigenvalue, and this can easily be done:

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\partial^\mu\Omega = 0$$

for any function Ω.

From the point of view of the path integral the problem is that the functional integral is taken over all A_μ including those related by a gauge transformation, leading to an infinite overcounting in the calculation of the generating functional, and hence an infinite overcounting for the Green's functions which are obtained from it by functional differentiation. To cure this problem we need to fix a particular gauge, and we do this by imposing the Lorentz gauge condition:

$$\partial_\mu A^\mu = 0 \tag{4.47}$$

Recall the Lagrangian with gauge fixing term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{4.48}$$

and the field equations:

$$\partial^2 A_\mu - (1 - \frac{1}{\xi})\partial_\mu(\partial_\nu A^\nu) = 0 \tag{4.49}$$

After partial integration and discarding surface terms we can now write the generating functional as

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x \frac{1}{2}A^\mu[g_{\mu\nu}\partial^2 + (\frac{1}{\xi}-1)\partial_\mu\partial_\nu]A^\nu + J^\mu A_\mu\right)} \tag{4.50}$$

and the operator in square brackets now has an inverse given by

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} - \frac{\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\epsilon} e^{-ik.(x-y)} \tag{4.51}$$

The Fourier transform of the Feynman propagator is thus

$$D_{\mu\nu}(k) = -\frac{\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\epsilon} \tag{4.52}$$

Amongst this class of gauge choices two common choices are Feynman gauge ($\xi = 1$) and Landau gauge ($\xi = 0$).

4.3 Feynman Rules of QED

QED involves the interaction of electrons and photons where the interaction corresponds to the Lagrangian

$$\mathcal{L}_{int} = -e\overline{\psi}\gamma^\mu A_\mu \psi. \tag{4.53}$$

Such an interaction may be introduced by the concept of "minimal substitution" familiar from classical electrodynamics. The momentum and energy become:

$$\mathbf{p} \to \mathbf{p} - e\mathbf{A}$$

$$E \to E - e\phi$$

or in four vector notation, the four momentum becomes:

$$p^\mu \to p^\mu - eA^\mu$$

Applying this classical concept of minimal substitution to the Dirac equation gives:

$$(i\not{D} - m)\psi = 0 \tag{4.54}$$

where we have introduced the covariant derivative notation

$$D_\mu \equiv \partial_\mu + ieA_\mu$$

The QED Lagrangian describing electrons, photons and their interactions is then given by,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 + \overline{\psi}(i\not{D} - m)\psi. \tag{4.55}$$

Here, $D_\mu = \partial_\mu + ieA_\mu$ is the electromagnetic covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $(\partial \cdot A)^2/2$ is the gauge fixing term for Feynman gauge.

The QED Lagrangian is invariant under a symmetry called *gauge symmetry*, which consists of the simultaneous gauge transformations of the photon field:

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{4.56}$$

and a phase transformation on the electron field

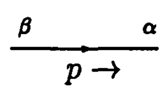
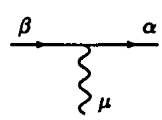
$$\psi \to e^{-ie\Lambda}\psi \tag{4.57}$$

The point is that the value of the phase transformation given by the same gauge function $\Lambda(x)$ as controls the photon gauge transformation. It is important to emphasise that $\Lambda(x)$ is a function of x so that the action of a derivative on $e^{-ie\Lambda}\psi$ will yield two terms by the product rule. However the simultaneous gauge transformation of the photon field means that the covariant derivative of ψ transforms like ψ itself under the combined gauge transformations above:

$$D_\mu\psi \to e^{-ie\Lambda} D_\mu\psi \tag{4.58}$$

Thus the QED Lagrangian is invariant under the simultaneous transformations above, referred to collectively as a gauge transformation.

In this section we are going to get some practice calculating cross sections and decay rates in QED. The starting point is the set of Feynman rules in Table 4.1 derived from the

For every ...	draw ...	write ...
Internal photon line		$\dfrac{-ig^{\mu\nu}}{q^2 + i\epsilon}$
Internal fermion line		$\dfrac{i(\not p + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$
Vertex		$-ie\gamma^\mu_{\alpha\beta}$
Outgoing electron		$\overline{u}^s_p$
Incoming electron		u^s_p
Outgoing positron		v^s_p
Incoming positron		$\overline{v}^s_p$
Outgoing photon		$\epsilon^{*\mu}$
Incoming photon		ϵ^μ

- Attach a directed momentum to every internal line
- Conserve momentum at every vertex

Table 4.1 Feynman rules for QED. μ, ν are Lorentz indices and α, β are spinor indices.

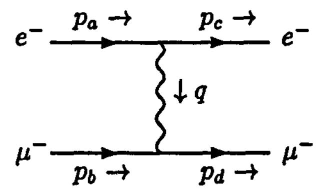


Figure 4.1 Lowest order Feynman diagram for electron–muon scattering.

QED Lagrangian above. The fermion propagator is (up to factors of i) the inverse of the operator, $\not{p}-m$, which appears in the quadratic term in the fermion fields, as discussed in Dave Dunbar's lectures. The derivation of the photon propagator, along with the need for gauge fixing, was discussed in section 4.2. The external line factors are easily derived by considering simple matrix elements in the operator formalism, where they are left behind from the expansions of fields in terms of annihilation and creation operators, after the operators have all been (anti-)commuted until they annihilate the vacuum. One could consider for example the process $\gamma \rightarrow e^+e^-$. In path integral language the natural objects to compute are Green functions, vacuum expectation values of time ordered products of fields: it takes a little more work to convert them to transition amplitudes and see the external line factors appear.

The spinor indices in the Feynman rules are such that matrix multiplication is performed in the opposite order to that defining the flow of fermion number. The arrow on the fermion line itself denotes the fermion number flow, *not* the direction of the momentum associated with the line: I will try always to indicate the momentum flow separately as in Table 4.1. This will become clear in the examples which follow. We have already met the Dirac spinors u and v. I will say more about the photon polarisation vector ϵ when we need to use it.

4.4 Electron–Muon Scattering

To lowest order in the electromagnetic coupling, just one diagram contributes to this process. It is shown in Figure 4.1. The amplitude obtained from this diagram is

$$i\mathcal{M}_{fi} = (-ie)\,\overline{u}(p_c)\gamma^\mu u(p_a)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(-ie)\,\overline{u}(p_d)\gamma_\nu u(p_b). \tag{4.59}$$

Note that I have changed my notation for the spinors: now I label their momentum as an argument instead of as a subscript, and I drop the spin label unless I need to use it. In constructing this amplitude we have followed the fermion lines backwards with respect to fermion flow when working out the order of matrix multiplication.

The cross-section involves the squared modulus of the amplitude, which is

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4}\, L^{\mu\nu}_{(e)} L_{(\mu)\,\mu\nu},$$

where the subscripts e and μ refer to the electron and muon respectively and,

$$L^{\mu\nu}_{(e)} = \overline{u}(p_c)\gamma^\mu u(p_a)\overline{u}(p_a)\gamma^\nu u(p_c),$$

with a similar expression for $L^{\mu\nu}_{(\mu)}$.

▷ Exercise 4.1
Verify the expression for $|\mathcal{M}_{fi}|^2$.

Usually we have an unpolarised beam and target and do not measure the polarisation of the outgoing particles. Thus we calculate the squared amplitudes for each possible spin combination, then average over initial spin states and sum over final spin states. Note that we square and then sum since the different possibilities are in principle distinguishable. In contrast, if several Feynman diagrams contribute to the same process, you have to sum the amplitudes first. We will see examples of this below.

The spin sums are made easy by the following results (I temporarily restore spin labels on spinors):

$$\begin{aligned} \sum_r u^r(p)\,\overline{u}^r(p) &= \not{p} + m \\ \sum_r v^r(p)\,\overline{v}^r(p) &= \not{p} - m \end{aligned} \tag{4.60}$$

▷ Exercise 4.2
Derive the spin sum relations in equation (4.60).

Using the spin sums we find,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{4q^4}\,\mathrm{tr}\left(\gamma^\mu(\not{p}_a+m_e)\gamma^\nu(\not{p}_c+m_e)\right)\mathrm{tr}\left(\gamma_\mu(\not{p}_b+m_\mu)\gamma_\nu(\not{p}_d+m_\mu)\right).$$

Since all calculations of cross sections or decay rates in QED require the evaluation of traces of products of gamma matrices, you will generally find a table of "trace theorems" in any quantum field theory textbook [1]. All these theorems can be derived from the fundamental anticommutation relations of the gamma matrices in equation (2.28) together with the invariance of the trace under a cyclic change of its arguments. For now it suffices to use,

$$\begin{aligned} \mathrm{tr}(\not{a}\not{b}) &= 4\,a\cdot b \\ \mathrm{tr}(\not{a}\not{b}\not{c}\not{d}) &= 4(a\cdot b\,c\cdot d - a\cdot c\,b\cdot d + a\cdot d\,b\cdot c) \\ \mathrm{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= 0 \quad \text{for } n \text{ odd} \end{aligned} \tag{4.61}$$

▷ Exercise 4.3
Derive the trace results in equation (4.61)

Using these results, and expressing the answer in terms of the Mandelstam variables of equation (3.35), we find,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{2e^4}{t^2}\left(s^2+u^2-4(m_e^2+m_\mu^2)(s+u)+6(m_e^2+m_\mu^2)^2\right).$$

This can now be used in the $2 \to 2$ cross section formula (3.34) to give, in the high energy limit, $s, u \gg m_e^2, m_\mu^2$,

$$\frac{d\sigma}{d\Omega^*} = \frac{e^4}{32\pi^2 s}\,\frac{s^2+u^2}{t^2}. \tag{4.62}$$

for the differential cross section in the centre of mass frame.

▷ Exercise 4.4
Derive the result for the electron–muon scattering cross section in equation (4.62).

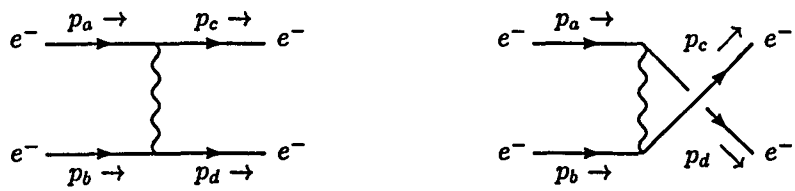


Figure 4.2 Lowest order Feynman diagrams for electron–electron scattering.

Other calculations of cross sections or decay rates will follow the same steps we have used above. You draw the diagrams, write down the amplitude, square it and evaluate the traces (if you are using spin sum/averages). There are one or two more wrinkles to be aware of, which we will meet below.

4.5 Electron–Electron Scattering

Since the two scattered particles are now identical, you can't just replace m_μ by m_e in the calculation we did above. If you look at the diagram of Figure 4.1 (with the muons replaced by electrons) you will see that the outgoing legs can be labelled in two ways. Hence we get the two diagrams of Figure 4.2.

The two diagrams give the amplitudes,

$$\begin{aligned} i\mathcal{M}_1 &= \frac{ie^2}{t}\overline{u}(p_c)\gamma^\mu u(p_a)\overline{u}(p_d)\gamma_\mu u(p_b), \\ i\mathcal{M}_2 &= -\frac{ie^2}{u}\overline{u}(p_d)\gamma^\mu u(p_a)\overline{u}(p_c)\gamma_\mu u(p_b). \end{aligned}$$

Notice the additional minus sign in the second amplitude, which comes from the anti-commuting nature of fermion fields. You should accept as part of the Feynman rules for QED that when diagrams differ by an interchange of two fermion lines, a relative minus sign must be included. This is important because

$$|\mathcal{M}_{fi}|^2 = |\mathcal{M}_1 + \mathcal{M}_2|^2,$$

so the interference term will have the wrong sign if you don't include the extra sign difference between the two diagrams.

4.6 Electron–Positron Annihilation

4.6.1 $e^+e^- \to e^+e^-$

For this process the two diagrams are shown in Figure 4.3, with the one on the right known as the annihilation diagram. They are just what you get from the diagrams for electron–electron scattering in Figure 4.2 if you twist round the fermion lines. The fact that the diagrams are related this way implies a relation between the amplitudes. The interchange of incoming particles/antiparticles with outgoing antiparticles/particles is called *crossing.* This is a case where the general results of crossing symmetry can be applied, and our diagrammatic calculations give an explicit realisation. Theorists spent a great deal of time studying such general properties of amplitudes in the 1960's when quantum field theory was unfashionable.

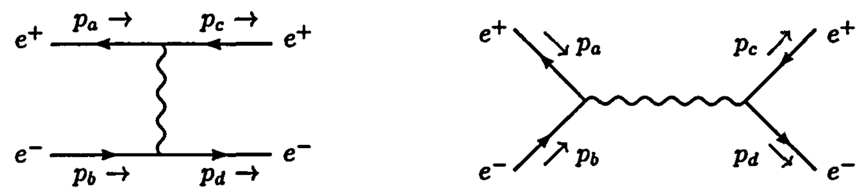


Figure 4.3 Lowest order Feynman diagrams for electron–positron scattering in QED.

4.6.2 $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to$ hadrons

If electrons and positrons collide and produce muon–antimuon or quark-antiquark pairs, then the annihilation diagram is the only one which contributes. At sufficiently high energies that the quark masses can be neglected, this immediately gives the lowest order QED prediction for the ratio of the annihilation cross section into hadrons to that into $\mu^+\mu^-$,

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_f Q_f^2, \tag{4.63}$$

where the sum is over quark flavours f and Q_f is the quark's charge in units of e. The 3 comes from the existence of three colours for each flavour of quark. Historically this was important: you could look for a step in the value of R as your e^+e^- collider's CM energy rose through a threshold for producing a new quark flavour. If you didn't know about colour, the height of the step would seem too large. Incidentally, another place the number of colours enters is in the decay of a π^0 to two photons. There is a factor of 3 in the amplitude from summing over colours, without which the predicted decay rate would be one ninth of its real size.

At the energies used today at LEP, of course, you have to remember the diagram with a Z replacing the photon. We will say some more about this later.

▷ **Exercise 4.5**

Show that the cross-section for $e^+e^- \to \mu^+\mu^-$ is equal to $4\pi\alpha^2/(3s)$, neglecting the lepton masses.

4.7 Compton Scattering

The diagrams which need to be evaluated to compute the Compton cross section for $\gamma e \to \gamma e$ are shown in Figure 4.4. For unpolarised initial and/or final states, the cross section calculation involves terms of the form

$$\sum_\lambda \epsilon_\lambda^{*\mu}(p)\,\epsilon_\lambda^\nu(p), \tag{4.64}$$

where λ represents the polarisation of the photon of momentum p. Since the photon is massless, the sum is over the two transverse polarisation states, and must vanish when contracted with p_μ or p_ν. In addition, however, since the photon is coupled to the electromagnetic current $J^\mu = \overline{\psi}\gamma^\mu\psi$ of equation (2.7), any term in the polarisation sum (4.64) proportional to p^μ or p^ν does not contribute to the cross section. This is because the current is conserved, $\partial_\mu J^\mu = 0$, so in momentum space $p_\mu J^\mu = 0$. The upshot is that in calculations you can use,

$$\sum_\lambda \epsilon_\lambda^{*\mu}(p)\,\epsilon_\lambda^\nu(p) = -g^{\mu\nu}, \tag{4.65}$$

Figure 4.4 Feynman diagrams for Compton scattering.

since the remaining terms on the right hand side do not contribute.

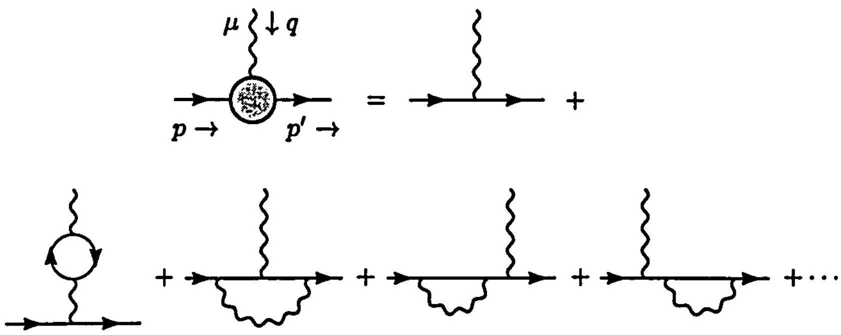


Figure 5.1 Diagrams for vertex renormalisation in QED up to one loop.

5 Introduction to Renormalisation

5.1 Renormalisation of QED

Let's start by considering how the electric charge is defined and measured. This will bring up the question of what happens when you try to compute higher loop corrections. In fact, the expansion in the number of loops is an expansion in Planck's constant $\hbar$, as you can show if you put back the factors of $\hbar$ for once.

The electric charge $\hat{e}$ is usually defined as the coupling between an on-shell electron and an on-shell photon: that is, as the vertex on the left hand side of Figure 5.1 with $p_1^2 = p_2^2 = m^2$, where m is the electron mass, and $q^2 = 0$. It is $\hat{e}$ and not the Lagrangian parameter e which we measure. That is,

$$\frac{\hat{e}^2}{4\pi} = \frac{1}{137}.$$

We call $\hat{e}$ the renormalised coupling constant of QED. We can calculate $\hat{e}$ in terms of e in perturbation theory. To one loop, the relevant diagrams are shown on the right hand side of Figure 5.1, and the result takes the form,

$$\hat{e} = e + e^3\Big[a_1 \ln \frac{M^2}{m^2} + b_1\Big] + \cdots \tag{5.1}$$

where a_1 and b_1 are constants obtained from the calculation. The e^3 term is divergent, so we have introduced a cutoff M to regulate it. This is called an ultraviolet divergence since it arises from the propagation of high momentum modes in the loops. The cutoff amounts to selecting only those modes where each component of momentum is less than M in magnitude. Despite the divergence in (5.1), it still relates the measurable quantity $\hat{e}$ to the coupling e we introduced in our theory. This implies that e itself must be divergent. The property of renormalisability ensures that in any relation between physical quantities the ultraviolet divergences cancel: the relation is actually independent of the method used to regulate divergences.

As an example, consider the amplitude for electron–electron scattering, which we considered at tree level in section 4.5. Some of the contributing diagrams are shown in Figure 5.2, where the crossed diagrams are understood (we showed the crossed tree level diagram explicitly in Figure 4.2). Ultraviolet divergences are again encountered when

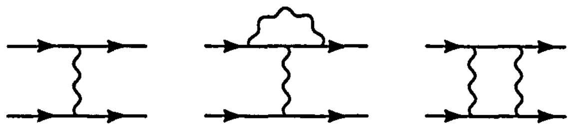


Figure 5.2 Some diagrams for electron–electron scattering in QED up to one loop.

the diagrams are evaluated, and the result is of the form,

$$i\mathcal{M}_{fi} = c_0 e^2 + e^4\Big[c_1 \ln\frac{M^2}{m^2} + d_1\Big] + \cdots \tag{5.2}$$

where c_0, c_1 and d_1 are constants, determined by the calculation. In order to evaluate $\mathcal{M}_{fi}$ numerically, however, we must express it in terms of the known parameter $\hat{e}$. Combining (5.1) and (5.2) yields,

$$i\mathcal{M}_{fi} = c_0 \hat{e}^2 + \hat{e}^4\Big[(c_1 - 2a_1 c_0)\ln\frac{M^2}{m^2} + d_1 - 2b_1 c_0\Big] + \cdots \tag{5.3}$$

where the ellipsis denotes terms of order $\hat{e}^6$ and above. Since $|\mathcal{M}_{fi}|^2$ is measurable, consistency (renormalisability) requires,

$$c_1 = 2a_1 c_0.$$

This result is indeed borne out by the actual calculations, and the relation between $\mathcal{M}_{fi}$ and $\hat{e}$ contains no divergences:

$$i\mathcal{M}_{fi} = c_0 \hat{e}^2 + \hat{e}^4(d_1 - 2b_1 c_0) + \mathcal{O}(\hat{e}^6). \tag{5.4}$$

To understand how this cancellation of divergences happened we can study the convergence properties of loop diagrams (although we shall not evaluate them). Consider the third diagram on the right hand side in Figure 5.1 and the middle diagram in Figure 5.2. These both contain a loop with one photon propagator, behaving like $1/k^2$ at large momentum k, and two electron propagators, each behaving like $1/k$. To evaluate the diagram we have to integrate over all momenta, leading to an integral,

$$I \sim \int_{\text{large } k} \frac{d^4k}{k^4}, \tag{5.5}$$

which diverges logarithmically, leading to the $\ln M^2$ terms in (5.1) and (5.2). Notice, however, that the divergent terms in these two diagrams must be the same, since the divergence is by its nature independent of the finite external momenta (the factor of two in equation (5.3) arises because there is a divergence associated with the coupling of each electron in the scattering process). In this way we can understand that at least some of the divergences are common in both (5.1) and (5.2). What about diagrams such as the third box-like one in Figure 5.2? Now we have two photon and two electron propagators, leading to,

$$I \sim \int_{\text{large } k} \frac{d^4k}{k^6}.$$

This time the integral is convergent.

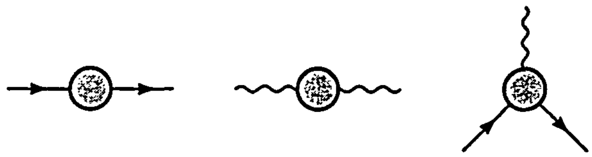


Figure 5.3 Primitive divergences of QED.

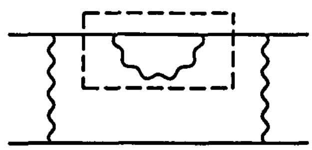
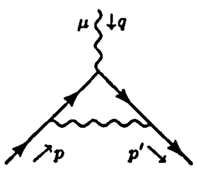


Figure 5.4 Diagram containing a primitive divergence.

Detailed study like this reveals that ultraviolet divergences always disappear in relations between physically measurable quantities. We discussed above the definition of the physical electric charge $\hat{e}$. A similar argument applies for the electron mass: the Lagrangian bare mass parameter m is divergent, but we can define a finite physical mass $\hat{m}$.

In fact you find that all ultraviolet divergences in QED stem from graphs of the type shown in Figure 5.3 and known as the *primitive divergences.* Any divergent graph will be found on inspection to contain a divergent subgraph of one of these basic types. For example, Figure 5.4 shows a graph where the divergence comes from the primitive divergent subgraph inside the dashed box. Furthermore, the primitive divergences are always of a type that would be generated by a term in the initial Lagrangian with a divergent coefficient. Hence by rescaling the fields, masses and couplings in the original Lagrangian we can make all physical quantities finite (and independent of the exact details of the adjustment such as how we regulate the divergent integrals). This is what we mean by renormalisability.

This should be made clearer by an example. Consider calculating the vertex correction in QED to one loop,



$$= \overline{u}(p')\left[A\gamma^\mu + B\sigma^{\mu\nu}q_\nu + Cq^2\gamma^\mu + \cdots\right]u(p).$$

The calculation shows that A is divergent. However, we can absorb this by adding a cancelling divergent coefficient to the $\overline{\psi}\not{A}\psi$ term in the QED Lagrangian (4.55). The B and C terms are finite and unambiguous. This is just as well, since an infinite part of B, for example, would need to be cancelled by an infinite coefficient of a term of the form,

$$\overline{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi,$$

which is not available in (4.55).

In fact, the B term gives the QED correction to the magnetic dipole moment, g, of the electron or muon (see page 160 of the textbook by Itzykson and Zuber [1]). These are predicted to be 2 at tree level. You can do the one-loop calculation (it was first done by Schwinger between September and November 1947 [3]) with a few pages of algebra to find,

$$g = 2\left(1 + \frac{\alpha}{2\pi}\right).$$

This gives $g/2 = 1.001161$, which is already impressive compared to the experimental values [2]:

$$\begin{aligned}(g/2)_{\text{electron}} &= 1.001159652193(10),\\ (g/2)_{\text{muon}} &= 1.001165923(8).\end{aligned}$$

Higher order calculations show that the electron and muon magnetic moments differ at two loops and above. Kinoshita and collaborators have devoted their careers to these calculations and are currently at the four loop level. Theory and experiment agree for the electron up to the 11th decimal place.

The C term gives the splitting between the $2s_{1/2}$ and $2p_{1/2}$ levels of the hydrogen atom, known as the Lamb shift. Bethe's calculation [4] of the Lamb shift, done during a train ride to Schenectady in June 1947, was an early triumph for quantum field theory. Here too, the current agreement between theory and experiment is impressive.

In discussing the vertex correction in QED, we said that the divergent part of the A term could be absorbed by adding a cancelling divergent coefficient to the $\overline{\psi}\not{A}\psi$ term in the QED Lagrangian (4.55). When a theory is renormalisable, *all* divergences can be removed in this way. Thus, for QED, if the original Lagrangian is (ignoring the gauge-fixing term),

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\not{\partial}\psi - e\overline{\psi}\not{A}\psi - m\overline{\psi}\psi,$$

then redefine everything by:

$$\begin{aligned}\psi &= Z_2^{1/2}\psi_R, & A^\mu &= Z_3^{1/2}A_R^\mu,\\ e &= Z_e\hat{e} = \frac{Z_1}{Z_2 Z_3^{1/2}}\,\hat{e}, & m &= Z_m\hat{m},\end{aligned}$$

where the subscript R stands for "renormalised." In terms of the renormalised fields,

$$\mathcal{L} = -\frac{1}{4}Z_3 F_{R\mu\nu}F_R^{\mu\nu} + iZ_2\overline{\psi}_R\not{\partial}\psi_R - Z_1\hat{e}\overline{\psi}_R\not{A}_R\psi_R - Z_m Z_2\hat{m}\overline{\psi}_R\psi_R.$$

Writing each Z as $Z = 1 + \delta Z$, reexpress the Lagrangian one more time as,

$$\mathcal{L} = -\frac{1}{4}F_{R\mu\nu}F_R^{\mu\nu} + i\overline{\psi}_R\not{\partial}\psi_R - \hat{e}\overline{\psi}_R\not{A}_R\psi_R - \hat{m}\overline{\psi}_R\psi_R + (\delta Z \text{ terms}).$$

Now it looks like the old lagrangian, but written in terms of the renormalised fields, with the addition of the δZ *counterterms.* Now when you calculate, the counterterms give you new vertices to include in your diagrams. The divergences contained in the counterterms cancel the infinities produced by the loop integrations, leaving a finite answer.

The old A and ψ are called the *bare* fields, and e and m are the bare coupling and mass.

Note that to maintain the original form of $\mathcal{L}$, you want $Z_1 = Z_2$, so that the $\not{\partial}$ and $\hat{e}\not{A}$ terms combine into a covariant derivative term. This relation does hold, and is a consequence of the electromagnetic gauge symmetry: it is known as the *Ward identity.*

5.2 Renormalisation in Quantum Chromodynamics

QCD is a theory of interactions between spin-1/2 quarks and spin-1 gluons. It is a nonabelian gauge theory based on the group $SU(3)$, with Lagrangian,

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_f \overline{\psi}_f (i\not{D} - m_f)\psi_f + \begin{array}{l}\text{gauge fixing and}\\ \text{ghost terms}\end{array} \tag{5.6}$$

Here, a is a colour label, taking values from 1 to 8 for $SU(3)$, and f runs over the quark flavours. The covariant derivative and field strength tensor are given by,

$$\begin{aligned} D_\mu &= \partial_\mu - igA^a_\mu T^a, \\ G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \end{aligned} \tag{5.7}$$

where the f^{abc} are the structure constants of $SU(3)$ and the T^a are a set of eight independent Hermitian traceless 3×3 matrix generators in the fundamental or defining representation (see the pre school problems and the quantum field theory course).

As in QED gauge fixing terms are needed to define the propagator and ensure that only physical degrees of freedom propagate. The gauge fixing procedure is more complicated in the nonabelian case and necessitates, for certain gauge choices, the appearance of Faddeev–Popov ghosts to cancel the contributions from unphysical polarisation states in gluon propagators. However, the ghosts first appear in loop diagrams, which we will not compute in this course.

There are no Higgs bosons in pure QCD. The only relic of them is in the masses for the fermions which are generated via the Higgs mechanism, but in the electroweak sector of the standard model.

A fundamental difference between QCD and QED is the appearance in the nonabelian case of interaction terms (vertices) containing gluons alone. These arise from the nonvanishing commutator term in the field strength of the nonabelian theory in equation (5.7). The photon is electrically neutral, but the gluons carry the colour charge of QCD (specifically, they transform in the adjoint representation). Since the force carriers couple to the corresponding charge, there are no multi photon vertices in QED but there are multi gluon couplings in QCD. This difference is crucial: it is what underlies the decreasing strength of the strong coupling with increasing energy scale.

In QCD, hadrons are made from quarks. Colour interactions bind the quarks, producing states with no net colour: three quarks combine to make baryons and quark–antiquark pairs give mesons. It is generally believed that the binding energy of a quark in a hadron is infinite. This property, called *confinement*, means that there is no such thing as a free quark. Because of asymptotic freedom, however, if you hit a quark with a high energy projectile it will behave in many ways as a free (almost) particle. For example, in deep inelastic scattering, or DIS, a photon strikes a quark in a proton, say, imparting a large momentum to it. Some strong interaction corrections to this part of the process can be calculated perturbatively. As the quark heads off out of the proton, however, the brown muck of myriad low energy strong interactions cuts in again and "hadronises" the quark into the particles you actually detect. This is illustrated schematically in Figure 5.5.

We now try to repeat the procedure we used for renormalising the coupling in QED, but this time in QCD, which is also a renormalisable theory. If we define the renormalised coupling $\hat{g}$ as the strength of the quark–gluon coupling, then in addition to the diagrams

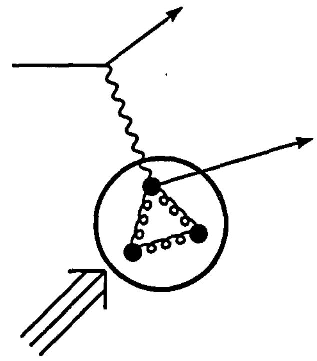


Figure 5.5 Schematic depiction of deep inelastic scattering. An incident lepton radiates a photon which knocks a quark out of a proton. The struck quark is detected indirectly only after hadronisation into observable particles.

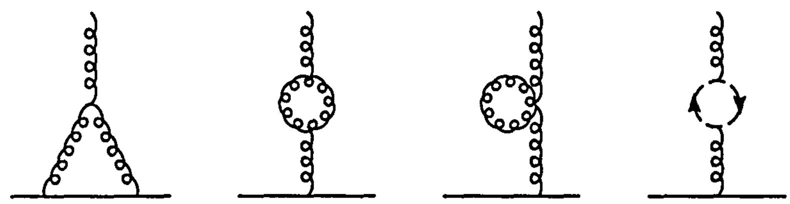


Figure 5.6 Additional diagrams for vertex renormalisation in QCD up to one loop. The dashed line denotes a ghost. For some gauge choices and some regularisation methods not all of these are required.

of Figure 5.1, with the photons replaced by gluons, there are more diagrams at one loop, shown in Figure 5.6. Looking at the second of these new diagrams, it is ultraviolet divergent (containing a $\ln M^2$ term), but also infrared divergent, since there is no mass to regulate the low momentum modes. In QED all the loop diagrams contain at least one electron propagator and the electron mass provides an infrared cutoff (you still have to worry when the electron is on-shell, but this is not our concern here). In the second diagram of Figure 5.6 there is no quark in the loop. Now suppose we choose to define the renormalised coupling off-shell at some non-zero q^2. The finite value of q^2 provides the infrared regulator and the diagram has a term proportional to $\ln(M^2/q^2)$.

Thus in QCD we can't define a physical coupling constant from an on-shell vertex. This is not really a serious restriction since the QCD coupling is not directly measurable anyway. Now the renormalised coupling depends on how we define it and therefore on at least one momentum scale (in almost all practical cases, only one momentum scale). The renormalised strong coupling is thus written,

$$\hat{g}(q^2).$$

When physical quantities are expressed in terms of $\hat{g}(q^2)$ the coefficients of the perturbation series are finite.

It would of course be possible to define the renormalised QED coupling to depend on some momentum scale. However, the on-shell definition used above is a natural one to pick.

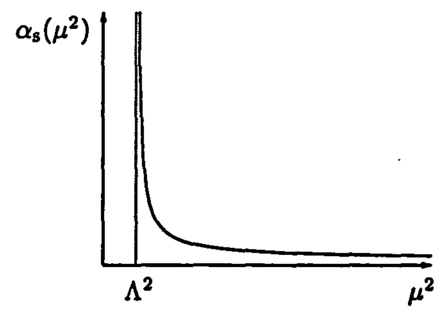


Figure 5.7 Running of the strong coupling constant with renormalisation scale.

You can define counterterms for QCD in the same way as was demonstrated for QED. Now the gauge coupling g enters in many terms where it could get renormalised in different ways. In fact, the gauge symmetry imposes a set of relations between the renormalisation constants, known as the *Slavnov–Taylor* identities, which generalise the Ward identity of QED.

We have just seen that the renormalised coupling in QCD, $\hat{g}(q^2)$, depends on the momentum at which it is defined. We say it depends on the *renormalisation scale*, and commonly refer to $\hat{g}$ as the "running coupling constant." We would clearly like to know just how $\hat{g}$ depends on q^2, so we calculate the diagrams in Figures 5.1 and 5.6, to get the first terms in a perturbation theory expansion:

$$\hat{g}(\mu) = g + g^3\Big[a_1 \ln\frac{M^2}{\mu^2} + b_1\Big] + \cdots \tag{5.8}$$

where a_1 and b_1 are constants and g is the "bare" coupling from the Lagrangian (5.6). I have switched to using μ^2 in place of q^2, and have written $\hat{g}$ as a function of μ for convenience. From this equation it follows that,

$$\mu\frac{\partial\hat{g}}{\partial\mu} \equiv \beta(\hat{g}) = -2a_1\,\hat{g}^3 + \cdots \tag{5.9}$$

The discovery by Politzer and by Gross and Wilczek, in 1973, that $a_1 > 0$ led to the possibility of using perturbation theory for strong interaction processes, since it implies that the strong interactions get weaker at high momentum scales — $\hat{g}(\infty) = 0$ is a stable solution of the differential equation (5.9). Keeping just the $\hat{g}^3$ term, we can solve (5.9) to find,

$$\alpha_s(\mu) \equiv \frac{\hat{g}^2(\mu)}{4\pi} = \frac{4\pi}{\beta_0\ln(\mu^2/\Lambda^2)}, \tag{5.10}$$

where Λ is a constant of integration and $\beta_0 = 32\pi^2 a_1$. Thus $\alpha_s(\mu)$ decreases logarithmically with the scale at which it is renormalised, as shown in Figure 5.7. If for some process the natural renormalisation scale is large, there is a chance that perturbation theory will be applicable. The value of β_0 is,

$$\beta_0 = 11 - \frac{2}{3}n_f, \tag{5.11}$$

where n_f is the number of quark flavours. The crucial discovery when this was first calculated was the appearance of the "11" coming from the self-interactions of the gluons via the extra diagrams of Figure 5.6. Quarks, and other non-gauge particles, always contribute negatively to β_0. Nonabelian gauge theories are the only ones we know where you can have asymptotic freedom (providing you don't have too much "matter" — providing the number of flavours is less than or equal to 16 for QCD).

What is the significance of the integration constant Λ? The original QCD Lagrangian (5.6) contained only a dimensionless bare coupling g (the quark masses don't matter here, since the phenomenon occurs for a pure glue theory), but now we have a dimensionful parameter. The real answer is that the radiative corrections (in all field theories except finite ones) break the scale invariance of the original Lagrangian. In QED there was an implicit choice of scale in the on-shell definition of $\hat{e}$. Lacking such a canonical choice for QCD, you have to say "measure α_s at $\mu = M_Z$" or "find the scale where $\alpha_s = 0.2$," so that a scale is necessarily involved. The phenomenon was called *dimensional transmutation* by Coleman. Λ is given by,

$$\Lambda = \mu \exp\left(- \int^{\hat{g}(\mu)} \frac{dg}{\beta(g)} \right), \tag{5.12}$$

and is μ-independent. The explicit μ dependence is cancelled by the implicit μ dependence of the coupling constant. Today it has become popular to specify the coupling by giving the value of Λ itself.

We've seen that the coupling depends on the scale at which it is renormalised. Moreover, there are many ways of defining the renormalised coupling at a given scale, depending on just how you have regulated the infinities in your calculations and which momentum scales you set equal to μ. The value of $\hat{g}(\mu)$ thus depends on the *renormalisation scheme* you pick, and with it, Λ. In practice, the most popular scheme today is called modified minimal subtraction, $\overline{\text{MS}}$, in which integrals are evaluated in $4-\epsilon$ dimensions and divergences show up as poles of the form ϵ^{-n} for positive integer n. In the particle data book [2] you will find values quoted for $\Lambda_{\overline{\text{MS}}}$ around 200 MeV (it also depends on the number of quark flavours). Don't buy a value of Λ unless you know which renormalisation scheme was used to define it.

In Figure 5.7 you see that the coupling blows up at $\mu = \Lambda$. This is an artifact of using perturbation theory. We can't trust our calculations if $\alpha_s(\mu) > 1$. In practice, you can perhaps use scales for μ down to about 1 GeV, but not much lower, and 2 GeV is probably safer. This region is a murky area where people try to match perturbative calculations onto results obtained from a variety of more or less kosher techniques.

▷ Exercise 5.1

Extending the expansion of $\hat{g}$ in terms of g in (5.8) to two loops gives

$$\hat{g}(\mu) = g + g^3\left[a_1 \ln\frac{M^2}{\mu^2} + b_1\right] + g^5\left[a_2 \ln^2\frac{M^2}{\mu^2} + b_2 \ln\frac{M^2}{\mu^2} + c_2\right],$$

with a similar equation for $\hat{g}(\mu_0)$ in terms of g. Renormalisability implies that $\hat{g}(\mu)$ can be expanded in terms of $\hat{g}(\mu_0)$,

$$\hat{g}(\mu) = \sum_{n=0}^{\infty} \hat{g}^{2n+1}(\mu_0) X_n,$$

where the X_n are finite coefficients. Show that this implies that a_2 is determined once the one loop coefficient a_1 is known. In fact a_1 determines all the terms $(\alpha_s \ln\mu)^n$, called the leading logarithms: from a one loop calculation, you can sum up all the leading logarithms.

For QED there is no positive contribution to the beta function, so the electromagnetic coupling has a logarithmic increase with renormalisation scale. However the effect is small even going up to LEP energies: α goes from $1/137$ to about $1/128$. The so called Landau pole, where α blows up, is safely hidden at an enormous energy scale.

Acknowledgements

I would like to thank Jonathan Flynn for donating to me his notes, and his beautiful latex files of this course from which I stole shamelessly. These notes are based heavily on those, and some sections are copied almost verbatim. According to Jonathan his notes owe a similar debt to his predecessors Chris Sachrajda and Tim Jones, who should therefore also be thanked. It is a pleasure to thank Steve Lloyd for organising and the school and Ann Roberts for keeping everything running smoothly. I would also like to thank my fellow lecturers, the tutors and the students for making the school so entertaining.

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THE STANDARD MODEL

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Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1996

The Standard Model.

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September, 1996.

Contents.

1. Introduction.

The **form** of the Lagrangian of the standard model is **completely determined** by three general requirements:

1) Invariance under the gauge group: $SU(3) \times SU(2) \times U(1)$.

2) The choice of matter fields (i.e. fermions and scalars) together with their transformation properties under the gauge group (i.e. the representation).

3) Perturbative renormalisability.

Do not worry if this is not obvious! I will shortly explain why the form of the Lagrangian is completely constrained by these requirements. I will try to point out where these requirements and choices are practically inevitable (for phenomenological or theoretical reasons) and where there are uncertainties.

Requirements (1) and (2) summarise all the qualitative information we have gleaned so far from experiment. Requirement (3) is a theoretical consideration, the motivation for which will be discussed in the last lecture, which leads to the following further constraint on the form of the Lagrangian:

<u>A perturbatively renormalisable Lagrangian is constructed by including in the Lagrangian all and only those couplings allowed by symmetries, with zero or positive mass dimension, and all possible mass terms allowed by the symmetries.</u>[1]

This further constraint thus leads to certain predictions that can be tested by experiment. In principle one could also require some global symmetries be satisfied, e.g. the continuous $U(1)$ groups corresponding to baryon number conservation, and conservation of the three separate Lepton numbers, or discrete symmetries such as CPT, but these symmetries turn out to be already automatically satisfied once the above constraints are imposed.

Notice that the above requirements determine only the **form** of the Lagrangian. They do not determine the parameters, i.e. physically the couplings, masses *etc.* We will see

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[1] Actually there is a rather important proviso to this rule: the kinetic terms must not hide any mass parameters. We will come back to this later.

that the standard model Lagrangian has 19 parameters in total, and at present we can only know the physical values of these parameters by determining them from precision experiments.

1.1. Perturbative Renormalisability.

As I have already stated, I will leave the motivation for this rule to the last lecture, but let me note here that this is really two issues, namely, "Why renormalisability?" and "Why perturbative?". The second of these issues is easy to answer, so I will do so here. For the electromagnetic interactions, the weak interactions (those mediated by the intermediate vector bosons), and at high energies also the strong interactions (as a consequence of asymptotic freedom), the strength of the interaction is small and therefore treating the interactions as perturbations (of a world with no interactions) seems sensible. The situation is much less clear for the Higgs sector.

1.2. The Field Content of the Standard Model.

I believe that by now, barring only the Higgs field, the qualitative features of the standard model can be regarded as so well experimentally tested and established that there can be little doubt that these features are correct. There could in principle be even other low energy physics beyond the standard model hiding away in the present data (if they are sufficiently difficult to detect *e.g.* neutrino masses, axions *etc.*) and certainly there has to be higher energy physics beyond the standard model, but all of these represent *additions* to the standard model and not modifications of the presently posited gauge group, field content and representations (*viz.* assignment of charges). Therefore I will for the most part simply state what these are, with the odd few ounces of justification, and leave the task of demonstrating just how solid the phenomenological evidence is, to my colleague Nigel Glover.

2. The Strong Interactions.

All the hadrons we have seen so far (with the possible exception of the odd resonance) correspond to one of two types: baryons with three quarks (and of course antibaryons with three anti-quarks) or mesons with a quark and an anti-quark. This pattern can be neatly understood if the quarks are assumed to come in three colours, i.e. each quark is a vector of three Dirac fermions

$$q = \begin{pmatrix} q_{red} \\ q_{green} \\ q_{blue} \end{pmatrix} \quad ,$$

forming a $\underline{3}$ of $SU(3)$, subject to a colour force mediated by 8 bi-coloured gluons – the gauge particles of local $SU(3)$. It is then energetically favourable for the bound states to be colour neutral, much as it is in QED for charged particles to be bound in electrically neutral atoms. The colour neutral combinations are nothing but

$$\bar{q}^i q_i$$

and

$$\epsilon_{ijk} q_i q_j q_k \quad ,$$

describing the mesons and baryons. Here i, j, k are colour indices.

Aside: A quick reminder how the 'construction kit' for the Lagrangian looks for any gauge group. (See Dave Dunbar's course.) If we want some non-Abelian symmetry to be a local symmetry so that the global symmetry

$$\psi(x) \mapsto \mathcal{U}\psi(x) \quad ,$$

where $\mathcal{U}$ is some 'rotation' (*i.e.* representation of a non-Abelian group), becomes instead

$$\psi(x) \mapsto \mathcal{U}(x)\psi(x) \quad ,$$

then wherever fields in different places are compared *e.g.* in $\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x)$, we need to introduce a gauge field $A_\mu(x)$ to soak up the remainder resulting from different amounts of rotation in the two different places. We do this by replacing ∂_μ by

$$D_\mu = \partial_\mu - igA_\mu^a T^a \quad ,$$

where T^a are the generators of the group – satisfying

$$[T^a, T^b] = if^{abc}T^c \quad ,$$

and f^{abc} are the structure constants. Now, since

$$\psi^\dagger(x) \mapsto \psi^\dagger(x)\mathcal{U}^\dagger(x) = \psi^\dagger(x)\mathcal{U}^{-1}(x) \quad ,$$

for a unitary group, $\bar{\psi}\gamma^\mu D_\mu \psi$ is invariant providing that

$$D_\mu \mapsto \mathcal{U}(x) D_\mu \mathcal{U}^{-1}(x) \quad . \tag{2.1}$$

Substituting in the formula for D_μ we see that A_μ *gauge transforms* as

$$A^a_\mu(x)T^a \mapsto \mathcal{U}(x)A^a_\mu T^a \mathcal{U}^{-1}(x) + \frac{i}{g}\mathcal{U}(x)\partial_\mu \mathcal{U}^{-1}(x) \quad .$$

For a general gauge transformation like this, there is not any simpler form, but for a small gauge transformation $\mathcal{U}(x) = \exp i\varepsilon^a(x)T^a$, the answer can be written down as

$$\begin{aligned} \delta A^a_\mu(x)T^a &= \frac{1}{g}\partial_\mu \varepsilon^a(x)T^a - i[A^b_\mu T^b, \varepsilon^c T^c] \\ \text{or} \quad \delta A^a_\mu(x) &= \frac{1}{g}\partial_\mu \varepsilon^a(x) + f^{bca}A^b_\mu(x)\varepsilon^c(x) \quad . \end{aligned}$$

We need something now to act as the *gauge invariant* kinetic term[2] for the gauge field. The *unique* answer lies in considering the *field strength*

$$T^a F^a_{\mu\nu}(x) = \frac{i}{g}[D_\mu, D_\nu] \quad .$$

Despite appearances, this is not a differential operator but a field. Indeed, if you use the fact that

$$\partial_\mu\{f(x)\psi(x)\} - f(x)\partial_\mu\psi(x) = \{\partial_\mu f(x)\}\psi(x) \quad ,$$

(where $f(x)$ here is any function), you see that

$$T^a F^a_{\mu\nu} = \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a - ig[A^b_\mu T^b, A^c_\nu T^c] \quad ,$$

so that

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{bca}A^b_\mu A^c_\nu \quad .$$

---

[2] *i.e.* a term in the Lagrangian with only two fields and with two or less space-time derivatives

Now, using equation (2.1) we have,

$$T^a F^a_{\mu\nu}(x) \mapsto \mathcal{U}(x) F^a_{\mu\nu}(x) T^a \mathcal{U}^{-1}(x)$$

and thus

$$\begin{aligned} F^a_{\alpha\beta}(x) F^a_{\gamma\delta}(x) &= 2\mathrm{Tr}\left\{(T^a F^a_{\alpha\beta})(T^b F^b_{\gamma\delta})\right\} \\ &\mapsto \mathrm{Tr}\left\{\mathcal{U}(x)(T^a F^a_{\alpha\beta})(T^b F^b_{\gamma\delta})\mathcal{U}^{-1}(x)\right\} \quad . \end{aligned} \tag{2.2}$$

We see that the $\mathcal{U}$'s cancel by cyclicity of the trace, and therefore $F^a_{\mu\nu}F^{a\mu\nu}$ is invariant as required. (*N.B.* it can be shown that f^{abc} is totally antisymmetric.[3]) *End of Aside.*

Quarks have been discovered in six flavours, u, d, c, s, t, b, but QCD is flavour blind and treats them all the same (hence the approximate strong interaction symmetries of isospin which 'rotates' u into d etc). Renormalisability only allows the field strength squared term and minimal coupling of the glue to the quarks, and thus (given the conventional[4] normalisation of the kinetic, bi-linear terms) we obtain uniquely

$$\mathcal{L}_{QCD} = \mathcal{L}_{glue} + \mathcal{L}_{\vartheta} + \sum_{\text{flavours}} \left(i\bar{q}\not{D}q - m_q \bar{q}q\right) \quad , \tag{2.3}$$

$$\text{where} \qquad \mathcal{L}_{glue} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} \tag{2.5}$$

$$\text{and} \qquad \mathcal{L}_{\vartheta} = \frac{1}{32\pi^2} \vartheta_{QCD} \epsilon_{\mu\nu\sigma\delta} G^{A\mu\nu} G^{A\sigma\delta} \quad . \tag{2.4}$$

The field strength and the covariant derivative are given by

$$\begin{aligned} G^A_{\mu\nu} &= \partial_\mu G^A_\nu - \partial_\nu G^A_\mu + g_3 f^{ABC} G^B_\mu G^C_\nu \qquad A, B, C = 1, \cdots, 8 \quad , \\ \text{and} \qquad D_\mu &= \partial_\mu - i g_3 G^A_\mu T^A \quad . \end{aligned}$$

Here the $(T^A)^j_i$'s are the eight 3×3 traceless Hermitian matrices, the generators of the $SU(3)$ group, conventionally normalised so that

$$\mathrm{Tr}\left(T^A T^B\right) = \frac{1}{2}\delta^{AB} \quad .$$

(Let me clear up some possible sources of confusion: I will for the most part understand the T^A's as matrices, the spinors (q) as column vectors, and the barred spinors ($\bar{q} = q^\dagger \gamma_0$)

[3] Exercise! Proof follows from $\frac{i}{2} f^{abc} = \mathrm{Tr}\left([T^a, T^b] T^c\right)$.

[4] except that they must of course have the right sign!

as row vectors in the colour space, and therefore suppress the explicit indices $i, j, k, \cdots$ (and so-on also for other internal groups). Of course I am also doing that for the Dirac indices so really $q \equiv q_{i\alpha}$, where i is a colour index and α a Dirac index. Once you get used to it, it does not take long to figure out from the context all the different indices that ought to be attached in various places. Secondly note that G^A_μ is the gluon gauge boson and $G^A_{\mu\nu}$ is its field strength. You can tell the two apart by the number of Lorentz indices, so it is helpful just to call them both G.)

Aside: We could have taken the flavours $j = u, d, c, s, t, b$, and written a mass matrix so that the fermion terms take the form

$$\mathcal{L}_{quark} = i\bar{q}_j \not{D} q_j - \bar{q}_j m_{jk} q_k \quad , \tag{2.6}$$

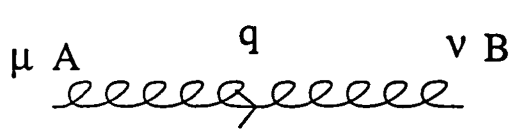
where the matrix m_{jk} is required Hermitian for $\mathcal{L}_{quark}$ to be real. But by a unitary transformation in flavour space $q_k \mapsto \Omega_{kk'} q_{k'}$, we can diagonalize the matrix m_{jk}, *i.e.* we choose Ω_{jk} so that $\left(\Omega^\dagger\right)_{ij} \Omega_{jk} = \delta_{ik}$, and

$$\Omega^\dagger m \Omega = \begin{pmatrix} m_u & 0 & 0 & 0 & 0 & 0 \\ 0 & m_d & 0 & 0 & 0 & 0 \\ 0 & 0 & m_c & 0 & 0 & 0 \\ 0 & 0 & 0 & m_s & 0 & 0 \\ 0 & 0 & 0 & 0 & m_t & 0 \\ 0 & 0 & 0 & 0 & 0 & m_b \end{pmatrix} .$$

Because this leaves the q's kinetic term ($\sim \delta_{ij}$ in flavour space) alone, this transformation turns this Lagrangian back into the original $\mathcal{L}_{QCD}$. Therefore Lagrangian (2.6) is effectively identical to that in (2.3). We should mention that these Lagrangians do not describe the real way the quarks get masses. They are allowed, in fact *required* by renormalisability, at the moment but they will not be allowed once we consider the charges under weak interactions.

You see that there are actually two sorts of field-strength squared. The second one, the *QCD ϑ term*, is a bit peculiar (although clearly allowed since any arrangement of Lorentz indices in (2.2) gave a gauge invariant term). It can be shown to be a total divergence $\mathcal{L}_\vartheta = \partial_\mu K_\mu$, and so it is tempting to throw it away since what actually enters in the path integral is the action $S = \int d^4x\, \mathcal{L}$. Therefore, $\mathcal{L}_\vartheta$ results in just a 'surface term'. But for certain special field configurations these surface terms cannot be ignored. For small g_3, the important configurations are called 'instantons' and lead to *non-perturbative* effects of order $\sim \exp -8\pi^2/g_3^2$. Now notice that $\mathcal{L}_\vartheta$ violates CP. (It is easiest to see this by noting that $\mathcal{L}_\vartheta \sim G_{01}G_{23}$ *etc.* must violate T invariance [$t \mapsto -t$ and $G_0 \mapsto -G_0$] and then use the fact that for all field theories CPT is conserved.) Experimentally however, CP appears to be conserved by QCD! The strongest bound on *strong* CP *violation* comes from putting limits on the neutron electric dipole moment implying $\vartheta_{QCD} < 10^{-7}$. Why is it so small? No-one knows. This is the *strong* CP *puzzle.* (For completeness, let me add –without explanation– that the bound above applies only once we have chosen our quark masses all to be real –otherwise these also result in CP violating effects.)

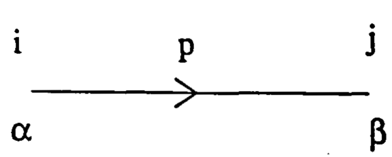
We can read off from the Lagrangian the QCD Feynman rules. The propagators are the same as have appeared in the other lectures (up to a few more Kronecker delta's):



$$= \frac{-i\delta^{AB}}{q^2 + i\epsilon} g_{\mu\nu}$$

Fig.1. The gluon propagator.

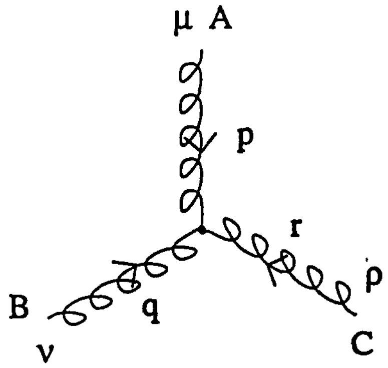
(Actually, this is only true in a certain gauge called Feynman gauge.)



$$= i\delta_i^{\ j} \left(\frac{1}{\not p - m_q + i\epsilon} \right)_{\beta\alpha} \equiv i\delta_i^{\ j} \frac{(\not p + m_q)_{\beta\alpha}}{p^2 - m_q^2 + i\epsilon}$$

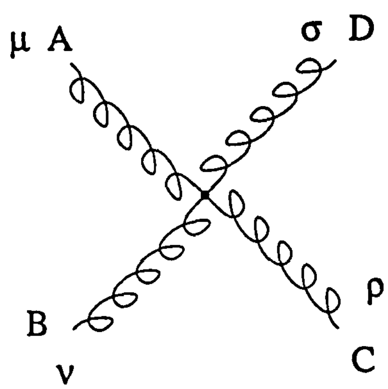
Fig.2. The quark propagator.

while the Feynman rules are



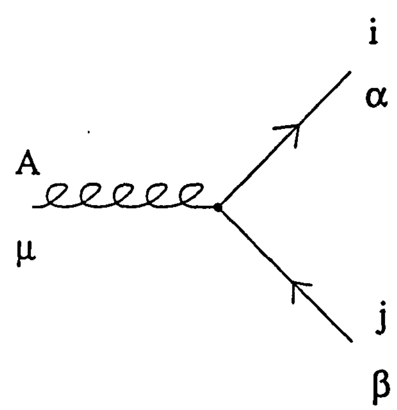
$$gf^{ABC}\left[g^{\mu\nu}(p-q)^{\rho}+g^{\nu\rho}(q-r)^{\mu}+g^{\rho\mu}(r-p)^{\nu}\right]$$

Fig.3. The gluon three-point interaction.



$$\begin{aligned}-ig^{2}\Big[&f^{EAB}f^{ECD}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}\right)\\&+f^{EAC}f^{EDB}\left(g^{\mu\sigma}g^{\rho\nu}-g^{\mu\nu}g^{\rho\sigma}\right)\\&+f^{EAD}f^{EBC}\left(g^{\mu\nu}g^{\sigma\rho}-g^{\mu\rho}g^{\sigma\nu}\right)\Big]\end{aligned}$$

Fig.4. The gluon four-point interaction.



$$ig\left(T^{A}\right)^{i}{}_{j}\gamma^{\mu}_{\alpha\beta}$$

Fig.5. The gluon-quark interaction.

Note that the ϑ_{QCD} term does not have Feynman rules because it is a surface term with effect only at $|x|=\infty$ (*i.e.* only *global* effects); it fails to appear at any order of perturbation theory! Also along with the gauge fixing one obtains *ghosts* and their propagators and interactions (see Dave Dunbar's course).

Of course, there are fermions other than the quarks in the standard model. These are the leptons which, by definition, do not feel the strong force. The gluons carry no other charges except colour, but the quarks do carry other charges. Their electromagnetic charges are $Q = -1/3$ for d, s, b and $Q = 2/3$ for u, c, t (as follows from the quark assignements and electromagnetic charges of the hadrons). In addition the quarks feel the weak interactions - which is our next subject.

3. Left Handed and Right Handed Fermion Fields.

We will very soon need to talk about left handed and right handed Fermi fields because the weak interactions are *chiral* (that is, they depend on the handedness). We can project out the left handed and right handed components of a (massless) fermion by introducing the 'projection operators'

$$P_L = \tfrac{1}{2}(1-\gamma_5) \qquad \text{and} \qquad P_R = \tfrac{1}{2}(1+\gamma_5) \quad .$$

We then define

$$\psi_L = P_L\psi \qquad \text{and} \qquad \psi_R = P_R\psi \quad ,$$

where ψ is a Dirac fermion. Because $P_L + P_R = 1$ (*c.f.* Problem 1), ψ can be split in two:

$$\psi = (P_L + P_R)\psi = \psi_L + \psi_R \quad .$$

Physically, these fields correspond to the following situation (*c.f.* Problem 1):

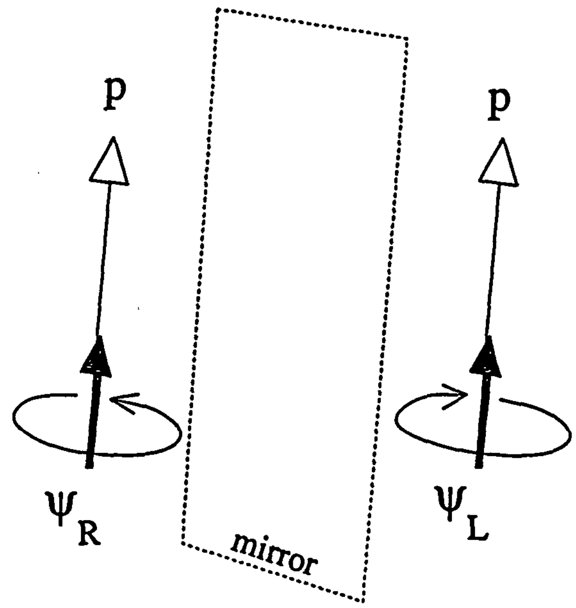


Fig.6. The physical meaning behind left handed and right handed helicity. Note that the "right handedness" refers to the fact that classically the spin turns as a right-handed screw.

These two guys are mirror images of each other, so you can see that chiral interactions must break parity.[5] Now, the kinetic term for the Dirac field also splits into two

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi = i\bar{\psi}_L\partial\!\!\!/\psi_L + i\bar{\psi}_R\partial\!\!\!/\psi_R \quad ,$$

because
$$\begin{aligned}\bar{\psi}_L\partial\!\!\!/\psi_R &= \psi^\dagger P_L\gamma^0\gamma^\mu\partial_\mu P_R\psi \\ &= \psi^\dagger\gamma^0 P_R\gamma^\mu\partial_\mu P_R\psi \\ &= \psi^\dagger\gamma^0\gamma^\mu\partial_\mu P_L P_R\psi \;= 0 \quad ,\end{aligned}$$

and similarly $\bar{\psi}_R\partial\!\!\!/\psi_L = 0$. (These steps follow using the results from Problems 1 and 2.) From this we see again that for massless fields we can just have left handed or right handed spinors – throwing the other half away. Not so, if they are massive since

$$m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$$

(*c.f.* Problem 2). We see that the mass, by coupling right handed spinors to left handed spinors acts as an amplitude for flipping the helicity; helicity is not conserved for massive fermions.[6]

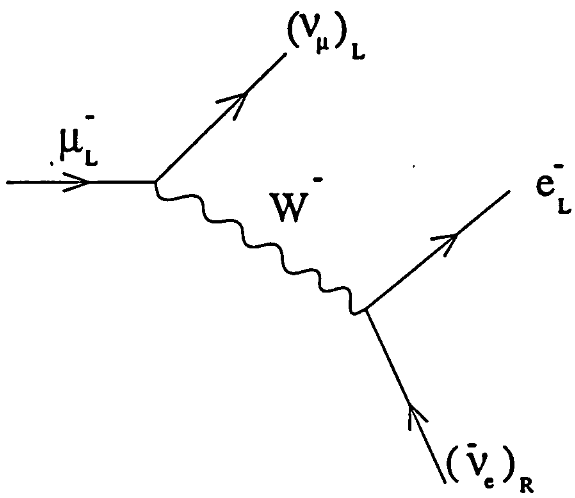
---

[5] Parity = a reflection + a rotation.

[6] This is fundamentally because spin is not conserved, only the total angular momentum $\underline{J} = \underline{L} + \underline{S}$ is conserved. See Steve King's course.

4. The Electroweak Interactions and The Leptons.

We start with the leptons because their interactions are easier to describe than those of the quarks, for a reason that will become clear later. By now, numerous experiments have shown that the weak interactions allow transmutation of the *left handed* component of the lepton into a *left handed* neutrino (and *vice versa*). (This started with the famous Cobalt 60 experiment of Mme. Wu et al [1956] in which the electron in the β decay was seen to be preferentially polarised.) Right handed neutrinos, so far as we can tell, are never produced or transmuted by the weak interactions. These interactions are well described by the process



$$\mu^- \to \nu_\mu e^- \bar{\nu}_e$$

Fig.7. Leptonic decay of the muon by weak interactions.

for example, which mathematically requires the introduction of the left handed and right handed spinors and assignement of the left handed lepton and corresponding left handed neutrino into $SU(2)$ doublets (*weak isospin*):

$$\underset{i=e,\mu,\tau}{L_i} = \begin{pmatrix} \nu_e \\ e_L^- \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu_L^- \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau_L^- \end{pmatrix}.$$

Here, you should understand that ν_e, ν_μ, ν_τ stand for the left handed fermion fields only. Since no evidence for right handed ν's exists, we do not introduce them. The right handed leptons $l_i = e_R^-, \mu_R^-, \tau_R^-$ (with $i = e, \mu, \tau$) do not couple to the charged weak vector bosons so they must be $SU(2)$ singlets. (Note the notation: a little l for a singlet and a big L for doublet. This pattern will be used consistently through the lectures.) The action of the weak isospin $SU(2)$ is defined by the generators which may be represented as $T_a \equiv \sigma_a/2$ *on the doublets*, where the σ_a are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

The action of the generators on the singlets is trivial:

$$T_a l_i = 0 \qquad i.e. \qquad T_a \equiv 0 \quad \text{for} \quad a = 1, 2, 3.$$

The electric charge matrix Q must be a generator of the gauge group, so that the photon and QED are incorporated. Notice that the entries in the doublets differ by unit charge and so eQ must have a component $eQ \sim eT_3$, but clearly this is not enough because the charges of the doublets would be $\pm e/2$, and the singlets l_i would have charge 0. On the other hand you see that the charges in the singlets and doublets are right if we shift by a constant charge ($-\frac{1}{2}$ for the doublets, -1 for the singlets). Thus

$$eQ = e\left\{T_3 + Y/2\right\} \quad , \tag{4.1}$$

where Y is the *hypercharge* and is just proportional to the unit matrix on both doublets and singlets – and thus is the generator of a *separate* $U(1)$ group: $[T_a, Y] = 0$. This explains the assignement of $SU(2) \times U(1)$ for the electroweak gauge group. We have the weak hypercharges $Y = -2$ for singlets and $Y = -1$ for doublets *i.e.*

$$Yl_i = -2l_i \qquad\qquad YL_i = -L_i \quad .$$

Now we are in a position to write down the Lagrangian

$$\begin{aligned} \mathcal{L}_{EW} = &-\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} + \frac{1}{32\pi^2}\vartheta_{weak}\varepsilon_{\mu\nu\sigma\delta}W^{a\mu\nu}W^{a\sigma\delta} \\ &+ i\sum_{j=e,\mu,\tau}\bar{L}_j\left(\partial_\mu - ig_2W^a_\mu\frac{\sigma^a}{2} + \frac{1}{2}ig_1B_\mu\right)\gamma^\mu L_j \\ &+ i\sum_{j=e,\mu,\tau}\bar{l}_j\left(\partial_\mu + ig_1B_\mu\right)\gamma^\mu l_j \quad . \end{aligned} \tag{4.2}$$

Here we have introduced the $U(1)$ gauge field B_μ which couples to weak hypercharge. Its corresponding coupling is g_1. The triplet of gauge fields W_μ^a couple to weak isospin - and their coupling is g_2. The field strengths are

$$\begin{aligned} W_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g_2 \varepsilon^{abc} W_\mu^b W_\nu^c \qquad a,b,c = 1,2,3 \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \quad . \end{aligned}$$

Once again, the Lagrangian is completely determined by the requirement of renormalisabity.

You see that there is also a ϑ term for the $SU(2)$ gauge fields. Just as in QCD, this leads to non-perturbative effects $\sim \exp -8\pi^2/g_2^2$ but because weak interactions are so small ($g_2 << 1$), these effects are so negligable (at least in normal situations) that it is not usual to include this term in the standard model. The B field does not have such a term: the reason is that certain topological arguments[7] show that unless the gauge group is 'bigger than or equal to' $SU(2)$ (strictly contains $SU(2)$ as a subgroup) then the surface terms can indeed be ignored.

Note that the Lagrangian contains no mass terms! Preservation of gauge invariance forbids us from writing down mass terms $\sim m^2 W_\mu^a W_\mu^a$ *etc.* for the gauge fields and we cannot write down masses for the fermions because the right handed guys transform differently from the left handed guys. (We would want to try to write $\sim m[\bar{L}l + \bar{l}L]$ but L is a vector and l is a scalar in weak isospin space so these indices do not match and nor are the combinations hypercharge neutral.) Obviously these masslessness properties are a phenomenological disaster: electrons, muons and taus *do* have masses, and there is no such thing as a long range ($\sim 1/r^2$) $SU(2)$ weak force so three gauge bosons must somehow get a mass also, leaving massless the one gauge boson coupling to the charge Q.

At the moment the electric charge is somewhat hidden in the couplings to the fermions:

$$g_2 T_3 W_\mu^3 + g_1 \frac{Y}{2} B_\mu$$

but because the kinetic terms for these guys

$$-\frac{1}{4}\left\{(\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3)^2 + (\partial_\mu B_\nu - \partial_\nu B_\mu)^2\right\}$$

[7] which would unfortunately take too long to explain

are just the canonical sum of squares we can rotate to a new basis

$$\begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} = \begin{pmatrix} \cos\vartheta_W & -\sin\vartheta_W \\ \sin\vartheta_W & \cos\vartheta_W \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \quad ,$$

leaving the kinetic terms still properly normalised. In this new basis the couplings look as

$$\left(g_2 \sin\vartheta_W T_3 + g_1 \cos\vartheta_W \frac{Y}{2}\right) A_\mu + \left(g_2 \cos\vartheta_W T_3 - g_1 \sin\vartheta_W \frac{Y}{2}\right) Z_\mu \quad .$$

Thus, if A_μ is to be the massless photon we need

$$\begin{array}{llll} & g_2 \sin\vartheta_W = e & \text{and} & g_1 \cos\vartheta_W = e \\ \text{so} & \tan\vartheta_W = g_1/g_2 & \text{and} & e = \dfrac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} \quad . \end{array}$$

This is illustrated in the little diagram below.

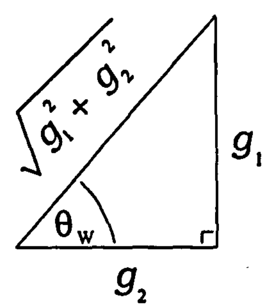


Fig.8. Definition of the Weinberg angle.

ϑ_W is the *Weinberg angle.* Z_μ is the Z boson that better somehow become heavy. We see that it couples to the fermions as

$$\begin{aligned} & eZ_\mu \left(\cot\vartheta_W T_3 - \tan\vartheta_W \frac{Y}{2}\right) \\ =& Z_\mu \frac{e}{\sin\vartheta_W \cos\vartheta_W} \left(T_3 - \sin^2\vartheta_W Q\right) \quad . \end{aligned}$$

In this last step I have used the formula (4.1) for Q. You see that Z_μ, unlike the photon, couples to neutrinos (through T_3) thus generating 'neutral current' weak interactions such as ν_μ e^- scattering:

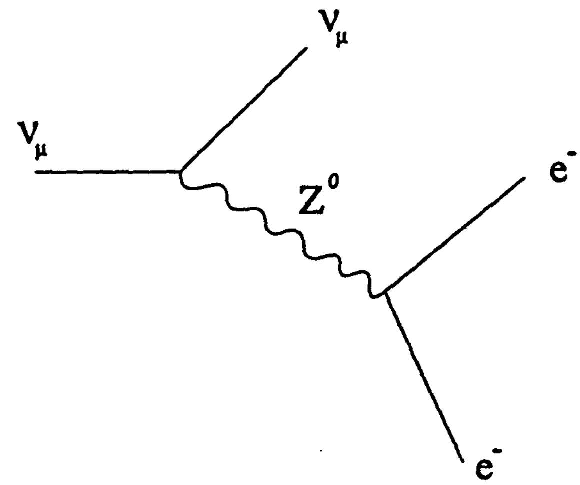


Fig.9. ν_μ e^- scattering via their neutral current weak interactions.

Let us look at just one of the fermions f (*i.e.* $f = e, \mu, \tau$, or one of the neutrinos) but both its left handed component and right handed component (if it has one). Both couplings of the Z_μ can be written then as

$$\mathcal{L}^Z_{int} = \frac{e}{\sin\vartheta_W \cos\vartheta_W} \left\{ \bar{f}_L Z^\mu \gamma_\mu (t_3 - \sin^2\vartheta_W) f_L + \bar{f}_R Z^\mu \gamma_\mu (-\sin^2\vartheta_W Q) f_R \right\} \quad ,$$

where I have written t_3 as the eigenvalue of T_3, used the fact that $T_3 f_R = 0$, and written eQ as the electric charge of f. Note that for neutrinos the f_R term disappears anyway because $Q = 0$ in this case. Now we reexpress this in terms of Dirac fermions,

$$\begin{aligned} \bar{f}_L \gamma^\mu f_L &= \bar{f}\gamma^\mu \tfrac{1}{2}(1-\gamma_5) f \\ \bar{f}_R \gamma^\mu f_R &= \bar{f}\gamma^\mu \tfrac{1}{2}(1+\gamma_5) f \quad , \end{aligned}$$

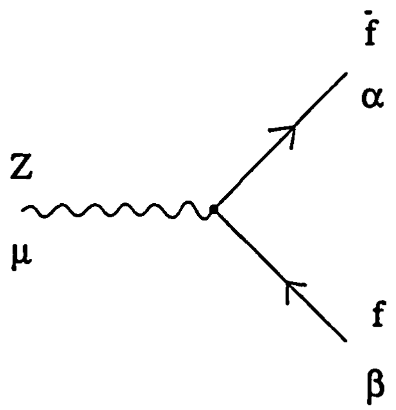
resulting in

$$\mathcal{L}^Z_{int} = \frac{e}{2\sin\vartheta_W \cos\vartheta_W} \bar{f} Z_\mu \gamma^\mu (C^f_V - C^f_A \gamma_5) f \quad ,$$

where

$$C^f_A = t_3 \qquad \text{and} \qquad C^f_V = t_3 - 2\sin^2\vartheta_W Q \quad .$$

Thus we can write the Feynman rule



$$\frac{-ie}{2\sin\vartheta_W\cos\vartheta_W}\gamma^\mu\left(C_V^f - C_A^f\gamma_5\right)$$

Fig.10. The Z-fermion interaction.

With this information, Problem 3 of the problems can now be attacked. We will see later that the same Feynman rule applies to quarks. You only need that they effectively form $SU(2)$ doublets $\sim \begin{pmatrix} u \\ d \end{pmatrix}$. With this information, Problem 4 may also be started.

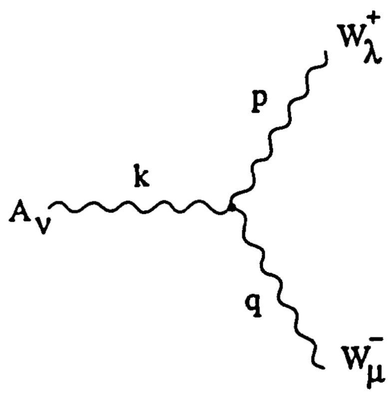
Note that the Z^0 is chargeless because $[Q, T_3] = 0$, while the couplings $T_1 W_\mu^1 + T_2 W_\mu^2$ do not commute with Q. If we change basis by writing this sum as

$$\underbrace{\frac{1}{\sqrt{2}}(T_1 + iT_2)}_{T_+}\underbrace{\frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}}}_{W_\mu^+} + \underbrace{\frac{1}{\sqrt{2}}(T_1 - iT_2)}_{T_-}\underbrace{\frac{W_\mu^1 + iW_\mu^2}{\sqrt{2}}}_{W_\mu^-} \quad ,$$

where $T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, are $SU(2)$ raising and lowering operators respectively, then we see that $W_\mu^\pm$ have charges $\pm e$ because

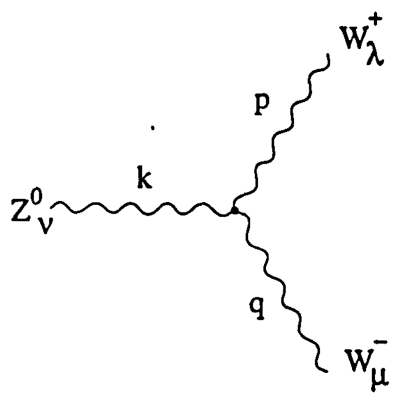
$$[Q, T_\pm] = [T_3, T_\pm] = \pm T_\pm \quad .$$

With the changes to physical vector particles all in place, one can read off the Feynman rules from $\mathcal{L}_{EW}$. The gauge boson self-couplings are



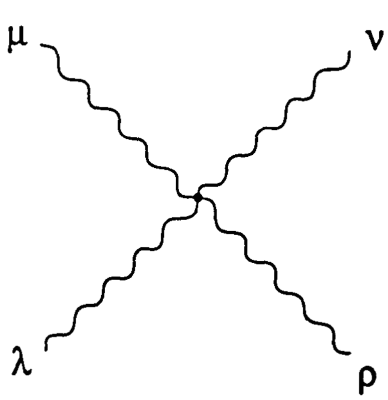
$$e\left[(k-q)_\lambda g_{\mu\nu} + (q-p)_\nu g_{\lambda\mu} + (p-k)_\mu g_{\lambda\nu}\right]$$

Fig.11. The AWW interaction.



$$e\cot\vartheta_W\left[(k-q)_\lambda g_{\mu\nu} + (q-p)_\nu g_{\lambda\mu} + (p-k)_\mu g_{\lambda\nu}\right]$$

Fig.12. The ZWW interaction.



$$g_{vector\ bosons}\left[2g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}\right]$$

Fig.13. The weak vector boson four-point interactions.

where $g_{vector\ bosons}$ are the following weights, depending on the vector bosons involved in the four-point interactions:

$$g_{W^+W^+W^-W^-} = ie^2/\sin^2\vartheta_W$$
$$g_{W^+W^-AA} = -ie^2$$
$$g_{W^+W^-ZZ} = -ie^2\cot^2\vartheta_W$$
$$g_{W^+W^-AZ} = -ie^2\cot\vartheta_W \quad .$$

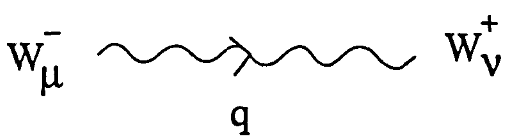
Let me cheat a little and put the masses M_Z and $M_{W^\pm}$ in anyway so that we can do some phenomenology. (Later we will see that the following steps are correct after all.) Let us look again at the charged current interaction of figure 7. Plugging the change of variables above, into the Lagrangian $\mathcal{L}_{EW}$ we see that the relevant interactions are [check it!]:

$$\frac{g_2}{\sqrt{2}}\bar{e}_L\gamma^\mu W_\mu^- \nu_e = \frac{e}{2\sqrt{2}\sin\vartheta_W}\bar{e}\gamma^\mu W_\mu^-(1-\gamma_5)\nu_e$$

and similarly

$$\frac{e}{2\sqrt{2}\sin\vartheta_W}\bar{\nu}_\mu\gamma^\mu W_\mu^+(1-\gamma_5)\mu \quad .$$

The propagator (Feynman gauge)



$$\frac{-ig_{\mu\nu}}{q^2 - M_W^2 + i\epsilon}$$

Fig.14. The weak vector boson propagator.

collapses to the simple form $ig_{\mu\nu}/M_W^2$ at low momenta and so the Feynman diagram in figure 7, is equivalent at low energies to an interaction

$$(-i)\left(\frac{ie}{2\sqrt{2}\sin\vartheta_W}\right)^2\left(\frac{i}{M_W^2}\right)\bar{\nu}_\mu\gamma^\mu(1-\gamma_5)\mu\,\bar{e}\,\gamma_\mu(1-\gamma_5)\nu_e \quad .$$

(Here the first three factors of i all come from the i in e^{iS}.) But this is the old Fermi current-current interaction, except that the coupling would be written $-G_F/\sqrt{2}$. Thus we deduce the Fermi constant:

$$G_F = \frac{\sqrt{2}e^2}{8M_W^2\sin^2\vartheta_W} \quad .$$

(Now you have all that is needed for Problem 4.)

The neutral current that couples to Z^μ is given above

$$J_\mu^{NC} = \bar{f}\gamma^\mu\left(C_V^f - C_A^f\gamma_5\right)f \quad ,$$

and evidently leads to the same effective form

$$-\left(\frac{e}{2\sin\vartheta_W\cos\vartheta_W}\right)^2\left(\frac{1}{M_Z^2}\right)J_\mu^{NC}J_\mu^{NC} \quad = -2\rho\frac{G_F}{\sqrt{2}}J_\mu^{NC}J^{NC\mu} \quad ,$$

where the ρ *parameter* is given as

$$\rho = \frac{M_W^2}{M_Z^2\cos^2\vartheta_W} \quad ,$$

and you see that 2ρ is the ratio of neutral and charged current interaction strengths.

5. Spontaneous Symmetry Breaking.

Why can we not just write down masses as we did above? It breaks the gauge invariance, but you should ask: do we really need gauge invariance? The real problem is not gauge invariance but the loss of renormalisability and/or unitarity. In the kinetic part of the field strength for say the Z boson, which in momentum space looks as $Z_{\mu\nu} \sim p_\mu Z_\nu - p_\nu Z_\mu$, the longitudinal component $Z_\mu^L \propto p_\mu$ slips through:

$$p_\mu Z_\nu^L - p_\nu Z_\mu^L = 0 \quad .$$

When we add a mass term

$$m^2 Z_\mu^2 \sim m^2\left(Z_\mu^L\right)^2 + \cdots$$

to the Lagrangian, it is the *only term* that appears for the longitudinal component and thus has to play the rôle that the momentum dependent kinetic term normally plays. To apply the requirements of perturbative renormalisability we then have to normalise this term by $Z_\mu^L \mapsto Z_\mu^L/m$ so that its 'kinetic' term appears as

$$S = \tfrac{1}{2}\int d^4x\,\left(Z_\mu^L\right)^2 \quad .$$

In this form Z_μ^L has mass dimension two, and so you will find it cannot have *any* interactions (*i.e.* cubic or higher in the fields) with itself or anything else without introducing perturbatively non-renormalisable couplings (*i.e.* with negative mass dimension).

(Perhaps you have asked yourself why you cannot simply add a term that *will* give a sensible kinetic term for Z_μ *e.g.* $\frac{1}{2}(\partial_\nu Z_\mu)^2$. If you have not, then ignore this paragraph! The answer is that adding such a term always results in a wrong sign for one of the components of the Z:

$$-\tfrac{1}{2}(\partial_\nu Z_\mu)^2 = -\tfrac{1}{2}(\partial_\nu Z_0)^2 + \tfrac{1}{2}(\partial_\nu Z_1)^2 + \tfrac{1}{2}(\partial_\nu Z_2)^2 + \tfrac{1}{2}(\partial_\nu Z_3)^2 \quad .$$

This then leads to severe unphysical behaviour such as negative probabilities *etc.*. Gauge invariance is there to eliminate this unphysical behaviour, which it does because the time component may always be gauged away to $Z_0 = 0$ – but remember we have now broken the gauge invariance.)

A way out of this apparently insuperable problem is to use spontaneous symmetry breaking.

5.1. A Global $U(1)$ Model.

The simplest example for our purposes is a single complex scalar field

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(|\varphi|^2) \quad .$$

The theory is invariant under a global $U(1)$ symmetry, namely phase redefinition

$$\varphi \mapsto e^{i\alpha} \varphi \quad .$$

By perturbative renormalisability, the interaction potential has the form

$$V(|\varphi|^2) = M^2 |\varphi|^2 + \lambda |\varphi|^4 \quad ,$$

but in principle there is nothing wrong with $M^2 < 0$. In this case the potential looks as

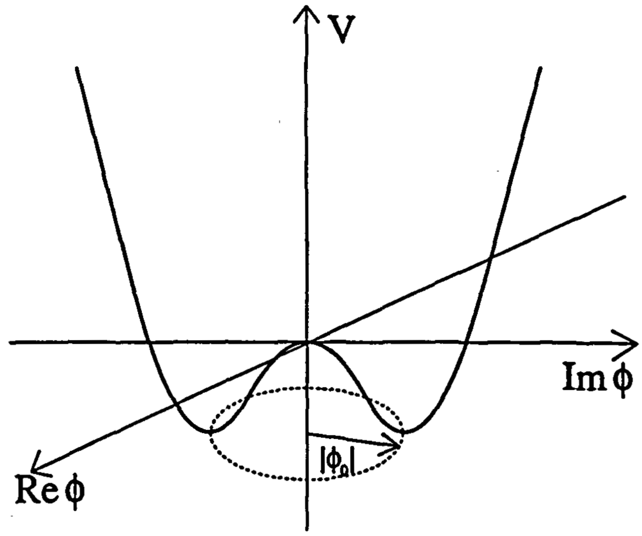


Fig.15. The potential $V(|\varphi|^2)$ with negative M^2.

We can think of this φ field as loads of little arrows,

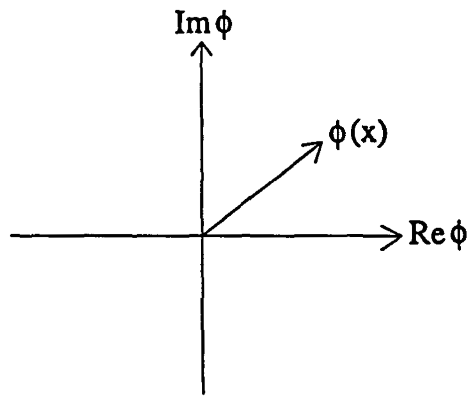


Fig.16. Complex $\varphi(x)$ field as coupled set of 'arrows'.

one for each position in space, nearest neighbours weakly coupled together through the kinetic term $\partial_\mu\varphi^*\partial^\mu\varphi$. Indeed, if we had chosen the arrows not to live in a plane but in three dimensional space (*i.e.* $\varphi \equiv \varphi^a$, $a = 1, 2, 3$ instead) then this would be a model for a ferromagnet. (φ would represent the local magnetization – the total spin in some small domain.) With the potential in the form above, the minimum energy of the system is not where $\varphi = 0$ but with φ equal to some value φ_0 on the circle

$$|\varphi_0|^2 = -\frac{M^2}{2\lambda} \quad .$$

As a result the system has 'spontaneously magnetized', and the vacuum (that is, the lowest energy state) has *spontaneously broken* the $U(1)$ symmetry. So far this discussion has been purely classical, but the same is true in the *quantum* field theory: although quantum fluctuations can alter the position of the 'spins' locally, they cannot 'pick up' the whole vacuum state and rotate it to a new position φ_0 because the whole vacuum has too much inertia. (In the ferromagnet analogy: although quantum fluctuations cause spins to fluctuate locally, they cannot cause a macroscopic change of direction for the overall magnetization, in any finite time interval, because this would cost a macroscopic amount of energy.) We can see intuitively what the fluctuations correspond to, also. There will be radial fluctuations σ (along the direction φ_0) that see a potential $\sim m^2\sigma^2$ and are therefore massive. On the other hand there are fluctuations where the 'spins' fluctuate away from φ_0 by going round the circle ('*spin waves*'). These cost no potential energy and are therefore massless. In fact in any situation where the energetics are such that the vacuum state spontaneously breaks a (global) symmetry of the theory, there will be alternative vacua (reached mathematically by applying the symmetry to the vacuum state) and there will be massless modes corresponding to local fluctuations along these directions. These massless modes are known as *Goldstone bosons.*

Returning to our example, let us use phase invariance to set φ_0 to be real:

$$\varphi_0 = \frac{v}{\sqrt{2}} \qquad \text{where} \qquad v = \sqrt{-\frac{M^2}{\lambda}} \quad .$$

(Here the factor of $1/\sqrt{2}$ has been taken out of the definition of v, just for convenience.) We write

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[v + \sigma(x) + i\pi(x)\right] \quad ,$$

where *sigma* and π are real, so that σ and π represent the fluctuation fields around the *vacuum expectation value* v. Substituting, one obtains, after some algebra,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\pi)^2 - \lambda v^2\sigma^2 + \mathcal{L}_{int} \quad ,$$

$$\text{where} \qquad \mathcal{L}_{int} = -\lambda v(\sigma^2+\pi^2)\sigma - \frac{\lambda}{4}(\sigma^2+\pi^2)^2 \quad .$$

We see that indeed the σ particle has a mass

$$m_\sigma^2 = 2\lambda v^2$$

and π is the massless Goldstone boson.

5.2. A Local U(1) Model.

Next consider what happens when we make the invariance local:

$$\mathcal{L}_{\varphi A} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + igA_\mu)\varphi^*(\partial^\mu - igA^\mu)\varphi - V(|\varphi|^2) \quad .$$

With the same potential as before, the field φ will again want to choose $|\varphi|^2 = -M^2/(2\lambda)$. But this time the Goldstone boson $\pi(x)$, corresponding to local phase fluctuations, is not entirely physical because $\varphi \mapsto e^{i\alpha(x)}\varphi(x)$ can be undone with a gauge transformation

$$A_\mu(x) \mapsto A_\mu(x) + \frac{1}{g}\partial_\mu\alpha(x) \quad ,$$

i.e. the π field can be considered instead to be part of the vector field! To see clearly what is going on, we need to pick the *unitary gauge* which corresponds to using the local phase invariance to fix everywhere $\varphi(x)$ to be real. In this gauge, fluctuations about the vacuum expectation value are just

$$\varphi(x) = [v + \sigma(x)]/\sqrt{2} \tag{5.1}$$

and

$$\mathcal{L}_{\varphi A} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}g^2v^2A_\mu^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \lambda v^2\sigma^2 + \text{ cubic and quartic interactions} \quad .$$

The gauge field has gained a mass,

$$m_A^2 = g^2v^2 \quad ,$$

and Goldstone's π field has disappeared. It has been '*eaten*' by the gauge field: the massive vector field now has three degrees of freedom (at rest, it points in some spatial direction) while the massless vector field had only two (two transverse polarizations of the photon). This is the so-called *Higgs mechanism.* It is worth remarking that this local $U(1)$ $\mathcal{L}$ is almost precisely the phenomenological Lagrangian used to describe superconductivity. Inside the superconductor, the photon is indeed massive (the Meissner effect). It is called 'phenomenological' because the field φ is used to describe the average effect of the condensate of Cooper pairs of electrons. All that is required is that the field be bosonic (so that its particles form a condensate) and have the right quantum numbers ($g = -2e$ in this case).

Now I want to finish this section with something strange (that does not have any interpretation in terms of superconductivity). Suppose that there existed a fermion whose right handed component ψ_R was electrically neutral, but whose left handed component ψ_L had charge $g = -2e$. In this case the constraints of gauge invariance and renormalisability uniquely determine the ψ part of the Lagrangian $\mathcal{L} = \mathcal{L}_{\varphi A} + \mathcal{L}_\psi$ so that

$$\mathcal{L}_\psi = i\bar{\psi}_L \not{D} \psi_L + i\bar{\psi}_R \not{\partial} \psi_R - \lambda_\psi \bar{\psi}_L \varphi \psi_R - \lambda_\psi \bar{\psi}_R \varphi^* \psi_L \quad .$$

(Notice that the two *Yukawa interaction* terms are complex conjugates of each other – as required by the reality of $\mathcal{L}$.) You see that no mass term is allowed because $-m_\psi(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ is not chargeless. But once the symmetry is broken as in eqn. (5.1), we obtain Yukawa interactions with the σ field, *and a mass term* for ψ with

$$m_\psi = \lambda_\psi v/\sqrt{2} \qquad (!)$$

We can use this relation to replace the Yukawa coupling λ_ψ in $\mathcal{L}_\psi$ by the ratio of the mass of the fermion and the vacuum expectation value of the scalar field:

$$\lambda_\psi = \frac{\sqrt{2}}{v} m_\psi \quad . \tag{5.2}$$

Now let us apply these ideas to solve the problem of mass in the standard model.

5.3. The Higgs.

Spontaneous symmetry breaking solves our previous problems because this way of gaining mass *is* renormalisable. The problems of renormalisability come from the very short wavelength high energy interactions and these modes could not care less that a zero energy-momentum infinitely long wavelength part has chosen some non-zero solution of the equations of motion. The properties of renormalisability are encoded in the Lagrangian – not the solutions of the equations of motion. (Nevertheless, the fact that renormalisability does now follow, is not at all obvious mathematically and it was 't Hooft's great acheivement in 1971, to prove that this is so.)

All we need now is something bosonic with the right quantum numbers and a choice of couplings that make it condense. It could be a single scalar Higgs field, *i.e.* a single representation, or it could be several representations. It could be just as fundamental as the other fields or it could be standing in for a condensate of bound states, like φ did for the Cooper pairs, or something more exotic.

While the Higgs these days is beginning to be constrained by the precision LEP experiments, the Higgs sector is still much the weakest in the standard model. The standard model corresponds to the simplest assumption: namely that the Higgs is a 'genuine' scalar field $H(x)$ in a single representation. The solution is then unique. Since we want to provide masses for the leptons, $H(x)$ better be equivalent to a weak isospin $SU(2)$ doublet with hypercharge $Y = 1$ so that the combination $\sim \lambda(\bar{L}Hl + \bar{l}H^\dagger L)$ is invariant and hypercharge neutral. [check it!] Remembering the formula (4.1) for the electric charge, we have,

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix} \quad ,$$

with the charges as indicated.

By global $SU(2) \times U(1)$ transformations, we can always choose the vacuum expectation value to be real and in the h^0 direction – which is what we want if we require Q to take its conventional form [*c.f.* (4.1)]:

$$Q = T_3 + Y/2 \qquad \Leftarrow \qquad Q\begin{pmatrix} 0 \\ v \end{pmatrix} = 0 \quad .$$

(In another basis Q is still defined to be the one unbroken generator. Things look different mathematically but the physics is just the same.)

Renormalisability now fixes uniquely the form of the Higgs and Yukawa sector of the standard model to be:

$$\begin{aligned} \mathcal{L}_{Higgs} = & \left[(\partial_\mu + ig_2 W_\mu^a \frac{\sigma^a}{2} + \frac{i}{2} g_1 B_\mu) H^\dagger\right] \left[(\partial^\mu - ig_2 W^{a\mu} \frac{\sigma^a}{2} - \frac{i}{2} g_1 B^\mu) H^\dagger\right] \\ & - M^2 H^\dagger H - \lambda \left(H^\dagger H\right)^2 - \sum_{i=e,\mu,\tau} \lambda_i \left(\bar{L}_i H l_i + \bar{l}_i H^\dagger L_i\right) \quad . \end{aligned} \tag{5.3}$$

For spontaneous symmetry breaking we require $M^2 < 0$, then the Higgs will choose a non-zero vacuum expectation value

$$\begin{aligned} h^+ &= 0 \\ h^0 &= v/\sqrt{2} \quad , \end{aligned}$$

where again the factor of $1/\sqrt{2}$ is introduced just for convenience, and

$$v^2 = -M^2/\lambda$$

(just as in the $U(1)$ model). In the unitary gauge, the Higgs field everywhere is chosen to have just a real h^0 component and so fluctuations are simply given by

$$H(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \sigma(x) \end{pmatrix} \quad . \tag{5.4}$$

The one real Higgs, $\sigma(x)$, survives with mass

$$m_\sigma^2 = 2\lambda v^2 = -2M^2 \quad .$$

(Again this is just the same equation as in the $U(1)$ model.) The three Goldstone bosons have been eaten and given $W_\mu^\pm$ and Z_μ^0 masses. The vacuum expectation value has provided the electron, mu and tau with their masses. To see this, substitute the above unitary gauge Higgs into $\mathcal{L}_{Higgs}$ and set $\sigma(x) = 0$. (We are not for the moment interested in σ and its interactions.) If you look at just the electron's Yukawa couplings, these become

$$\begin{aligned} \lambda_e \left(\bar{L}_e H l_e + \bar{l}_e H L_e\right) \mapsto & \lambda_e \left\{ \begin{pmatrix} \bar{\nu}_e & \bar{e}_L \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} e_R + \bar{e}_R \begin{pmatrix} 0 & v/\sqrt{2} \end{pmatrix} \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \right\} \\ = & \lambda_e \frac{v}{\sqrt{2}} \left(\bar{e}_L e_R + \bar{e}_R e_L\right) \quad . \end{aligned}$$

The same manipulations hold for μ and τ, and thus we identify their masses as

$$m_i = \lambda_i \frac{v}{\sqrt{2}} \qquad\qquad i = e, \mu, \tau \quad . \tag{5.5}$$

Similarly you can show [*i.e.* check it!] that the mass terms for the gauge bosons read

$$\frac{v^2}{8} \left\{ g_2^2 \left[(W_\mu^1)^2 + (W_\mu^2)^2 \right] + \left(g_1 B_\mu - g_2 W_\mu^3 \right)^2 \right\} \quad .$$

Now using the fact that the terms in square brackets can be written $2W_\mu^+ W^{-\mu}$ and that, from inverting our previous rotation,

$$\begin{aligned} & Z_\mu = \sin\vartheta_W B_\mu - \cos\vartheta_W W_\mu^3 \\ i.e. \qquad & \sqrt{g_1^2 + g_2^2}\, Z_\mu = g_1 B_\mu - g_2 W_\mu^3 \quad , \end{aligned}$$

we see that the mass terms read,

$$M_W^2 W_\mu^+ W^{-\mu} + \tfrac{1}{2} M_Z^2 Z_\mu Z^\mu$$

with

$$\begin{aligned} M_{W^\pm} &= \frac{1}{2} g_2 v \\ &= \frac{1}{2} \frac{ev}{\sin\vartheta_W} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} M_Z &= \frac{1}{2} v \sqrt{g_1^2 + g_2^2} \\ &= \frac{1}{2} \frac{ev}{\sin\vartheta_W \cos\vartheta_W} \quad . \end{aligned} \tag{5.7}$$

In particular we have

$$\cos\vartheta_W = M_W / M_Z \quad .$$

All these relations are classical (tree level) relations and get corrected (a little) quantum mechanically. A common convention is to define the 'physical' value of ϑ_W to be *given* by the above relation. The measured values of M_W and M_Z are

$$\begin{aligned} M_W &= 80.338 \pm .040 \begin{matrix} +.009 \\ -.018 \end{matrix} \; GeV \\ M_Z &= 91.1863 \pm .002 \; GeV \end{aligned}$$

and thus

$$\sin^2\vartheta_W = 1 - M_W^2 / M_Z^2 = .2238 \pm .0008 \begin{matrix} +.0003 \\ -.0002 \end{matrix} \quad .$$

Being totally ahistorical, we can use these numbers together with $\alpha = e^2/(4\pi) = 1/137$, to obtain a tree level estimate of the Fermi constant

$$G_F = \frac{\sqrt{2} e^2}{8 M_W^2 \sin^2\vartheta_W} \approx 1.12 \times 10^{-5} \; (GeV)^{-2} \quad ,$$

which should be compared to the measured value

$$G_F = 1.16637(2) \times 10^{-5} \; (GeV)^{-2} \quad .$$

On the other hand we have at tree level that (twice) the ratio of neutral to charged current interactions

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\vartheta_W} = 1 \qquad\qquad (!)$$

This is not an accident but a consequence of a hidden approximate symmetry (broken only when the electric charge $e \neq 0$) called *custodial* $SU(2)$. Not all Higgs representations have this symmetry so the fact that ρ is indeed measured to be very close to 1 is strong indirect evidence for this class of representations.

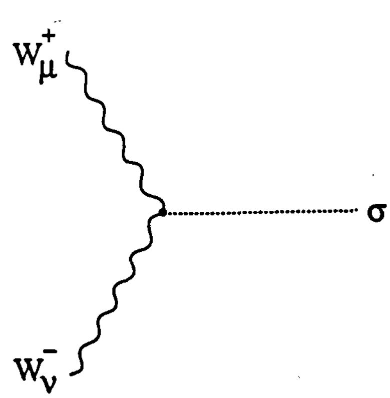
Substituting these numbers into the equations for the Z and $W^{\pm}$ masses, (5.6) and (5.7), one obtains the tree level estimate for the *electroweak breaking scale*:

$$v \approx 250\ GeV \quad .$$

This fixes the ratio $m_\sigma^2/\lambda = 2v^2$ but neither the Higgs mass nor its self coupling is known separately (if indeed the Higgs exists as such!). Some weak constraints on m_σ^2 are being deduced indirectly from precision LEP experiments (see Nigel). Note that, from (5.4) and (5.5), the Higgs Yukawa couplings are

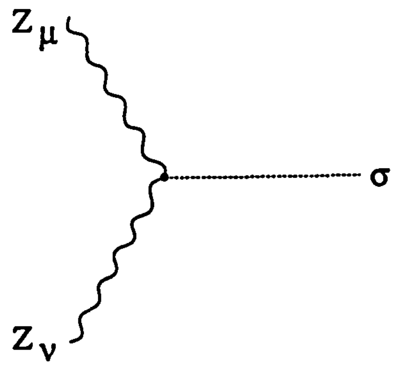
$$-\frac{m_f}{v}\bar{f}\sigma f \qquad\qquad f = e, \mu, \tau \quad ,$$

so the Higgs couples strongest to the most massive particle. (This will be true for the quarks also.) As well as these Yukawa interactions, substitution of the unitary gauge Higgs, (5.4), leads to many interactions of the Higgs with the gauge bosons. In the standard way these turn into Feynman rules (which I have chosen to write in terms of e, M_W, ϑ_W and m_σ):



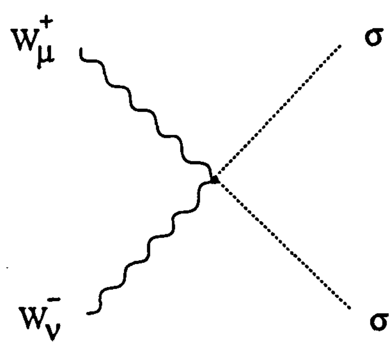
$$\frac{ieM_W}{\sin\vartheta_W} g_{\mu\nu}$$

Fig.17. The Higgs-WW interaction.



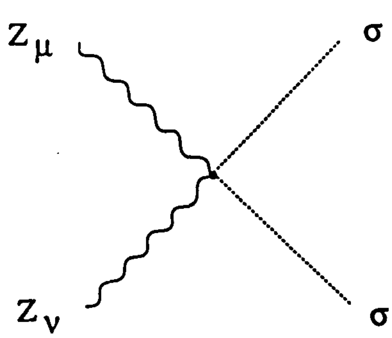
$$\frac{ieM_W}{\sin\vartheta_W \cos^2\vartheta_W} g_{\mu\nu}$$

Fig.18. The Higgs-ZZ interaction.



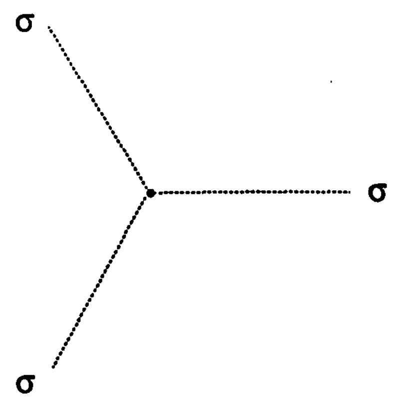
$$\frac{i}{2}\frac{e^2}{\sin^2\vartheta_W} g_{\mu\nu}$$

Fig.19. The Higgs-Higgs-WW interaction.



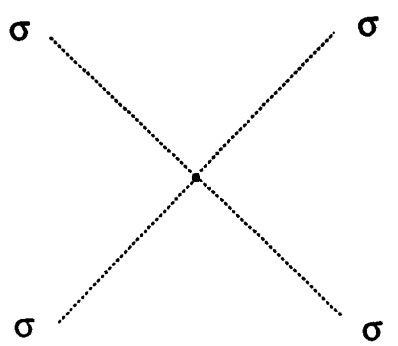
$$\frac{i}{2}\frac{e^2}{\sin^2\vartheta_W \cos^2\vartheta_W} g_{\mu\nu}$$

Fig.20. The Higgs-Higgs-ZZ interaction.



$$-\frac{3i}{2}\frac{e}{\sin\vartheta_W}\frac{m_\sigma^2}{M_W}$$

Fig.21. The Higgs three-point self-interaction.



$$-\frac{3i}{4}\frac{e^2}{\sin^2\vartheta_W}\frac{m_\sigma^2}{M_W^2}$$

Fig.22. The Higgs four-point self-interaction.

Just as in the QCD case much earlier, we could have used mass matrices:

$$-\frac{\sqrt{2}}{v}\sum_{i,j}\left\{m_{ij}\bar{L}_i H l_j + m^*_{ji}\bar{l}_j H^\dagger L_i\right\} \quad .$$

Here they are really Yukawa coupling matrices λ_{ij}, but anticipating formulae like (5.2) and (5.5), we write $\lambda_{ij} = \sqrt{2}\, m_{ij}/v$, and so defining the mass matrix m_{ij} in the process. Note that now the mass matrix need not be Hermitian. But once again this is entirely equivalent to the previous Lagrangian, because now we can use *different* unitary transformations in flavour space

$$L_j \mapsto (\Omega_R)_{jj'} l_{j'} \qquad\qquad L_i \mapsto (\Omega_L)_{ii'} L_{i'}$$

and use the fact that *any* matrix can be diagonalised by two unitary transformations:

$$\Omega_L^\dagger m \Omega_R = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \quad .$$

You can check that this leaves the rest of the lepton Lagrangian invariant, because these bits are still proportional to unit matrices in flavour space (*i.e.* $\bar{l}_i\delta_{ij}\cdots l_j$ and $\bar{L}_i\delta_{ij}\cdots L_j$) and couple only $\bar{l}$ to l and $\bar{L}$ to L.

5.4. Weak Interactions of Quarks.

The weak interactions for the quarks take a very similar form to those for the leptons, for example β decay $n \to pe^-\bar{\nu}_e$, or $\pi^- \to e^-\nu_e$, is described at the quark level by

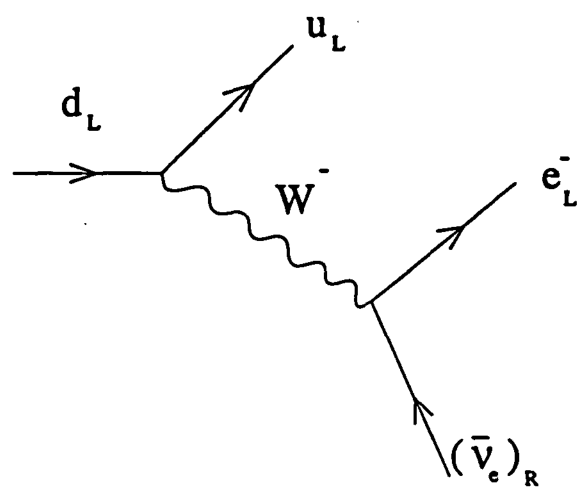


Fig.23. A d quark weak decay.

and similarly *e.g.* charm decay $D^+ \to \bar{K}^0\pi^+$ is described as

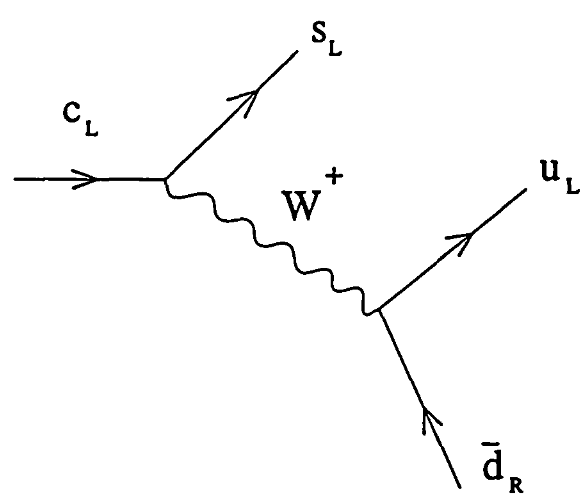


Fig.24. A c quark weak decay.

so that quarks also form weak isospin doublets according to the generation, *i.e.*

$$\underset{i=u,c,t}{Q_i} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} .$$

Actually we know this is not quite correct: the weak interactions mix generations *e.g.* in the strangeness-changing decay $K^+ \to \mu^+\nu_\mu$

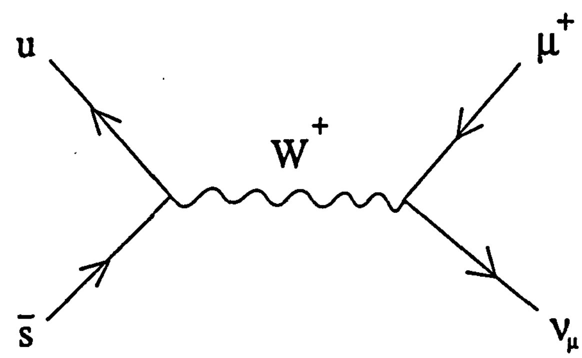


Fig.25. The strangeness changing weak decay: $K^+ \to \mu^+\nu_\mu$.

but let us persist with the above assignments and see what happens. First of all we see that the hyper-charge of the Q_i doublets is

$$Y = \frac{1}{3}$$

so that (4.1) comes out right. The right handed partners must all exist, if all the quarks are to have masses, but since the right handed quarks do not have charged current interactions they must be assigned to singlets. Thus the right handed hypercharges are just twice their charges:

$$\begin{aligned}
q_i^u \quad &= u_R, c_R, t_R \qquad Y = \frac{4}{3} \\
i = &u, c, t \\
q_i^d \quad &= d_R, s_R, b_R \qquad Y = -\frac{2}{3} \quad . \\
i = &d, s, b
\end{aligned}$$

We have already defined all the other parts of the standard model so now we can write down uniquely (by renormalisability) the quark sector:

$$
\begin{aligned}
\mathcal{L}_{quarks} = \; & i \sum_{i=u,c,t} \bar{Q}_i \left(\partial_\mu - ig_3 G^A_\mu T^A - ig_2 W^a_\mu \frac{\sigma^a}{2} - \frac{1}{6} ig_1 B_\mu \right) \gamma^\mu Q_i \\
& + i \sum_{i=u,c,t} \bar{q}^u_i \left(\partial_\mu - ig_3 G^A_\mu T^A - \frac{2}{3} ig_1 B_\mu \right) \gamma^\mu q^u_i \\
& + i \sum_{i=d,s,b} \bar{q}^d_i \left(\partial_\mu - ig_3 G^A_\mu T^A + \frac{1}{3} ig_1 B_\mu \right) \gamma^\mu q^d_i \\
& - \frac{\sqrt{2}}{v} \sum_{\substack{i=u,c,t \\ j=d,s,b}} \left\{ m^d_{ij} \bar{Q}_i H q^d_j + m^{d\,*}_{ji} \bar{q}^d_j H^\dagger Q_i \right\} \\
& - \frac{\sqrt{2}}{v} \sum_{i,j=u,c,t} \left\{ m^{u\,*}_{ji} \bar{q}^u_j \left(Q_i \times H \right) + m^u_{ij} \left(\bar{Q}_i \times H^* \right) q^u_j \right\} \quad .
\end{aligned}
\tag{5.8}
$$

I have put in the gluon interactions because the quark part of $\mathcal{L}_{QCD}$ I gave you much earlier, in (2.3) and (2.6), was wrong and should be replaced with the above. (It was wrong because I had not of course included the $SU(2) \times U(1)$ gauge fields, and you see that quark masses are now forbidden – by the $SU(2) \times U(1)$ charges.) The funny looking interactions on the last line are a consequence of a little serendipitous accident of $SU(2)$. Rotations in the two dimensional plane of the vectors $\underline{u}$ and $\underline{v}$ leave $(\underline{u} \times \underline{v})$ invariant *i.e.* $\underline{u} \times \underline{v}$ is a two dimensional scalar

$$\underline{u} \times \underline{v} \equiv u_1 v_2 - v_1 u_2 \quad .$$

This is true for complex vectors u_a, v_b and $SU(2)$ rotations ${U_a}^b$ too because in both cases the 'rotations' have unit determinant:

$$\varepsilon^{ab} u_a v_b \mapsto \underbrace{\varepsilon^{ab} {U_a}^c {U_b}^d}_{\underbrace{(\det U)}_{1} \varepsilon^{cd}} u_c v_d \quad .$$

Since

$$\bar{Q} \times H^* = \bar{Q} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} H^* = \bar{Q} \left(i\sigma_2 H^* \right)$$

is $SU(2)$ invariant, it follows that

$$\tilde{H} = i\sigma_2 H^*$$

transforms as an $SU(2)$ doublet. (The proof is simple – can you see it?) Now we can write the up Yukawa terms in a more normal way

$$-\frac{\sqrt{2}}{v}\sum_{i,j=u,c,t}\left\{m^u_{ij}\bar{Q}_i\tilde{H}q^u_j + m^{u\,*}_{ji}\bar{q}^u_j\tilde{H}^\dagger Q_i\right\} \quad .$$

(*Exercise:* check that you understand that $\mathcal{L}_{quarks}$ is invariant under $SU(2)$ weak isospin and check that it is indeed hypercharge neutral.)

The full Lagrangian of the standard model has now been written! It is

$$\mathcal{L}_{SM} = \mathcal{L}_{glue} + \mathcal{L}_{\theta} + \mathcal{L}_{EW} + \mathcal{L}_{Higgs} + \mathcal{L}_{quarks} \quad ,$$

the formulae for the various parts appearing in (2.5), (2.4), (4.2), (5.3), (5.8), respectively. (*Another exercise:* Convince yourself that there are no other renormalisable interaction term consistent with the local $SU(3) \times SU(2) \times U(1)$ symmetry, the chosen field content and their representations! *i.e.* that this is indeed the unique solution to the three principles we started with.)

5.5. The CKM Matrix.

In unitary gauge, recal that H takes the form (5.4). This implies that

$$\tilde{H}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} v+\sigma(x) \\ 0 \end{pmatrix} \quad .$$

Putting these into the quark Yukawa interaction terms you see that we obtain

$$-\bar{D}_L m^d D_R - \bar{U}_L m^u U_R - \frac{\sigma}{v}\bar{D}_L m^d D_R - \frac{\sigma}{v}\bar{U}_L m^u U_R + \text{complex conjugates,}$$

where I have introduced the notations

$$U_L = \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} \quad U_R = \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} \quad D_L = \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} \quad D_R = \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} \quad .$$

This is a more helpful notation now that $SU(2)$ invariance is broken. The first two terms are the mass matrices and we can diagonalize them, like in the lepton case, by separate unitary transformations on each flavour vector:

$$
\begin{aligned}
U_L &= \Omega_L^u U_L' \\
U_R &= \Omega_R^u U_R' \\
D_L &= \Omega_L^d D_L' \\
D_R &= \Omega_R^d D_R'
\end{aligned}
\tag{5.9}
$$

chosen so that

$$
\begin{aligned}
(\Omega_L^u)^\dagger\, m^u\, \Omega_R^u &= \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \\
(\Omega_L^d)^\dagger\, m^d\, \Omega_R^d &= \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}
\end{aligned}
$$

and thus the Yukawa terms simply read

$$
-\sum_{\text{flavours}} \left(1+\frac{\sigma}{v}\right) m_q \bar{q}' q' \quad .
$$

Now for the quarks, this is not the end of the story. Unlike for the leptons our flavour transformations $\Omega_L^u \neq \Omega_L^d$ are *different* for the top and bottom part of $SU(2)$ doublets! Therefore this transformation messes up those interactions in $\mathcal{L}_{SM}$ that 'cared' that $Q \sim \begin{pmatrix} U_L \\ D_L \end{pmatrix}$ was an $SU(2)$ doublet. (In the lepton case this problem does not arise because the standard model has no right handed neutrinos and therefore the analogous second mass term $\sim m_{ij}^\nu \bar{L}_i \tilde{H}\,(\nu_R)_j$ is missing.) We must distinguish, then, between the primed basis U' and D' that propagate with well defined masses – and are thus called the *mass eigenstates*, and the original basis $\begin{pmatrix} U_L \\ D_L \end{pmatrix}$, U_R, D_R which form (irreducible) representations of the $SU(2) \times U(1)$ algebra and are thus called *weak eigenstates.* Roughly speaking, the weak eigenstates do the interacting, while the mass eigenstates do the propagating (*e.g.* outwards into the detector). We see that there is a (non-trivial) unitary transformation between them.

The parts of $\mathcal{L}_{SM}$ that do not care, are all the terms diagonal in flavour space. Therefore the quark kinetic terms are left invariant by (5.9). Also the Z_μ and A_μ couplings are left invariant since T_3 is diagonal in flavour space. (You easily see from $\mathcal{L}_{quarks}$, $\mathcal{L}_{EW}$ and the analysis given there, that these couplings take the same form as for the leptons – only the values of t_3 and Q [*i.e.* Y] differ.) Therefore there are *no flavour changing neutral currents* (part of the GIM mechanism – after Glashow, Iliopoulos and Maiani).

The only parts that do care about the doublet structure are the charged current interactions:

$$\frac{e}{\sqrt{2}\sin\vartheta_W} \underbrace{\bar{U}_L\gamma^\mu W^+_\mu D_L}_{\frac{1}{2}\bar{U}\gamma^\mu W^+_\mu(1-\gamma_5)D} \quad + \text{complex conjugate}$$

(compare $\bar{\nu}_\mu$–μ interaction earlier). These become

$$\frac{e}{2\sqrt{2}\sin\vartheta_W}\bar{U}'\gamma^\mu W^+_\mu(1-\gamma_5)VD' + \text{complex conjugate},$$

where

$$V = (\Omega^u_L)^\dagger\,\Omega^d_L \equiv \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

is a unitary matrix – the famous *Cabibbo-Kobayashi-Maskawa matrix* (and in the standard model is responsible for the cross-generational decays we have so far been ignoring).

The CKM matrix, being a unitary 3×3 matrix, has 9 'angles' [the 8 rotations of $SU(3)$ and an overall phase. This last allowed because $\det V$ is in general a phase, rather than 1 as for $SU(3)$]. On the other hand not all these angles are physical. If we make a phase redefinition on each quark $u \mapsto e^{i\varphi_u}u$ *etc.*, nothing further changes in the $\mathcal{L}_{quarks}$ except that

$$V \mapsto \begin{pmatrix} e^{-i\varphi_u} & 0 & 0 \\ 0 & e^{-i\varphi_c} & 0 \\ 0 & 0 & e^{-i\varphi_t} \end{pmatrix} V \begin{pmatrix} e^{i\varphi_d} & 0 & 0 \\ 0 & e^{i\varphi_s} & 0 \\ 0 & 0 & e^{i\varphi_b} \end{pmatrix} .$$

If all these phases were equal, V would not change at all, so we can only define away 5 phases this way, leaving 4 physical parameters in V. If V had been real it would be a three dimensional rotation and described by three Euler angles, therefore the full complex V has 3 Euler angles and one phase. A popular parametrisation[8] is:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\,e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\,e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}\,e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\,e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\,e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}$$

[8] There are many possible parametrisations – all physically equivalent.

$c_{ij} = \cos\vartheta_{ij}$, $s_{ij} = \sin\vartheta_{ij}$, i, j being generation labels. Experimentally, the *magnitudes* $|V_{ij}|$ are

$$|V_{ij}| = \begin{pmatrix} 0.9747-0.9759 & 0.218-0.224 & 0.002-0.005 \\ 0.218-0.224 & 0.9738-0.9752 & 0.032-0.048 \\ 0.004-0.015 & 0.030-0.048 & 0.9988-0.9995 \end{pmatrix} \quad ,$$

(taken from the 1994 Particle Data Book). δ_{13} is known to lie between 0 and 2π (!). Since $|V_{tb}|$ is so close to 1, $c_{13} = c_{23} = 1$ to an excellent approximation, so $V_{us} \approx s_{12}$. This is the sine of the *Cabbibo angle* – the only mixing angle there would be if there were just two generations. What is the significance of δ_{13}? The interactions with δ_{13} in, are again –

$$\frac{e}{2\sqrt{2}\sin\vartheta_W}\{\bar{U}_i'\gamma^\mu W_\mu^+(1-\gamma_5)V_{ij}D_j' + \bar{D}_j'\gamma^\mu W_\mu^-(1-\gamma_5)V_{ij}^* U_i'\} \quad .$$

Under CP:

$$\psi \overset{P}{\mapsto} \gamma^0\psi$$

$$\psi \overset{C}{\mapsto} C\bar{\psi}^T \qquad \text{where} \qquad C = i\gamma^2\gamma^0 \quad ,$$

and so-on, the two terms are *interchanged* but *without* changing V_{ij} into V_{ij}^*, therefore if there is irremovable complexity in the CKM matrix, it is a signal of *CP violation.* This is the case, if $\delta \neq 0$ or π.

Final exercise: Count the number of parameters in the standard model and verify it is nineteen.

6. Why Renormalisability?

I want to stress this has ultimately nothing to do with the voodoo idea of "sweeping infinities under the carpet"! This was Dirac's pejorative comment, but this 1940's way of apologising for renormalisation was replaced by a complete intuitive understanding of the meaning of renormalisation after Wilson's work of the 1970's. (Incidentally the prime-movers behind perturbative renormalisation, namely Feynman and Schwinger, had already an understanding much closer to that of Wilson than Dirac.)

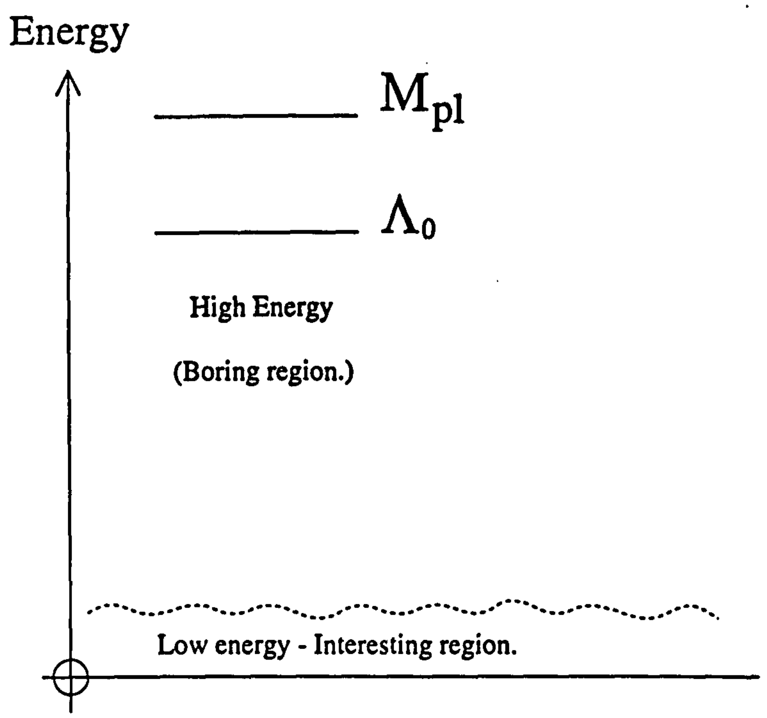


Fig.26. The energy scales relevant to the issue of renormalisability. The 'Interesting region' includes all accessible physics – in particular all the masses of the particles in the standard model. The 'Boring region' covers a range of energy scales very much larger than the 'Interesting region'. The bare Lagrangian $\mathcal{L}_0$ is defined at the ultra-high energy scale Λ_0, and contains parameters tuned to ensure that all physical masses m are very much less than Λ_0.

The real reason for renormalisability has to do physically with very high energies, for example the Planck mass $M_{Planck} = \sqrt{\hbar c/G_{Newton}} \sim 10^{19} GeV$: the energy where quantum gravity effects must become important. We know gravity exists but we do not know how to describe it quantum mechanically, and in particular it is not part of the standard model. (Actually, any very high energy physics which is not included in the standard model will do for this argument.) Therefore we have to conclude that the standard model is only an effective description: just a good approximation up to some energy scale $\Lambda_0 < M_{Planck}$. *Let us assume however that this energy scale Λ_0 is still very much higher than all the particle masses in the standard model.* (We will mention later what happens if this is not the case.)

To construct our effective description we supply a 'bare' Lagrangian $\mathcal{L}_0$ and compute the field theory 'path integral' by restricting by hand, in some way, all the energies that arise so that they are in the well described region, i.e. less than Λ_0. In other words we use Λ_0 as an ultraviolet *cutoff.*

Now we have a very peculiar situation: It is not at all the generic situation in quantum field theory that the particle masses m measure much less than the highest allowed energy. On the contrary, the particles experience so much inertia from 'wading' through the sea of virtual particles that, unless there is some very special reason, their effective masses are all $m \sim \Lambda_0$ (or they could be even heavier than this in which case they fail to propagate at all. Recall, virtual particles of energy E continually pop out of the vacuum for some time $\Delta t \lesssim \hbar/E$, as allowed by Heisenberg's uncertainty principle. If you prefer to draw Feynman diagrams, then they take the sort of form shown below. Nevertheless the result above simply follows from dimensional analysis.)

Fig.27. Self-energy diagrams are interactions of real particles with the virtual sea and result in effective masses $m_{eff} \sim \Lambda_0$ unless something very special happens. Shown are a fermion interacting with a virtual gauge boson, and a scalar self-interacting through a virtual pair.

This "very special reason" for small effective masses, could be symmetries for special sorts of particles (gauge bosons of unbroken local symmetry are always massless – like the photon; unbroken chiral symmetry can ensure massless fermions – as we will see later), but for bog-standard particles it can only come about from tuning values of the parameters in the bare Lagrangian $\mathcal{L}_0$ so that the classical values in the Lagrangian almost precisely cancel out the big $\sim \Lambda_0$ effects arising quantum mechanically. (Roughly speaking we choose the classical values to be almost the negative of the resulting quantum effects.) When this is done, the interesting part of the quantum field theory is governed by very long wavelength excitations $\lambda \sim 1/m$ compared to the small length scale of the bare Lagrangian $\sim 1/\Lambda_0$. Consequentially, nearly all these small length details are washed out, i.e. are invisible to the interesting long wavelength excitations: we are left with a *universal* quantum description (up to very tiny corrections $\sim m^2/\Lambda_0^2$ which for the moment we ignore) that depends on only finitely many parameters. These finitely many parameters are nothing but the fine-tuned differences remaining after cancelling the large quantum effects. We see that the fine-tuned differences just correspond physically *to all the couplings with zero or positive mass dimensions, and all the masses*, because only these parameters can receive large quantum corrections (by dimensional arguments again: any coupling of dimension d will get a high energy quantum correction of $\sim \Lambda_0^d$, which is large if and only if $d > 0$. Zero mass dimensions must be included because dimensionless couplings can suffer logarithmic divergences $\sim \ln(\Lambda_0/m)$.) These fine-tuned differences are the *real* couplings and masses, i.e. the parameters that are measured by experiment. They are called *renormalised* to distinguish them from the *bare* parameters in the bare Lagrangian $\mathcal{L}_0$.

It is a very important and deep fact that the resulting quantum field theory is universal, so it is worth repeating this with different words. The precise choice of $\mathcal{L}_0$ (and also the precise way the theory is cutoff) is entirely irrelevant: there are infinitely many different bare Lagrangians (and cutoffs) which yield exactly the same answer once a sufficient number of parameters are fine-tuned. (This infinite set of choices is called the "universality class".) The real physics is the universal physics we obtain at energies much lower than Λ_0: all the rest is theoretical scaffolding (usually) mathematically required in order to construct the real physics – but finally when the real physics is produced, it is independent of how the scaffolding was constructed.

The simplest bare Lagrangian we can take is one in which we include all and only those parameters which need fine-tuning to cancel large quantum effects. *This is the modern understanding of* what it means to have *a renormalisable Lagrangian*: it is just the simplest bare Lagrangian capable of producing (by fine tuning) the universal low energy behaviour we actually want.

So which parameters must we include? For a theory in which quantum effects could be strongly interacting throughout the region of interest, this is a hard theoretical problem and no general answer is known. Fortunately, if we assume that the standard model is weakly interacting, and therefore treatable by perturbation theory – as has already been argued, the answer is easy: to lowest order we can ignore the quantum effects entirely and the renormalised couplings observed in the low energy region are then simply the same as the bare couplings appearing in $\mathcal{L}_0$. But as underlined above the renormalised couplings are all the parameters with non-negative mass dimensions and so we have the following general rule, already quoted at the very beginning of these lectures, for constructing a perturbatively renormalisable bare Lagrangian:

A perturbatively renormalisable Lagrangian is constructed by including in the Lagrangian all and only those couplings with zero or positive mass dimension and all possible mass terms.

All these couplings and masses are known as the *(perturbatively) renormalisable couplings.* All the couplings, infinite in number, that we are leaving out, are known as *(perturbatively) non-renormalisable couplings.* We can now see why the requirements (1), (2) and (3) mentioned in the first lecture, fix the Lagrangian completely. We do not have any freedom over the choice of couplings once the symmetries and field content are fixed.

Let me finish this section with some paranthetical comments.

Where are the infinities that have been swept under the carpet? Theoretically it is often helpful to imagine that the standard model (or other partial theory) is actually valid up to infinite energies – even though we know this is not true. To obtain these circumstances we send $\Lambda_0 \to \infty$, and as a result we find that unless we fine-tune the bare couplings with *infinite* precision, interactions with the virtual sea over the infinite range of energy will result in infinite answers.

Note well that our choice of bare Lagrangian is by itself very little connected to reality, rather most of it (as a result of universality) is a figment of the theorists imagination! We can add any, or as many, of the so-called non-renormalisable couplings to the Lagrangian, as we wish, but after the requisite amount of fine-tuning, the experimentally measurable results are guaranteed – to very good accuracy – to be unchanged, by universality. Put another way, universality tells us that we are guaranteed to be almost totally blind to physics with very high energies $\sim \Lambda_0$. The blind-ness of the low energy, long wavelength, physics to the ultra-high energy, short distance, cutoff-scale physics is not complete: effects of order $\sim m^2/\Lambda_0^2$ can seep through, but if Λ_0 is sufficiently large these will be practically unobservable *unless* they correspond to some process which would be forbidden by the renormalisable $\mathcal{L}_0$.

We have all the while been assuming that $\Lambda_0 >> m$ *i.e.* the cutoff scale is much greater than energy scale of the interesting region. What happens if the opposite is assumed true and the standard model is assumed only valid to slightly higher energies than the interesting region? In this case we must assume Λ_0 is quite low, and the corrections to universality that we have just been discussing become very important. In this case we can predict virtually nothing because all the infinitely many couplings we can add to $\mathcal{L}_0$ crucially affect the answer, and obviously it is useless to attempt an infinite number of experiments to fix them all! On the other hand nor *should* we be able to predict anything in this case because if the standard model breaks down at this low energy scale then by the same token the new physics (be it new particles with masses $\gtrsim \Lambda_0$ or whatever) crucially affects the physics at energies below Λ_0 through virtual effects. (This was very much the situation with strong interaction physics during the 1960's, before the underlying theory of QCD was hypothesised, and led some people to disbelieve in quantum field theory for a while.)

Acknowledgements

I have assimilated parts from so many sources that it would be too self-indulgent for me to explain it in any detail. Of course the material is now so standard that all forms of presentation are equivalent. In presenting mine, I would like to acknowledge helpful conversations with Steve King, Jonathon Flynn, Graham Shore, Doug Ross and Cliff Burgess. The lecture notes I referred to, are from Weinberg, Jan Ambjørn, Doug Ross and Jonathon Flynn. The books I consulted were the introductory chapters of Graham Ross' *Grand Unified Theories*, Pais' book *Inward Bound* (a historical account) and Georgi's book *Weak Interactions* (thanks Steve King!). This last book I particularly recommend, however it is occasionally idiosyncratic and is unfortunately out of print!

It is a pleasure to thank everyone responsible for making lecturing at the school such an enjoyable and rewarding experience. I would like to thank Steve Lloyd for keeping the show rolling with such good humour, Ann Roberts for all the behind the scenes organisation (and with particular regard to my writeup – her patience), and my fellow lecturers and tutors. Most of all, I would like to thank the students for their goodwill and enthusiasm.

7. Problems.

1. Check that $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma_\mu\} = 0$. Show that $P_L = \frac{1}{2}(1 - \gamma_5)$ and $P_R = \frac{1}{2}(1 + \gamma_5)$ are projection operators, *i.e.*

$$P_L^2 = P_L \qquad P_R^2 = P_R \qquad P_L P_R = P_R P_L = 0 \qquad P_L + P_R = 1 \quad .$$

Consider a massless fermion with $p_\mu = (E, 0, 0, E)$. Show that $P_L u(p)$ and $P_R u(p)$ are eigenstates of the helicity h with eigenvalues $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively.

$$h = \frac{1}{2}\frac{\underline{\sigma} \cdot \underline{p}}{|\underline{p}|} = -\frac{1}{2}\frac{\gamma_0 \gamma_5 \underline{\gamma} \cdot \underline{p}}{E} \quad .$$

2. Show $P_L \gamma^\mu = \gamma^\mu P_R$ and $P_R \gamma^\mu = \gamma^\mu P_L$. Show $\bar{\psi}_R \psi_R = \bar{\psi}_L \psi_L = 0$ (so these cannot be included in a Lagrangian) and hence

$$m\bar{\psi}\psi = m\left(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R\right) \quad .$$

3. The decay rate for the two-body decay $Z \to f\bar{f}$ is

$$\Gamma = \frac{1}{2M_Z}\int D|M|^2 = \frac{1}{64\pi^2 M_Z}\int d\Omega |M|^2 \quad ,$$

where D denotes the phase space measure. Recall that the $Zf\bar{f}$ vertex is

$$\frac{-ie}{2\sin\vartheta_W \cos\vartheta_W}\gamma^\mu\left(C_V^f - C_A^f \gamma_5\right) \quad .$$

First show that, summing over the fermion and averaging over the boson spins,

$$|M|^2 = \frac{1}{12}\frac{e^2}{\sin^2\vartheta_W \cos^2\vartheta_W}\left[(C_V^f)^2 + (C_A^f)^2\right](-g_{\mu\nu})\mathrm{Tr}\left(\gamma^\mu \gamma \cdot k_1 \gamma^\nu \gamma \cdot k_2\right) \quad ,$$

where k_1, k_2 are the fermion momenta and the gauge boson polarisation sum is

$$\sum_\lambda \epsilon_\mu^{(\lambda)*}\epsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2} \quad .$$

Hence show that the decay rate is

$$\Gamma = \frac{1}{48\pi} \frac{e^2}{\sin^2\vartheta_W \cos^2\vartheta_W} \left[(C_V^f)^2 + (C_A^f)^2\right] M_Z \quad .$$

4. Using the explicit forms for C_V and C_A in the standard model, derive expressions for the decay rates $Z \to \nu_e \bar{\nu}_e$, $Z \to e^+e^-$, $Z \to u\bar{u}$ and $Z \to d\bar{d}$ in terms of $\sin^2\vartheta_W$.

   What is the total width of the Z in the standard model?

   [Take $G_F = 1.2 \times 10^{-5}\,(GeV)^{-2}$, $\sin^2\vartheta_W = .23$, and $M_Z = 91\,GeV$.]

5. By carefully comparing the form of the relevant current-vector interactions, show that the decay rate of $W^+ \to e^+\nu_e$ is

$$\Gamma_W^{(e)} = \frac{1}{48\pi} \frac{e^2}{\sin^2\vartheta_W} M_W \quad .$$

   Hence *predict* the total width of the W^+ (before LEP measures it!).

   (Hints: Use the calculations of Problems 3 and 4. Use *weak eigenstates*!)

TOPICS IN STANDARD MODEL PHENOMENOLOGY

By E W N Glover
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Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1996

Topics in Standard Model Phenomenology

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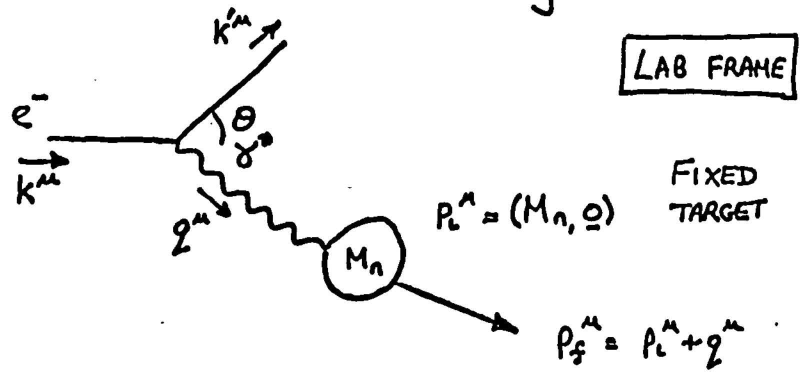
- Momentum Sum Rule and Scaling Violations

Structure of Proton

Electron-Nucleus Scattering



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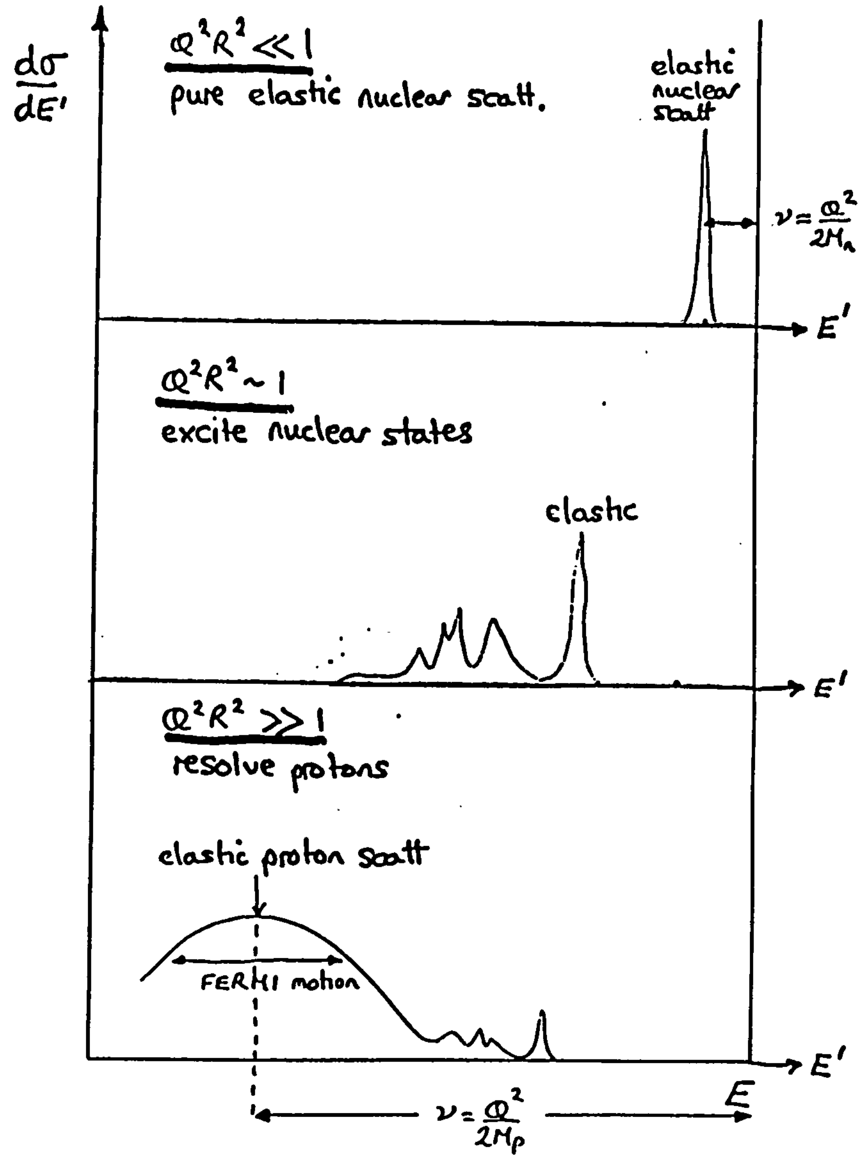
$$k^{\mu} = (E, 0, 0, E)$$
$$k'^{\mu} = (E', 0, E'\sin\theta, E'\cos\theta)$$
$$q^{\mu} = (E - E', \underline{q}) = (\nu, \underline{q})$$

For Elastic scattering $\qquad Q \sim \frac{1}{\lambda}$

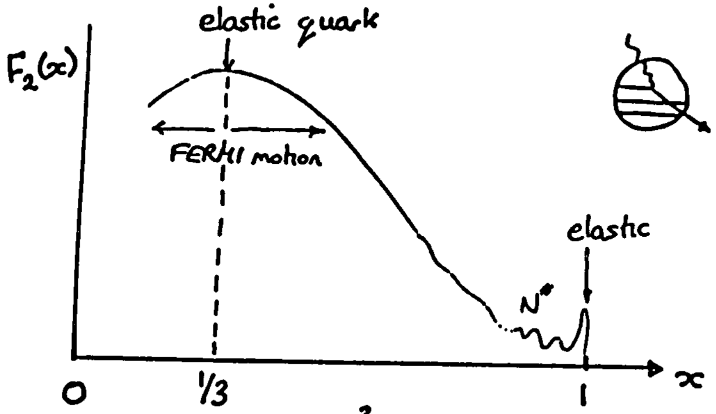
$$p_f^{\mu} p_{f\mu} = m_n^2$$
$$= (p_i + q)_{\mu} (p_i + q)^{\mu}$$
$$= m_n^2 + \underbrace{2 p_i . q}_{2 m_n \nu} + \underbrace{q^2}_{-Q^2}$$

$\Rightarrow \nu = \dfrac{Q^2}{2m_n}$

e - NUCLEUS SCATT.



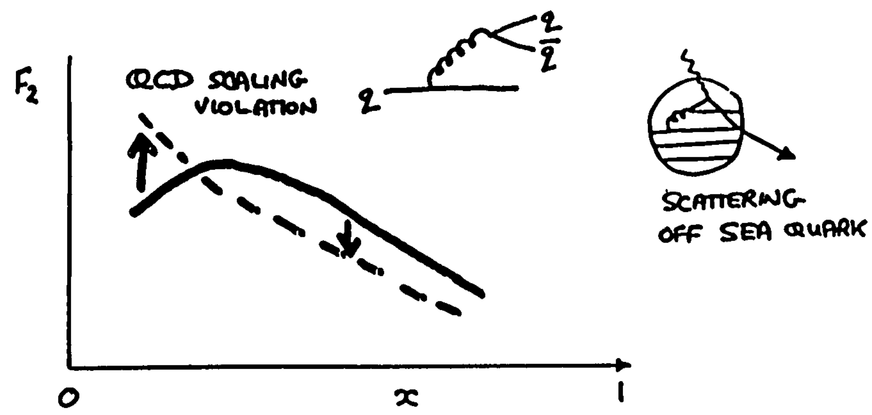
- e - PROTON SCATTERING



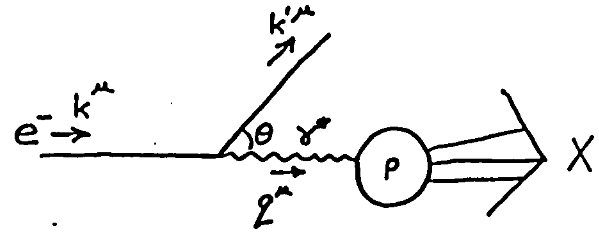
$$x = \frac{Q^2}{2M\nu}$$

- SAME SORT OF BEHAVIOUR AS e-nucleus SCATTERING

- AS Q^2 INCREASES, START RESOLVING SEA QUARKS



DEEP INELASTIC SCATTERING



$k^\mu = (E, 0, 0, E)$ $\qquad q^\mu = k^\mu - k'^\mu$

$k'^\mu = (E', 0, E'\sin\theta, E'\cos\theta)$

$p^\mu = (M, 0, 0, 0)$

Usual variables

$$Q^2 = -q_\mu q^\mu = 2EE'(1-\cos\theta) \gg M_p^2 \quad \text{DEEP}$$

$$x = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{2M(E-E')}$$

$$s = 2ME$$

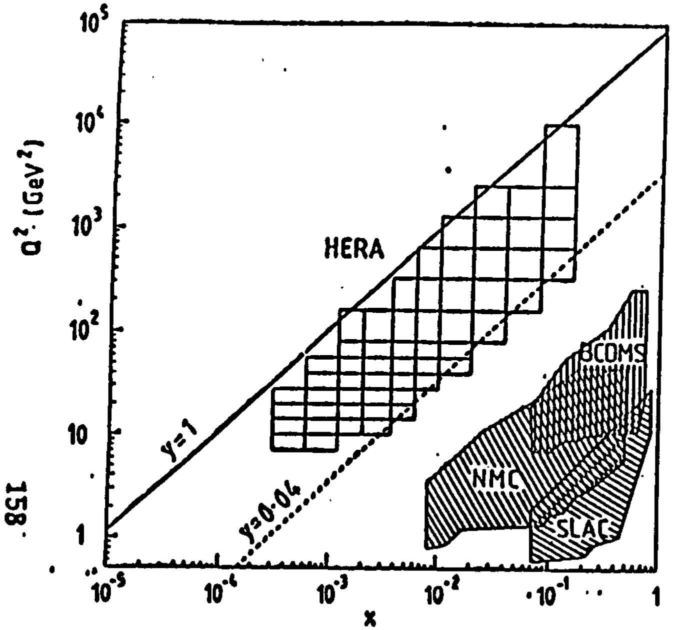
$$W^2 = (p+q)^2 = Q^2\frac{(1-x)}{x} \gg M_p^2 \quad \text{INELASTIC}$$

$$y = \frac{E-E'}{E} = \frac{Q^2}{xs}$$

- for given s, only 2 independent variables

$$E', \theta \iff x, Q^2$$

(x, Q^2) regions probed by DIS expts.



$$Q^2 = s\,x\,y$$

$$\log Q^2 = \log x + \log y + \log s$$

GENERAL FORM OF CROSS SECTION

$$d\sigma = \underbrace{\frac{1}{2s}}_{\text{FLUX}} \sum_X \underbrace{\int d\Phi}_{\text{PHASE SPACE}} \underbrace{\frac{1}{4}\sum_{\text{spins}} |M|^2_{ep\to eX}}_{\text{SPIN AVERAGED MATRIX ELEMENTS}}$$

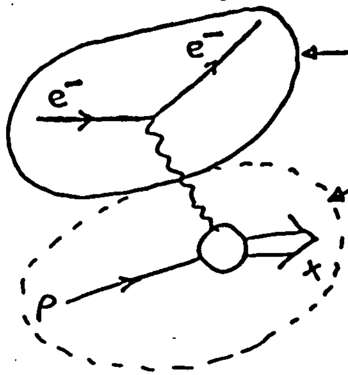
Phase space

$$\int d\Phi = \underbrace{\frac{1}{(2\pi)^3}\frac{d^3k'}{2E'}}_{\text{Scattered electron}} \; d\Phi_X$$

phase space of hadronic final state

$$\frac{d^3k'}{2E'} = \pi E' dE' d\cos\theta = \frac{\pi Q^2}{2 s x^2}\, dQ^2 dx$$

Matrix elements



Described by QED $L^{\mu\nu}$

Described by NON-PERTURBATIVE strong interactions $h_{X,\mu\nu}$ – form factors

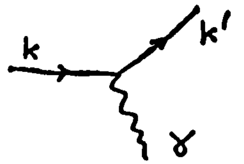
$$\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{e^4}{Q^4} L^{\mu\nu} h_{X,\mu\nu}$$

hadron tensor

photon propagator

Lepton tensor

Lepton tensor



$$H_I = e\bar{\psi} A^\mu \gamma_\mu \psi$$

$$\to \mathcal{M}_\mu = \bar{u}(k')\gamma_\mu u(k)$$

$$L_{\mu\nu} = \frac{1}{2}\sum_{spins} \mathcal{M}_\mu \mathcal{M}_\nu^+ = \frac{1}{2}\mathrm{Tr}\, \not{k}\gamma_\mu \not{k}'\gamma_\nu$$

$$= 2(k_\mu k'_\nu + k_\nu k'_\mu - k.k' g_{\mu\nu})$$

Hadron tensor

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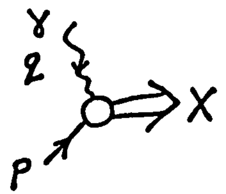

CAN'T CALCULATE!

<u>define</u> $H^{\mu\nu} = \sum_X \int d\Phi_X \, h_X^{\mu\nu}$

– can only be a function of $q^2, q\cdot p$ $(p^2 \sim 0)$

<u>OR</u> x, Q^2.

Putting pieces together

$e^2 = 4\pi\alpha$

$Q^2 = sxy$

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{1}{2s}\frac{1}{(2\pi)^3}\frac{\pi Q^2}{2sx^2}\frac{(4\pi\alpha)^2}{Q^4} L_{\mu\nu}H^{\mu\nu}$$

$$= \frac{\alpha^2 y^2}{2Q^6} L_{\mu\nu} H^{\mu\nu}$$

Most general form of $H_{\mu\nu}$ (symmetric fn of $g_{\mu\nu}, p_\mu, q_\mu$)

$$H_{\mu\nu} = -H_1 g_{\mu\nu} + H_2 \frac{p_\mu p_\nu}{Q^2} + H_4 \frac{q_\mu q_\nu}{Q^2} + H_5 \frac{(p_\mu q_\nu + q \ldots}{Q^2}$$

vanish when contracted with $L^{\mu\nu}$

$$\Rightarrow L^{\mu\nu}H_{\mu\nu} = 4k.k'\,H_1 + 4\frac{p.k\,p.k'}{Q^2}H_2$$

$$= 2Q^2 H_1 + \frac{Q^2(1-y)}{x^2y^2}H_2$$

<u>STANDARD DEFINITIONS</u>

$H_1 = 4\pi F_1 \qquad H_2 = 8\pi x F_2$

$$\boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{xQ^4}\left[y^2 x F_1 + (1-y)F_2\right]}$$

$\sqrt{s} \gg M$

F_1, F_2 contain all information about proton

– no assumption of partons made

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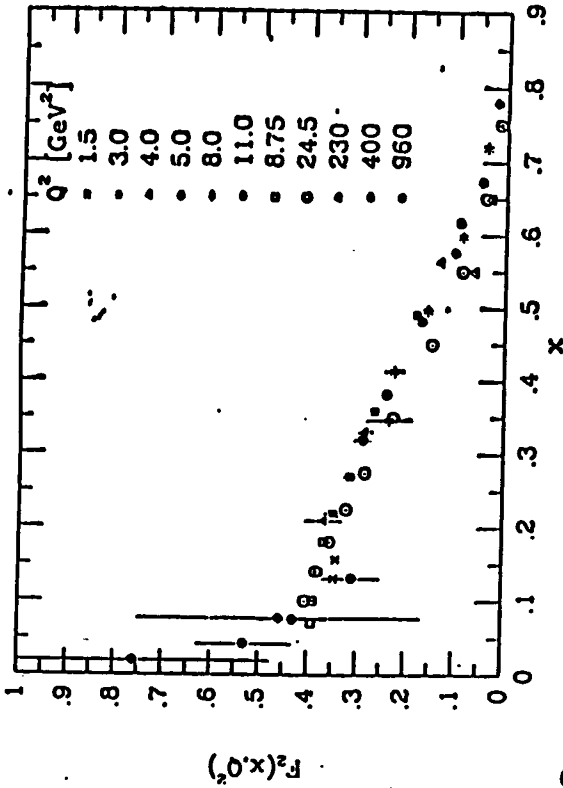


BJORKEN SCALING.

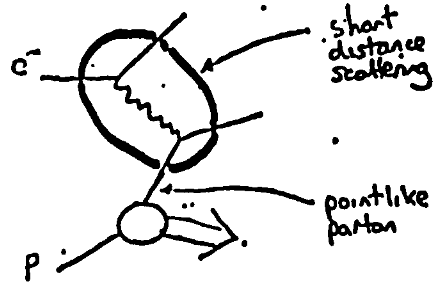
F_2 INDEP OF Q^2 $\Rightarrow$ SCATTERING OFF POINTLIKE CONSTITUENTS

— OTHERWISE VARIATION WITH Q/Q_0 WITH SIZE SCALE Q_0^{-1}

NAIVE PARTON MODEL

$\gamma^* p$ interaction at large Q^2 can be expressed as sum of incoherent scatt from point like quarks.

Over short time scale $1/\sqrt{Q^2}$ photon "sees" non-interacting quarks. Final hadronize occurs long after



$f_q(\xi)\,d\xi$ represents prob. that quark carries momentum fraction between ξ and $\xi + d\xi$

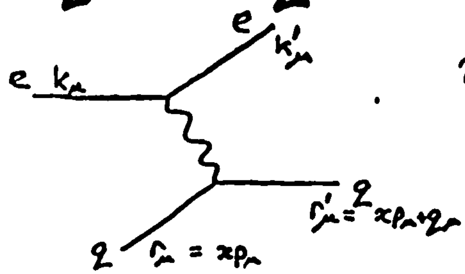
$$\Rightarrow \quad \frac{d^2\sigma}{dx\,dQ^2} = \sum_q \int_0^1 d\xi\, f_q(\xi)\, \frac{d^2\sigma}{dx\,dQ^2}(e_q + ex)$$

$$(q + \xi p)^2 = 2p.q\,\xi - Q^2 = 0$$

ie $\quad \boxed{\xi = x} \qquad\qquad Q^2 \gg m_q^2$

fractional mom = Bjorken x

$eq \longrightarrow eq$



$$\mathcal{M} = \frac{e^2 e_q}{Q^2}\, \bar{u}(k')\gamma_\mu u(k)\; \bar{u}(r')\gamma^\mu u(r)$$

$$\Rightarrow \quad \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = e^4 e_q^2 \; 2\, \frac{(k.r)^2 + (k'.r)^2}{(k.k')^2}$$

$$k.r = x\, p.K = xs/2$$

$$k'.r = k.r - q.r = xs(1-y)/2$$

$$k.k' = Q^2/2 = xsy/2$$

$$\boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{xQ^4} \sum_q f_q^p(x)\, e_q^2\, \frac{x}{2}\left[1 + (1-y)^2\right]}$$

$\Rightarrow$ COMPARING WITH GENERAL FORMULA

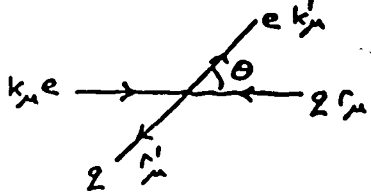
$$\boxed{\begin{aligned} F_2^{ep} &= 2x F_1^{ep} \\ &= \sum_q e_q^2\, x f_q^p(x) \end{aligned}}$$

CALLAN GROSS RELATION – SPIN 1/2

SCALING $F_2(x, \cancel{Q^2})$

INSIGHT INTO y DEPENDENCE

eq scattering in CM frame



$$k_\mu = E(1,0,0,1)$$
$$r_\mu = E(1,0,0,-1)$$
$$k'_\mu = E(1,0,\sin\theta,\cos\theta)$$

$$\Rightarrow \quad k.r = \frac{xs}{2} = 2E^2$$

$$k'.r = \frac{xs}{2}(1-y) = E^2(1+\cos\theta)$$

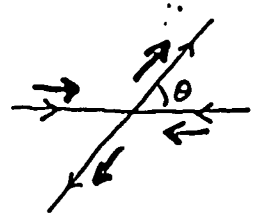
$$k.k' = \frac{xs}{2}y = E^2(1-\cos\theta)$$

$$\Rightarrow \quad y = \frac{1}{2}(1-\cos\theta)$$

$y=0$ forward scatt

$y=1$ backward scatt

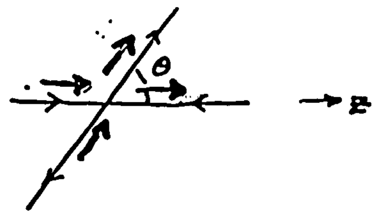
But $|\mathcal{M}|^2 \propto \left(1 + (1-y)^2\right)$



same helicity

$RR \longrightarrow RR$

$J_z^{in} = 0 = J_z^{out}$



opposite helicity

$RL \longrightarrow RL$

MUST VANISH AS $\theta \to \pi$

$J_z^{in} = 1$

Parton density functions

proton $=$ uud $+$ $q\bar{q}$ pairs
"valence" "sea"

$$f_u^p(x) = u(x) = u_V(x) + u_{sea}(x)$$
$$f_{\bar{u}}^p(x) = \bar{u}(x) = u_{sea}(x)$$

$$\Rightarrow \int_0^1 (u - \bar{u})\,dx = \int_0^1 u_V\,dx = 2$$

prob. of finding u

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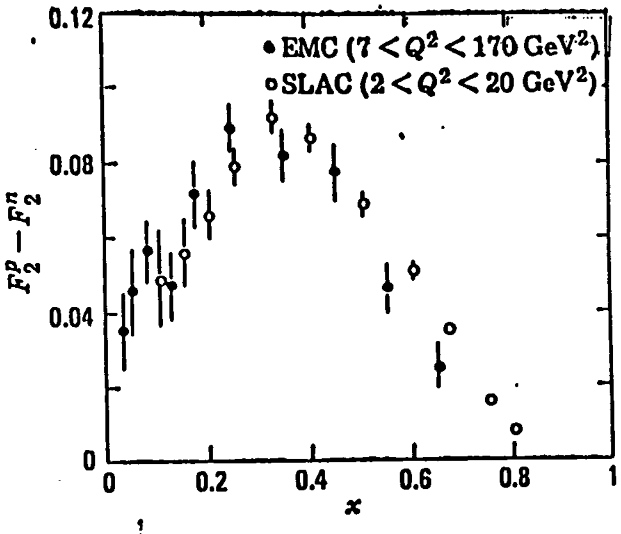
$$F_2^{ep}(x) = x\left(\frac{4}{9}u + \frac{1}{9}d + \frac{1}{9}s + \ldots + \frac{4}{9}\bar{u} + \frac{1}{9}\bar{d} + \frac{1}{9}\bar{s} + \ldots\right)$$

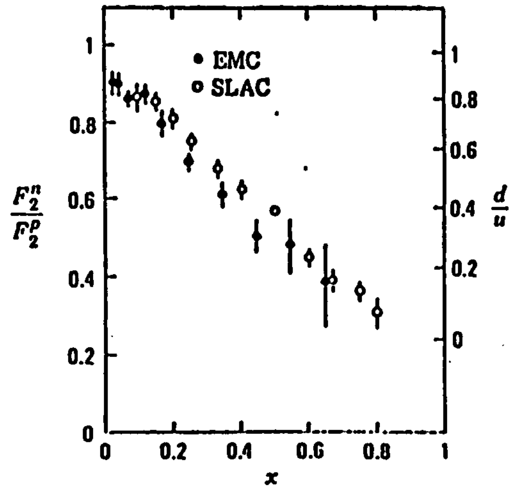
Usually write all $f_2^X(x)$ in terms of proton distributions

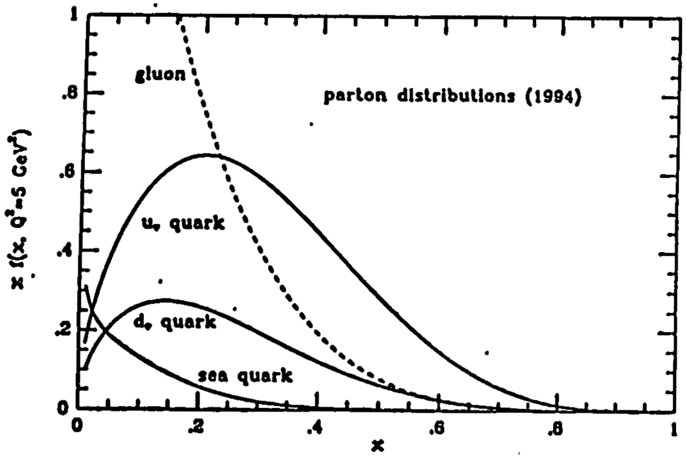
$$F_2^{en}(x) = x\left(\frac{4}{9}d + \frac{1}{9}u + \frac{1}{9}s + \ldots + \frac{4}{9}\bar{d} + \frac{1}{9}\bar{u} + \frac{1}{9}\bar{s} + \ldots\right)$$

$u \leftrightarrow d$, $\bar{u} \leftrightarrow \bar{d}$ by ISOSPIN

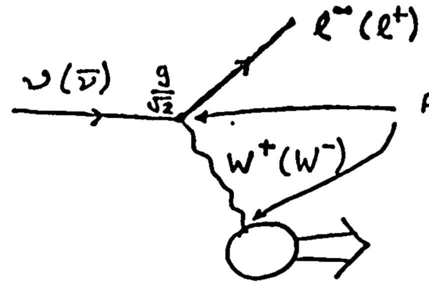
$$\Rightarrow F_2^{ep} - F_2^{en} = \frac{x}{3}\left[u + \bar{u} - d - \bar{d}\right] \approx \frac{x}{3}\left[u_V - d_V\right]$$







$$\nu N \longrightarrow \mu X$$



PARITY VIOLATION

$$\bar u_\ell(k')\,\gamma_\mu \frac{(1-\gamma_5)}{2}\, u_\nu(k)$$

FROM γ_5 IN TRACE

$$\Rightarrow \quad L^{\nu(\bar\nu)}_{\mu\nu} = L^{e}_{\mu\nu} \pm 2i\,\varepsilon_{\mu\nu\rho\sigma}\, k^\rho k'^\sigma$$

$$H^{\mu\nu} = \;\dots\dots\; - \frac{i}{Q^2}\,\varepsilon^{\mu\nu\lambda\kappa}\, p_\lambda q_\kappa\, H_3$$

$= 0$

$$\Rightarrow \text{NEW TERM IN } L^{\nu(\bar\nu)}_{\mu\nu} H^{\mu\nu} = \pm \frac{2H_3}{Q^2}\left(p.k\,q.k' - p.k'\,q.k\right)$$

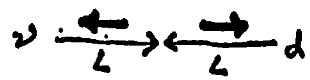
$$= \pm \frac{H_3 Q^2}{xy}\left(1 - y/2\right)$$

$$H_3 = 8\pi x F_3$$

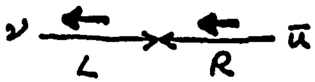
$Q^2 \ll M_W^2$

$$\boxed{\begin{aligned}\frac{d^2\sigma^{\nu(\bar\nu)}}{dx\,dQ^2} &= \frac{G_F^2}{\pi x}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 \left\{ y^2 x F_1^\nu + (1-y)F_2^\nu \pm y(1-y/2)\,xF_3^\nu \right\} \\ &= \frac{G_F^2}{\pi x}\left\{ \frac{y^2}{2}\left(2xF_1^\nu - F_2^\nu\right) + \frac{1}{2}\left(1+(1-y)^2\right)F_2^\nu \pm \frac{1}{2}\left(1-(1-y)^2\right)xF_3^\nu \right\}\end{aligned}}$$

$\nu q \to \nu q$



$$\frac{d\sigma}{dx\,dQ^2}(\nu d \to \mu u) = \frac{G_F^2}{\pi}$$



$$\frac{d\sigma}{dx\,dQ^2}(\nu \bar{u} \to \mu \bar{d}) = \frac{G_F^2}{\pi}(1-y)^2$$

so
$$\boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{G_F^2}{\pi x}\left\{ x(d+s) + x(\bar{u}+\bar{c})(1-y)^2 \right\}}$$

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$$\boxed{\begin{aligned} F_2^{\nu p} &\doteq 2x\,(d+\bar{u}+s+\bar{c}) \\ xF_3^{\nu p} &= 2x\,(d-\bar{u}+s-\bar{c}) \end{aligned}}$$

for scattering off neutrons $u \leftrightarrow d$, $\bar{u} \leftrightarrow \bar{d}$

FOR ISOSCALAR TARGET N

$$F_2^{\nu N} = \frac{1}{2}\left(F_2^{\nu p} + F_2^{\nu n}\right) = x\,(u+\bar{u}+d+\bar{d}+2s+2c)$$

$$xF_3^{\nu N} \qquad = x\,(u-\bar{u}+d-\bar{d}+2s-2c)$$

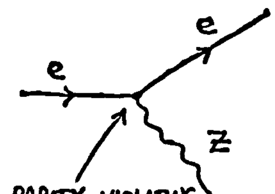
Together with F_2^{ep}, F_2^{ed} "fix"

$u+\bar{u}$, $d+\bar{d}$, $\bar{u}+\bar{d}$, s

STRUCTURE FUNCTIONS @ HERA

- NEUTRAL CURRENT

f	v_f	a_f
e	$-1+4\sin^2\theta$	-1
u	$1-8/3\sin^2\theta$	1
d	$-1+4/3\sin^2\theta$	-1



$$\frac{-ig\gamma_\mu}{\cos\theta_W}(v - a\gamma_5)$$

$$\boxed{\frac{d^2\sigma^{e^\pm}}{dx\,dQ^2} = \frac{4\pi\alpha^2}{xQ^4}\left[xy^2F_1 + (1-y)F_2 \mp xy\left(1-\frac{y}{2}\right)F_3\right]}$$

$$F_2 = 2xF_1 = \sum_q \left[xq + x\bar{q}\right] A_q$$

$$xF_3 = \sum_q \left[xq - x\bar{q}\right] B_q$$

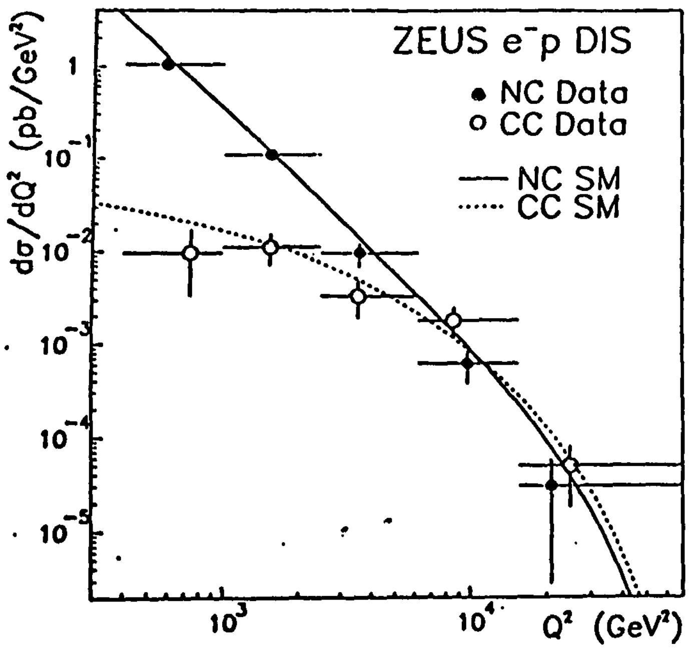
$$A_q = \underbrace{e_q^2}_{\gamma-\gamma} - \underbrace{2e_q v_e v_q \chi(Q^2)}_{\gamma-Z} + \underbrace{(v_e^2+a_e^2)(v_q^2+a_q^2)\chi^2(Q^2)}_{Z-Z}$$

$$B_q = -2e_q a_e a_q \chi(Q^2) + 4v_e a_e v_q a_q \chi^2(Q^2)$$

$$\chi(Q^2) = \left(\frac{\sqrt{2}G_F M_Z^2}{16\pi\alpha}\right)\left(\frac{Q^2}{M_Z^2+Q^2}\right) \text{ — SMALL UNLESS } Q^2 \text{ LARGE}$$

- CHARGED CURRENT (LIKE νN SCATTERING)

$$\boxed{\frac{d^2\sigma^{e^-}}{dx\,dQ^2} = \frac{G_F^2}{\pi x}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2\left\{ x(u+c) + x(\bar{d}+\bar{s})(1-y)^2 \right\}}$$



MORE NEEDED

1) <u>MOMENTUM SUM RULE</u>

$\int_0^1 x\,q(x)\,dx$ measures momentum carried by q

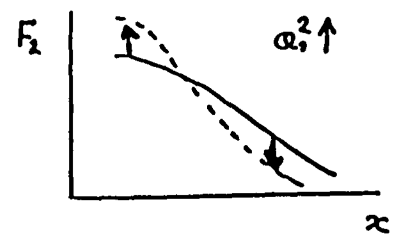
$\Rightarrow$ expect

$$\sum_{q,\bar{q}} \int_0^1 x\,q(x)\,dx = 1$$

~ 0.5

– rest of momentum carried by GLUONS

2) <u>BJORKEN SCALING VIOLATED</u>



see small "logarithmic" violations

$\dashrightarrow$ QCD

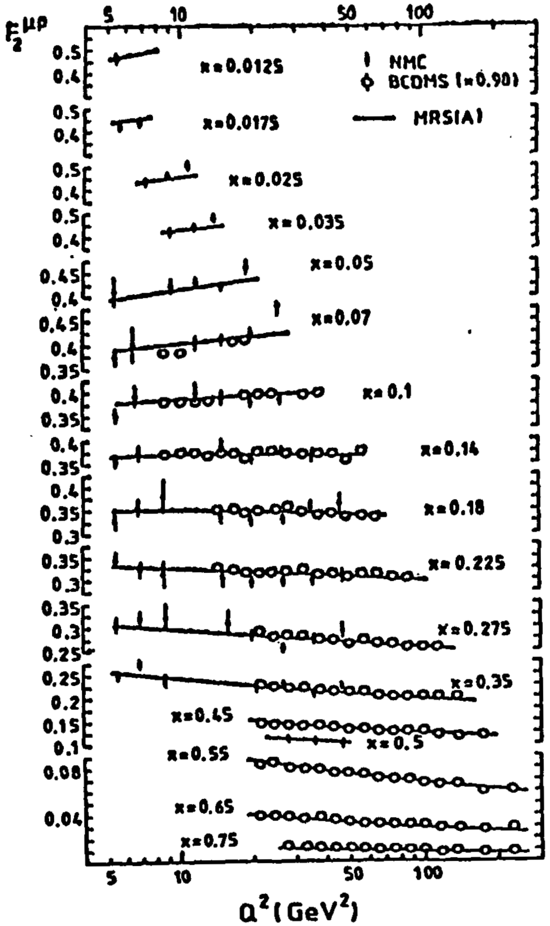


Fig. 6

2. $\sigma(e^+e^- \to \text{hadrons})$

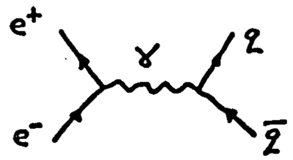
- $e^+e^- \to q\bar{q}$ and $R_{e^+e^-}$
- $R_{e^+e^-}$ at LEP/SLC
- $\mathcal{O}(\alpha_s)$ corrections to $R_{e^+e^-}$
- Real radiation: $e^+e^- \to q\bar{q}$ + gluon
  - Singularities
  - Soft and Collinear Limits
  - Radiation Patterns
- First measurement of α_s
- Running Coupling Constants
- Second measurement of α_s and renormalisation scale
- World average α_s

TOTAL HADRONIC CROSS SECTION e^+e^-

- compute $\sigma(e^+e^- \to \text{hadrons})$ from $e^+e^- \to$ quarks, gluons

  since Probability (quarks, gluons $\to$ hadrons) $= 1$

- At LOWEST ORDER $\quad e^+(p_1) + e^-(p_2) \to q(q_1) + \bar{q}(q_2)$



(for Z exchange see later)

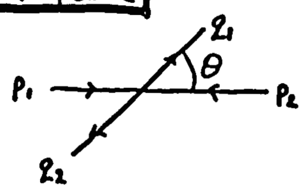
ASSUME $M_q \ll \sqrt{s}$

$$\mathcal{M} = e^2 e_q \, \delta_{ij} \, \bar{v}(p_1)\gamma_\mu u(p_2) \, \frac{g^{\mu\nu}}{(p_1+p_2)^2} \, \bar{u}_i(q_1)\gamma_\nu v_j(q_2)$$

δ_{ij}: COLOUR OF $q, \bar{q}$ SAME

$$\to \frac{1}{4}\sum_{\substack{\text{spins}\\ \text{colours}}} |\mathcal{M}|^2 = e^4 e_q^2 N \; 2 \; \frac{(p_1\cdot q_1)^2 + (p_1\cdot q_2)^2}{(p_1\cdot p_2)^2}$$

$\boxed{\text{CM frame}}$

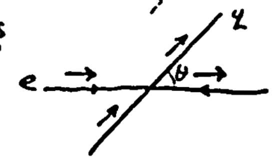


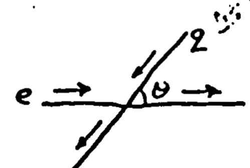
$$p_1^\mu = \frac{\sqrt{s}}{2}(1,0,0,1)$$
$$p_2^\mu = \frac{\sqrt{s}}{2}(1,0,0,-1)$$
$$q_1^\mu = \frac{\sqrt{s}}{2}(1,0,\sin\theta,\cos\theta)$$
$$q_2^\mu = \frac{\sqrt{s}}{2}(1,0,-\sin\theta,-\cos\theta)$$

$$\Rightarrow \quad p_1\cdot q_1 = \frac{s}{4}(1-\cos\theta) \qquad p_1\cdot p_2 = \frac{s}{2} = q_1\cdot q_2$$
$$p_1\cdot q_2 = \frac{s}{4}(1+\cos\theta)$$

$$\Rightarrow \quad \frac{1}{4}\sum |\mathcal{M}|^2 = \frac{e^4}{2} e_q^2 N\left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right]$$

Helicities





$J_z = +1$ — VANISHES $\theta \to \pi$, $\sim (1+\cos\theta)^2$ — VANISHES $\theta \to 0$, $\sim (1-\cos\theta)^2$

QED doesn't distinguish LH and RH

$$\frac{1}{4}\sum |\mathcal{M}|^2 = (\underbrace{4\pi\alpha}_{=e^2})^2 e_q^2 \cdot N\,(1+\cos^2\theta)$$

- Phase Space

$$d\Phi = \frac{d^3q_1}{(2\pi)^3 2E_1}\frac{d^3q_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^4(p_1+p_2-q_1-q_2)$$
$$= \frac{1}{16\pi^2}\frac{E_1^2 dE_1}{E_1E_2}\, d\cos\theta \, d\phi \;\; \delta(\sqrt{s}-E_1-E_2)$$
$$= \frac{1}{8\pi} dE_1 \; d\cos\theta \; \frac{1}{2}\delta(\sqrt{s}/2 - E_1)$$

$$d\Phi = \frac{d\cos\theta}{16\pi}$$

$$\int d\sigma = \frac{1}{2s}\int \frac{1}{4}\sum |M|^2 \, d\Phi \qquad \int_{-1}^{1} = \frac{8}{3}$$

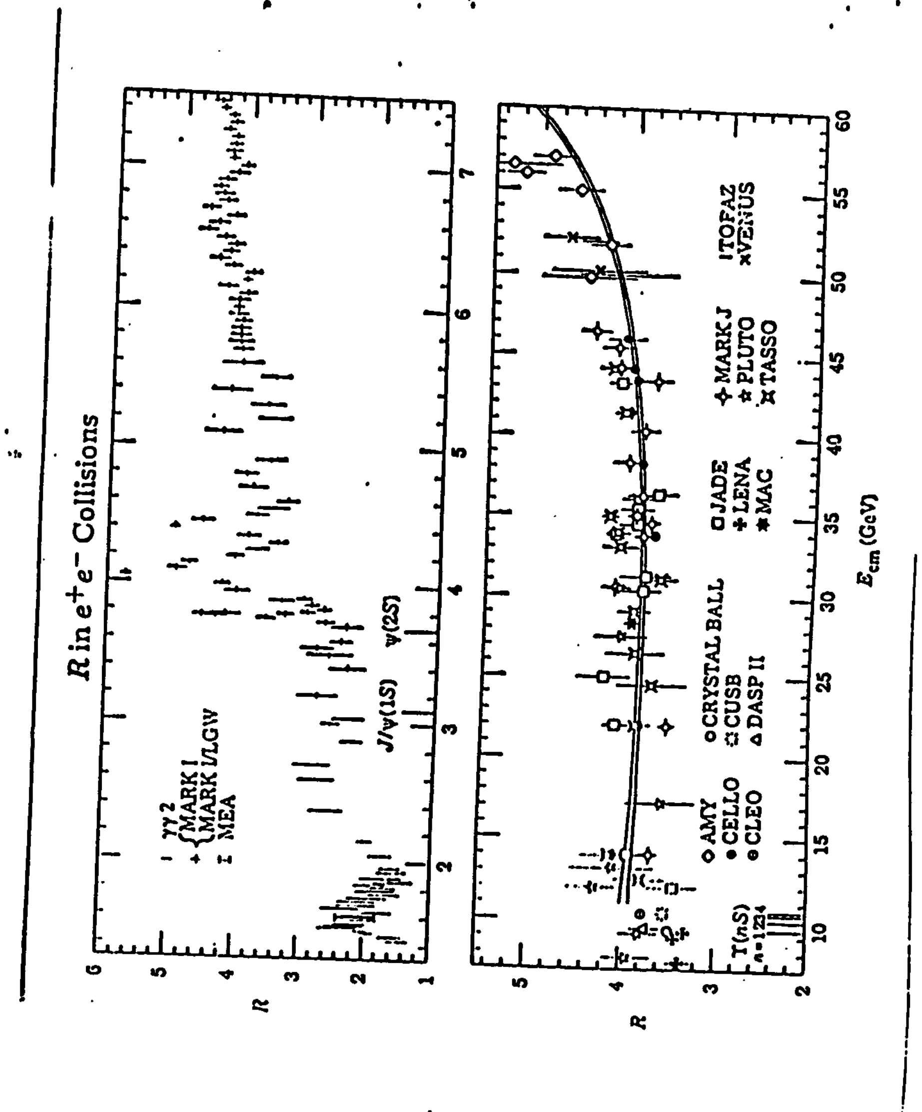
$$= \frac{1}{2s} \; 16\pi^2\alpha^2 e_q^2 N \; \overbrace{(1+\cos^2\theta)\,\frac{d\cos\theta}{16\pi}}$$

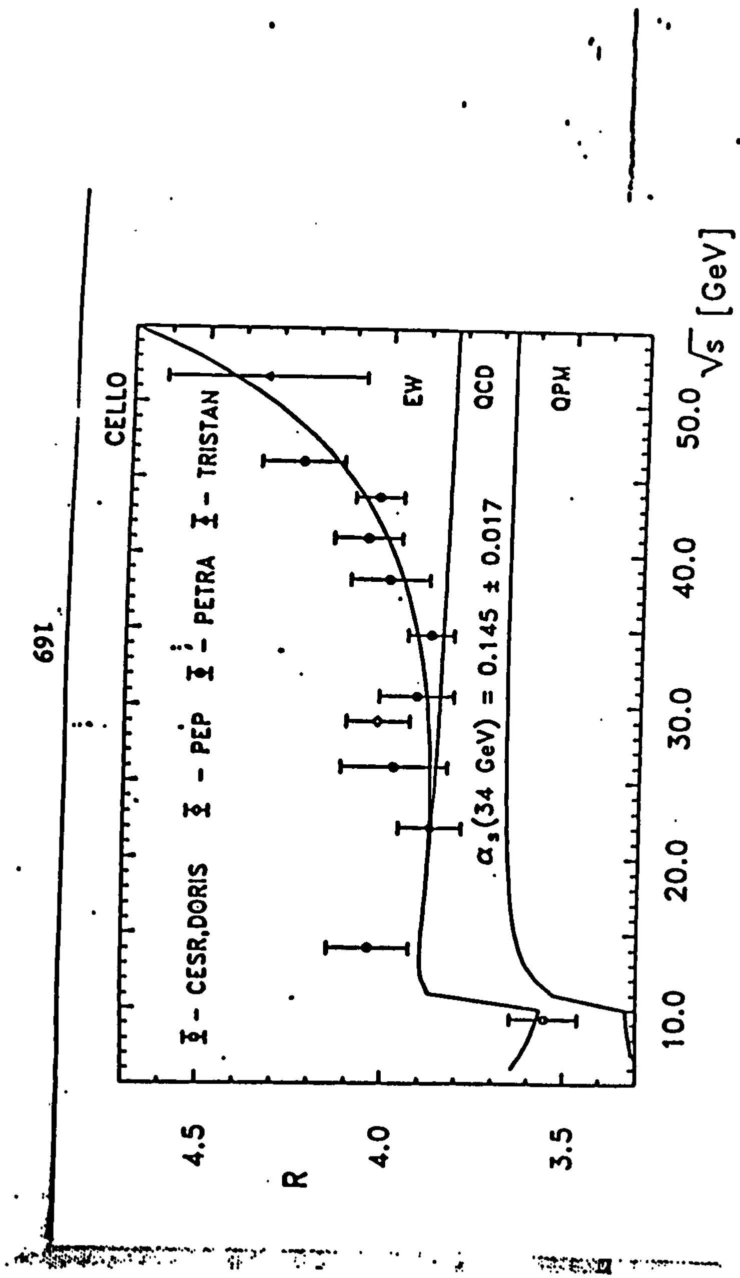
$$\sigma(e^+e^- \to q\bar{q}) = \underbrace{\frac{4\pi\alpha^2}{3s}}_{= \sigma(e^+e^- \to \mu^+\mu^-)} \; e_q^2 \, N$$

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx \underbrace{\sum_q e_q^2 N}_{= \frac{11}{3}}$$

e.g. at $\sqrt{s} = 34$ GeV, $q = u, d, s, c, b$

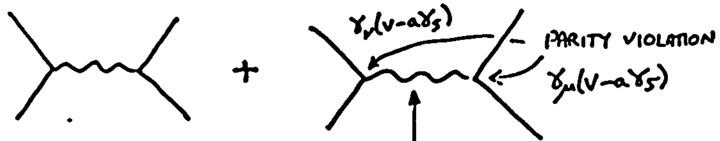
$$R^{EXP} = 3.88 \pm 0.06$$





$R^{e^+e^-}$ at LEP/SLC.

- include effect of Z exchange



resonating propagator
$\sim g^{\mu\nu}/(s-M_Z^2+i\Gamma_Z M_Z)$

$$\Rightarrow \quad \frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} N \left[(1+\cos^2\theta)\, A_q(s) + \underbrace{2\cos\theta\, B_q(s)}_{\text{integrates to } 0} \right]$$

$$A_q(s) = \underbrace{e_q^2}_{\gamma-\gamma} - 2 e_q \underbrace{v_q v_e \chi_1(s)}_{\gamma-Z} + \underbrace{(v_e^2+a_e^2)(v_q^2+a_q^2)\chi_2(s)}_{Z-Z}$$

$$B_q(s) = -2 e_q a_q a_e \chi_1(s) + 4 v_e a_e v_q a_q \chi_2(s)$$

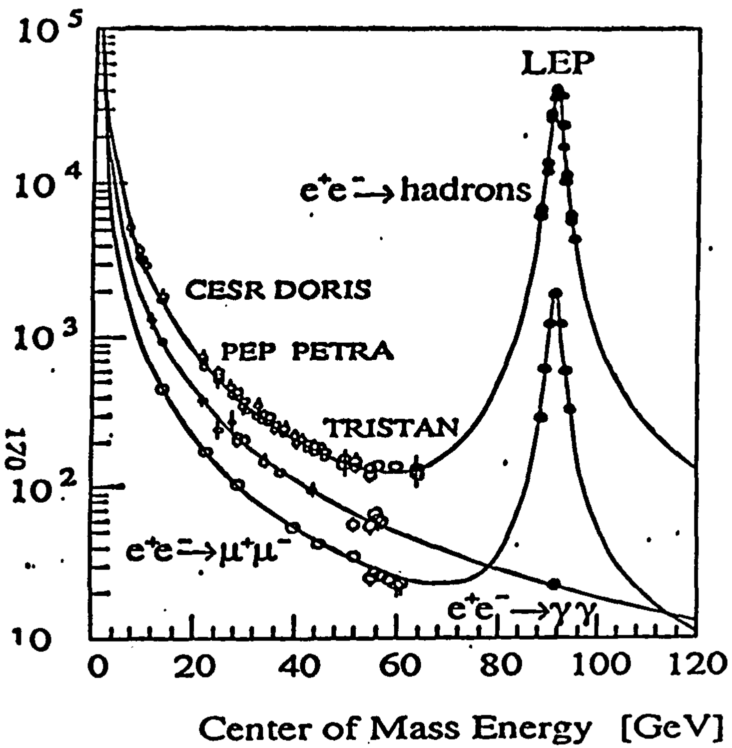
$$\chi_1(s) = \frac{\sqrt{2} G_F M_Z^2}{16\pi\alpha} \frac{s(s-M_Z^2)}{(s-M_Z^2)^2+\Gamma_Z^2 M_Z^2}$$

$$\chi_2(s) = \left(\frac{\sqrt{2} G_F M_Z^2}{16\pi\alpha}\right)^2 \frac{s^2}{(s-M_Z^2)^2+\Gamma_Z^2 M_Z^2}$$

$$\Rightarrow \quad R_0 = N \sum e_q^2 \longrightarrow \boxed{N \frac{\sum A_q(s)}{A_\mu(s)} = R_0(s)}$$

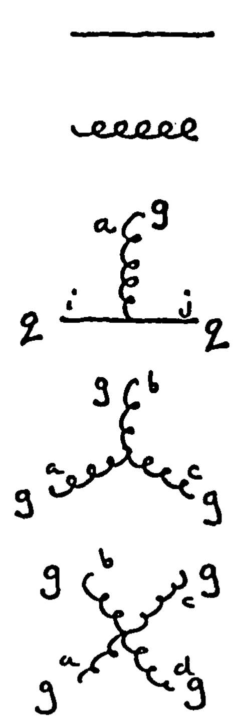
at $\sqrt{s} = 34$ GeV $\qquad R_0 \rightarrow \frac{11}{3} + 0.05 = 3.716$

cf $R^{DATA} = 3.88 \pm 0.05$



igure 1: Total cross section for e^+e^- annihilation into hadrons and ıuon pairs as function of the centre-of-mass energy. Also given is ıe two-photon cross section. Experimental data are compared with redictions from the standard model.

QCD INTERACTIONS



quark propagator $\overline{\psi}\psi$

gluon propagator $G_\mu G_\nu$

$q\bar{q}g$ interaction

$\propto g_s T^a_{ij}$

ggg interaction

$\propto g_s f_{abc}$

$gggg$ interaction

$\propto g_s^2 f_{abe} f_{cde}$

i 1 ... N

a 1 ... N^2-1

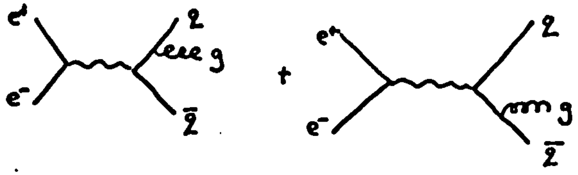
$[T^a, T^b] = i f^{abc} T^c$

$T^a_{ij} T^b_{ji} = \frac{\delta^{ab}}{2}$

$O(\alpha_s)$ QCD CORRECTIONS

- REAL GLUON EMISSION

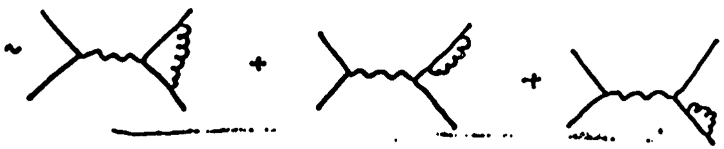
$\mathcal{M}_{gq\bar q} \sim$



$$\mathcal{M} \propto g_s \Rightarrow |\mathcal{M}|^2 \propto \alpha_s$$

- VIRTUAL GLUON EMISSION

$\mathcal{M} \sim$

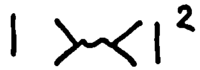


$\sim g_s^2$

$\mathcal{M}_{q\bar q} \sim$

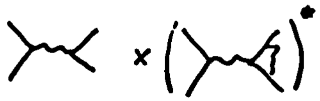
$+$ ...

$|\mathcal{M}_{q\bar q}|^2 \sim$



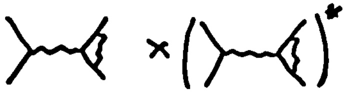
$O(1)$

$+$



$O(\alpha_s)$

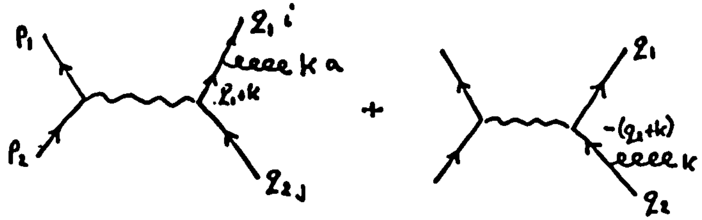
$+$



$O(\alpha_s^2)$

- ONLY INTERESTED IN INTERFERENCE OF TREE AND ONE LOOP GRAPHS AT $O(\alpha_s)$

1) $e^+e^- \to q\bar q g$



$$\mathcal{M} = e^2 e_q g_s T^a_{ij} \quad \bar v(p_1)\gamma_\mu u(p_2)\,\frac{g^{\mu\nu}}{(p_1+p_2)^2}$$

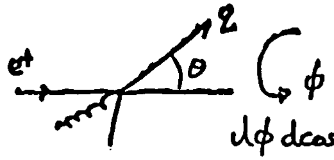
$$\bar u_i(q_1)\left[\frac{\gamma^\alpha(\not q_1+\not k)\gamma_\nu}{(q_1+k)^2} - \frac{\gamma_\nu(\not q_2+\not k)\gamma^\alpha}{(q_2+k)^2}\right] v_j(q_2)\; \underbrace{\varepsilon^a_\alpha(k)}_{\text{GLUON POLARISATION VECTOR}}$$

$$\Rightarrow \boxed{\frac{1}{4}\sum_{\substack{\text{spins}\\ \text{colours}}} |\mathcal{M}|^2 = 2e^4 e_q^2 \left(\frac{N^2-1}{2}\right) g_s^2 \cdot \frac{\left[(q_1\cdot p_1)^2+(q_1\cdot p_2)^2+(q_2\cdot p_1)^2+(q_2\cdot p_2)^2\right]}{q_1\cdot k\;\; k\cdot q_2\;\; p_1\cdot p_2}}$$

<u>Note</u>

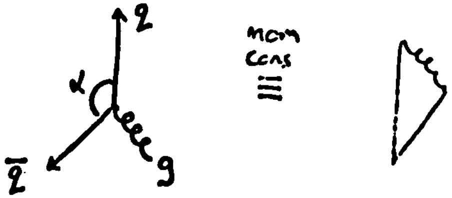
$$\sum_{\substack{a=1,8\\ i,j=1,3}} T^a_{ij}(T^a_{ij})^* = \sum T^a_{ij}T^a_{ji} = \mathrm{Tr}\, T^a T^a = \frac{N^2-1}{2}$$

Phase Space



$d\phi \, d\cos\theta$

- Because of momentum conservation $q\bar{q}g$ lie in plane



$dx_1 dx_2 d\alpha$

FRACTIONAL ENERGIES OF QUARKS

$$x_1 = \frac{2E_q}{\sqrt{s}} \quad x_2 = \frac{2E_{\bar{q}}}{\sqrt{s}} \quad x_3 = \frac{2E_g}{\sqrt{s}} \qquad x_1+x_2+x_3 = 2$$

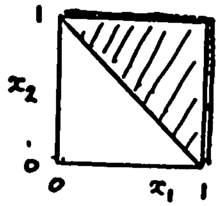
$$d\Phi = \frac{s}{2^{10}\pi^5} \, d\phi \, d\cos\theta \, d\alpha \, dx_1 dx_2 \qquad \begin{array}{l} k.q_1 = \frac{s}{2}(1-x_2) \\ k.q_2 = \frac{s}{2}(1-x_1) \end{array}$$

$\Rightarrow$ integrating out Euler angles and summing over quark flavours

$$d\sigma_{q\bar{q}g} = \underbrace{\frac{4\pi\alpha^2}{3s} N \sum e_q^2}_{\sigma^0_{q\bar{q}}} \cdot \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right) \int dx_1 dx_2 \, \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

integration region

DALITZ



see that integrand diverges as $x_1, x_2 \to 1$

INFRARED SINGULARITIES

- divergences occur when the particles are unresolvable

e.g. COLLINEAR $\theta_{qg} \to 0$ or $\theta_{\bar{q}g} \to 0$ ($x_1 \to 1$ OR $x_2 \to 1$)

SOFT $E_g \to 0$ ($x_1 \to 1$ AND $x_2 \to 1$)

- both limits, look like tree level.

Collinear

Soft

- IMPORTANT BECAUSE
  1) OCCUR IN ALL PROCESSES
  2) GENERATE LARGE LOGARITHMS

COLLINEAR LIMIT of $\frac{1}{4}\sum|M|^2$

q_1 parallel to k $\qquad \theta_{qg}=0$

LET $\quad q_1 = (1-z)Q_1 \qquad Q_1$ is momentum

$\quad k = z\,Q_1 \qquad$ of pair $Q_1^2=0$

$\Rightarrow \quad q_1\cdot p_1 \rightarrow (1-z)\,Q_1\cdot p_1 \qquad q_2\cdot p_1 \rightarrow Q_1\cdot p_2$

$\quad q_1\cdot p_2 \rightarrow (1-z)\,Q_1\cdot p_2 \qquad q_2\cdot p_2 \rightarrow Q_1\cdot p_1$

$\quad k.q_2 \rightarrow z\,Q_1\cdot q_2 = z\,p_1\cdot p_2$

173 $\quad [\ldots] \longrightarrow 2\,[1+(1-z)^2]\left((Q_1\cdot p_1)^2+(Q_1\cdot p_2)^2\right)$

$$\frac{1}{4}\sum|M_{q\bar{q}g}|^2 \longrightarrow \frac{1}{4}\sum|M_{q\bar{q}}|^2$$

$$\times.\ \frac{g_s^2}{2k.q_1}\left(\frac{N^2-1}{2N}\right)2\left(\frac{1+(1-z)^2}{z}\right)$$

divergence

ALTARELLI-PARISI
splitting function

SOFT LIMIT $\quad k \rightarrow 0$

$q_1\cdot p_1 \rightarrow q_2\cdot p_2 \qquad q_2\cdot p_1 \rightarrow q_1\cdot p_2$

$[\ldots] \longrightarrow 2\,[(q_1\cdot p_1)^2+(q_1\cdot p_2)^2]$

so $\quad \frac{1}{4}\sum|M_{q\bar{q}g}|^2 \longrightarrow \frac{1}{4}\sum|M_{q\bar{q}}|^2$

$$\times.\ g_s^2\left(\frac{N^2-1}{2N}\right)\frac{2\,q_1\cdot q_2}{q_1.k\ k.q_2}$$

EIKONAL FACTOR

– DICTATES FLOW OF SOFT PARTICLES

e.g ANTENNA RADIATION PATTERNS

$$P \sim \frac{1-\cos\theta_{q\bar{q}}}{(1-\cos\theta_{qg})(1-\cos\theta_{\bar{q}g})\,E_g^2}$$

for fixed E_g, can ask in which direction is it more likely to radiate a soft gluon.

the q and $\bar{q}$ represent any colour connected pair!

$\Theta_{q\bar{q}} = \pi$

$\Theta_{qg} = \pi - \Theta_{\bar{q}g}$

- more soft particles (P bigger) radiated in q and $\bar{q}$ directions (INSIDE JET)
- JUST LIKE QED – photon radiation from charged line
= QCD radiation from coloured line

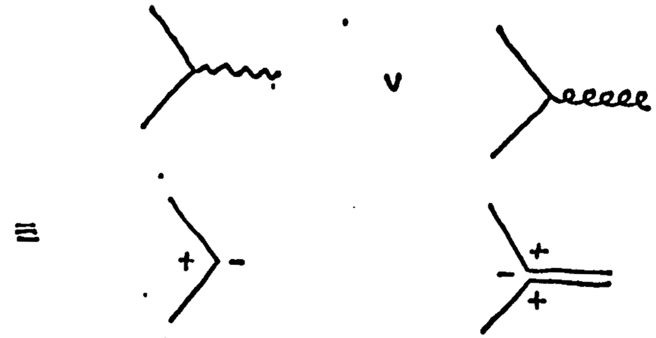
ess
idiation

more radiation bent dipole

- CAN get idea of where particle flux is likely to be small by drawing all colour connected pairs (strings)

As quark-antiquark separate string breaks and hadrons form.

RADIATION IN $q\bar{q}\gamma$ v $q\bar{q}g$



single colour antena double colour antena

- if soft gluons related to hadronic particle directions

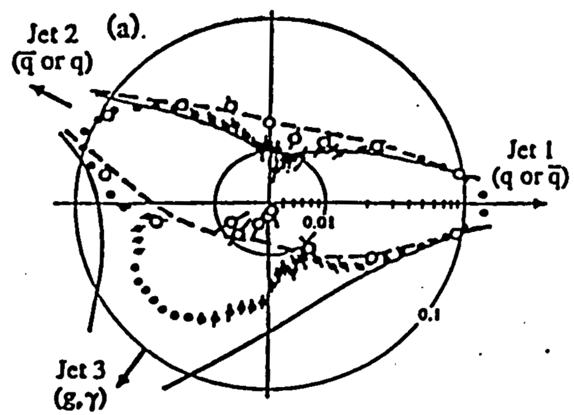
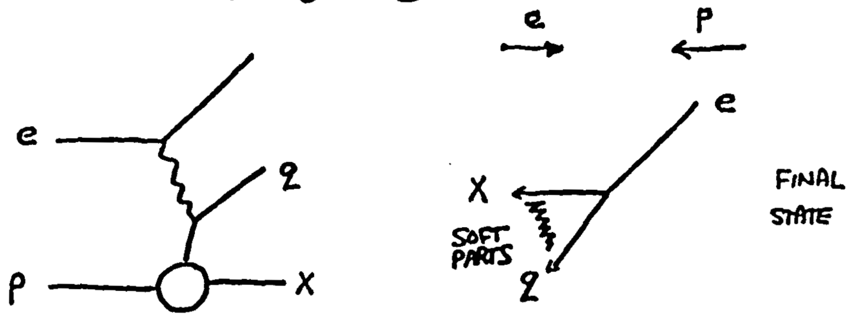
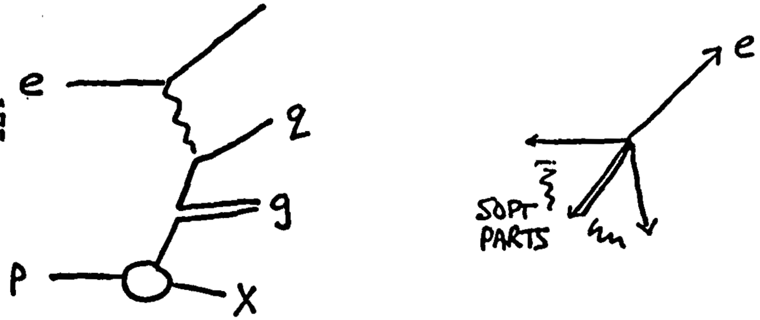


Figure 42: Particle flow on a logarithmic scale as a function of angle in the pl the event for $q\bar{q}\gamma$, (open points) and $q\bar{q}g$, (closed points).

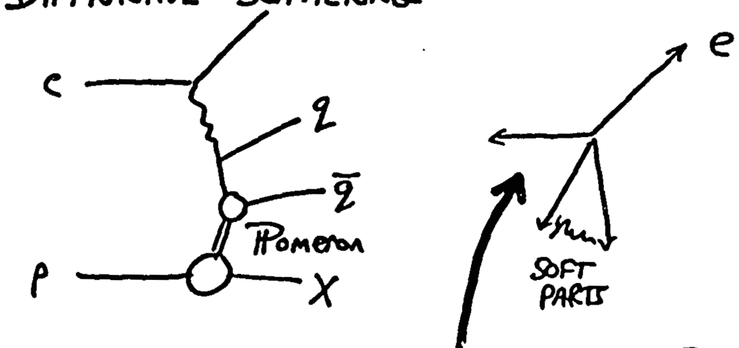
RAPIDITY GAPS



e.g 2 JET PRODUCTION IN DIS



DIFFRACTIVE SCATTERING



RAPIDITY GAP

3. Structure of hadronic events

- Event shapes
- Thrust Distribution
  - Spin of the Gluon
  - $\mathcal{O}(\alpha_s^2)$ Corrections
  - Resumming large logs
- Jet algorithms
- Measuring QCD group parameters
- Jet production, Local Parton Hadron Duality and Hadronisation
- A simple Hadronisation model
- Hadronisation and Thrust

$\sigma_{q\bar{q}g}$ CONTINUED

- TO DO INTEGRAL OVER x_1, x_2 NEED SOME REGULARISATION PROCEDURE
  - DIMENSIONAL REGULARISATION work in $n = 4-2\varepsilon$ dimensions and divergences appear as $1/\varepsilon^2, 1/\varepsilon$
  - BOTH $|M|^2$ and $d\Phi$ change. Get integrals like

$$\int_0^1 \frac{dx_1}{(1-x_1)} (1-x_1)^{-\varepsilon} \longrightarrow -\frac{1}{\varepsilon} \qquad C^{-\varepsilon} \equiv C$$

$$\Rightarrow \quad \sigma_{q\bar{q}g} = \sigma^0_{q\bar{q}} \frac{\alpha_s}{2\pi} \left(\frac{N^2-1}{2N}\right) H(\varepsilon) \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} + O(\varepsilon)\right]$$

$H(\varepsilon) = 1 + C(\varepsilon)$

virtual corrections

$$\sigma_{q\bar{q}(g)} = \sigma^0_{q\bar{q}} \frac{\alpha_s}{2\pi} \left(\frac{N^2-1}{2N}\right) H(\varepsilon) \left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + O(\varepsilon)\right]$$

- WHEN ADDED TOGETHER SINGULARITIES IN ε CANCEL AND TOTAL CORRECTION TO CROSS SECTION IS FINITE AS $\varepsilon \to 0$ (4-D)
- NOT ACCIDENT :- POWERFUL THEOREMS (Bloch, Nordsieck, Kinoshita, Lee, Nauenberg) GUARANTEE THIS FOR SUITABLY DEFINED INCLUSIVE CROSS SECTIONS

e.g. $\boxed{\sigma_{TOT} = \sigma^0_{q\bar{q}} \left(1 + \frac{\alpha_s}{\pi}\right)}$ OK $\qquad \sigma_{q\bar{q}}$ X

1st MEASUREMENT OF α_s

$$R = \frac{\sigma_{TOT}}{\sigma_{\mu^+\mu^-}} = \underbrace{R_0(s)}_{3.72} \left(1 + \frac{\alpha_s}{\pi}\right) = 3.88 \pm 0.06$$

$\sqrt{s} = 34$

$$\Rightarrow \quad \boxed{\alpha_s = 0.135 \pm 0.05}$$

BUT α_s DEPENDS ON SCALE!.

RUNNING COUPLING CONSTANTS

- PROBLEM IS FEYNMAN GRAPHS LIKE

CONTAIN ULTRAVIOLET DIVERGENCES WHEN LOOP MOMENTA ARE LARGE

- UNLIKE INFRARED DIVERGENCES, THESE ARE NOT CANCELLED BY REAL RADIATION
- SOLUTION IS TO RENORMALIZE COUPLINGS AT A RENORMALISATION SCALE

ie $$g_s^2 \longrightarrow g_s^2(\mu) - \infty(\mu)$$

g_s^2: BARE COUPLING IN $\mathcal{L}$

$g_s^2(\mu)$: FINITE COUPLING AT SCALE μ

$\Rightarrow$ PHYSICAL QUANTITIES COMPUTED IN TERMS OF $g_s^2(\mu)$ ARE FINITE.

- BECAUSE μ DOES <u>NOT</u> APPEAR IN $\mathcal{L}$, PHYSICAL QUANTITIES COMPUTED TO ALL ORDERS IN PERTURBATION THEORY ARE INDEPENDENT OF μ ie $\dfrac{\partial\sigma}{\partial\mu} = 0$

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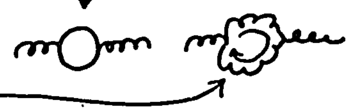
- HOWEVER, CROSS SECTIONS COMPUTED ONLY TO A FINITE ORDER IN PERTURBATION THEORY ARE NOT INDEPENDENT OF μ.

- COUPLINGS AT DIFFERENT SCALES ARE RELATED BY PERTURBATION THEORY

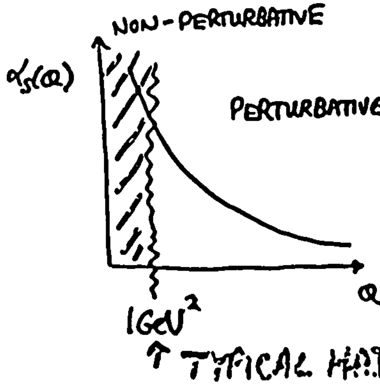
$$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = \beta(\alpha_s) = -b_0 \alpha_s^2 + \ldots\ldots$$



- IN QCD $\quad b_0 = \dfrac{11N - 2n_f}{12\pi}$

  QED $\quad b_0 = -\dfrac{1}{3\pi}$

$\Rightarrow$ RUNNING DIFFERENT (α_{QED} INCREASES WITH Q^2)



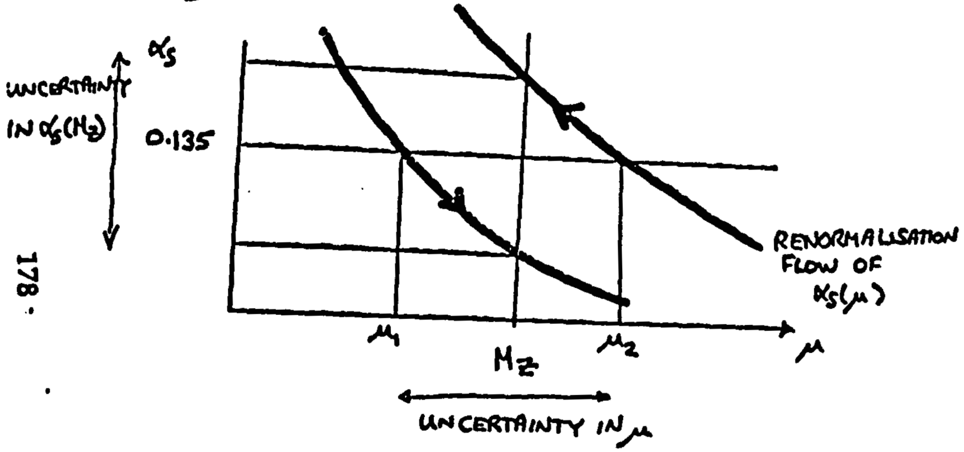
Λ IS SCALE WHERE α_s BECOMES ∞

CONFINEMENT

RETURNING TO MEASUREMENT OF α_s

$$\alpha_s(\mu) = 0.135 \pm 0.05$$

- RELATE TO $\alpha_s(M_Z)$.
  DEPENDS ENTIRELY ON CHOICE FOR μ

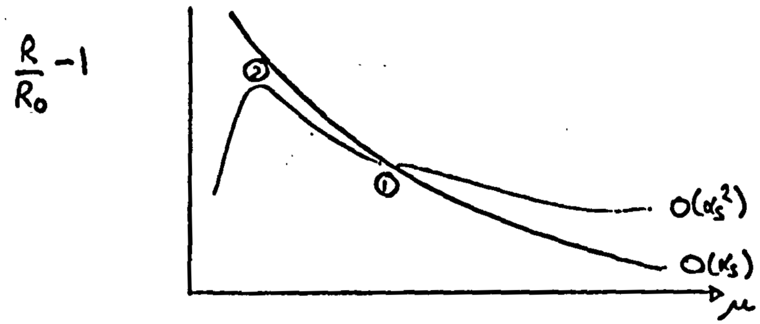


$\Rightarrow$ DEPENDING ON CHOICE OF μ GET ANY VALUE FOR $\alpha_s(M_Z)$

- SOME CHOICES "MORE SENSIBLE" THAN OTHERS, BUT STILL UNCERTAIN
- PROBLEM REDUCED AT HIGHER ORDER IN α_s

eg
$$\frac{R}{R_0} = 1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[1.41 + b_0 \log \frac{\mu^2}{s}\right].$$

as $\mu \uparrow$ $\quad\downarrow\quad\downarrow\quad\uparrow$



TYPICAL CHOICES

1. FAC ; fastest apparent convergence
2. PMS ; principle of minimal sensitivity
3. $\mu \sim \sqrt{s}$ physical scale

– then vary by factor either way.

- VARYING μ IS ATTEMPT TO GUESS HOW BIG HIGHER ORDER CORRECTIONS ARE

α_s MEASURED AT $\sqrt{s} \sim M_Z$

$$R_0(M_Z^2) = N \frac{A_q(M_Z^2)}{A_\mu(M_Z^2)} \approx N \frac{\sum_q (v_q^2 + a_q^2)}{v_\mu^2 + a_\mu^2}$$

$$= 19.937 \quad \text{for } m_t = 174, \; m_H = 300$$

$$R^{LEP} = 20.788_{7} \pm 0.032_{29} \qquad \text{BRUSSELS } 95$$

SOLVE (LET $\mu = M_Z$)

$$R^{LEP} = R_0(M_Z^2)\left(1 + 1.06\frac{\alpha_s}{\pi} + 0.85\left(\frac{\alpha_s}{\pi}\right)^2 - 15\frac{\alpha_s^3}{\pi^3}\right)$$

INCLUDING b, t MASSES ETC, ETC

$\Rightarrow$ at $O(\alpha_s)$ $\quad \alpha_s|^1 = 0.126$

$O(\alpha_s^2)$ $\quad \alpha_s|^2 = 0.123$

$O(\alpha_s^3)$ $\quad \alpha_s|^3 = \underline{\underline{0.125}}$

DECREASING ERROR FROM μ DEP

$\Delta\alpha_s^{\mu} = 0.002$

$M_Z/4 < \mu < M_Z$

$\Delta\alpha_s^{EXP} = 0.005$

$\Delta\alpha_s^{EW} = 0.002$ $\quad Z \to b\bar{b}$

$\Delta\alpha_s^{H} = 0.002$ $\quad 60 < M_H < 1000$

$\Delta\alpha_s^{t} = 0.001$ $\quad 100 < M_t < 200$

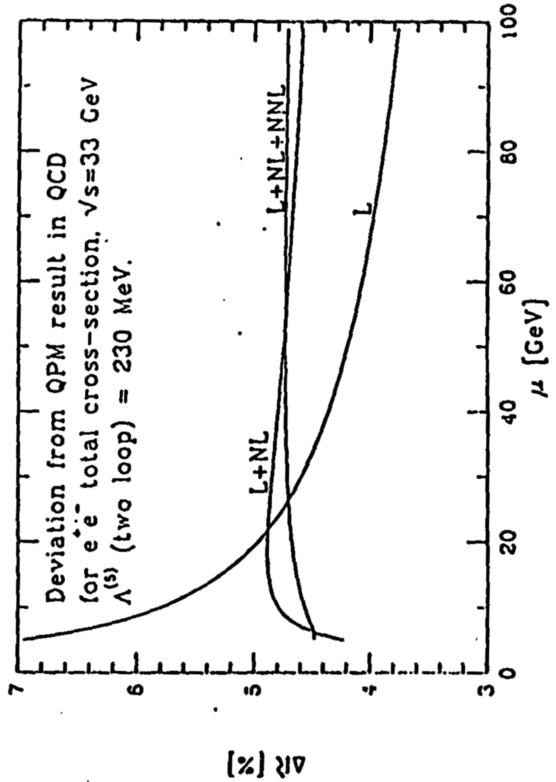


Figure 10: The effect of higher order QCD corrections to R, as a function of the renormalization scale μ.

SLAC/BCDMS 0·113 ± 0·004

CCFR 0·111 ± 0·005 → 0·117 Bethke, Moriond 95

Process	Ref.	Q [GeV]	$\alpha_s(Q)$	$\alpha_s(M_{Z^0})$	$\Delta\alpha_s(M_{Z^0})$ exp.	$\Delta\alpha_s(M_{Z^0})$ theor.	Theory
GLS (CCFR)	[15]	1.73	0.24 ± 0.047	$0.107^{+0.007}_{-0.009}$	$^{+0.006}_{-0.007}$	$^{+0.004}_{-0.006}$	NNLO
R_τ (CLEO)	[16]	1.78	0.302 ± 0.024	0.116 ± 0.003	0.002	0.002	NNLO
R_τ (ALEPH)	[17]	1.78	0.355 ± 0.021	0.122 ± 0.003	0.002	0.002	NNLO
R_τ (OPAL)	[17]	1.78	$0.375^{+0.032}_{-0.035}$	0.123 ± 0.003	0.002	0.002	NNLO
R_τ (Raczka)	[18]	1.78	0.333 ± 0.021	0.120 ± 0.003	0.002	0.002	NNLO
$\eta_c \to \gamma\gamma$ (CLEO)	[16]	2.98	0.187 ± 0.029	0.101 ± 0.010	0.008	0.006	NLO
$Q\overline{Q}$ states	[19]	5.0	0.188 ± 0.018	0.110 ± 0.006	0.000	0.006	q LGT
$b\bar{b}$ states	[19]	5.0	0.203 ± 0.007	0.115 ± 0.002	0.000	0.002	LGT
$\Upsilon(1S)$ (CLEO)	[16]	9.46	0.164 ± 0.013	0.111 ± 0.006	0.001	0.006	NLO
$e^+e^- \to jets$ (CLEO)	[16]	10.53	0.164 ± 0.015	0.113 ± 0.006	0.002	0.006	NLO
$ep \to jets$ (H1)	[20]	5 - 60		0.123 ± 0.018	0.014	0.010	NLO
$p\bar{p} \to W\ jets$ (D0)	[21]	80.6	0.123 ± 0.015	0.121 ± 0.014	0.012	0.005	NLO
$e^+e^- \to Z^0$:							
scal. viol. (ALEPH)	[17]	91.2		0.127 ± 0.011	-	-	NLO
ev. shapes (SLD)	[22]	91.2		0.120 ± 0.008	0.003	0.008	resum.
$\Gamma(Z^0 \to had.)$ (LEP)	[23]	91.2		0.127 ± 0.006	0.005	$^{+0.003}_{-0.004}$	NNLO

→ 0·116 ± 0·003

0·118 ± 0·003

→ 0·124 ± 0·005

Table 1. Summary of most recent measurements of α_s, presented at this conference. Abbreviations: GLS-SR = Gross-Llewellyn-Smith sum rules; (N)NLO = (next-)next-to-leading order perturbation theory; LGT = lattice gauge theory (q stands for quenched approximation); resum. = resummed next-to-leading order. Most results are still preliminary.

- taking α_s as parameter in EW fits with M_t M_H

$\Rightarrow$ $\boxed{\begin{array}{l} \alpha_s(M_Z) = 0.120 \pm 0.003 \\ M_H = 149^{+190}_{-82}\ \text{GeV} \end{array}}$

Now input 175 ± 6 GeV

Blondel Warsaw

α_s WORLD AVERAGE

- measured in many processes with high precision at many scales
  - all results consistent within errors
- recent compilation

$$\boxed{\alpha_s(M_Z^2) = 0.117 \pm \Delta}$$

$\Delta \approx 0.006$

ESTIMATE, NOT RIGOUROUS ERROR SINCE ERRORS NON GAUSSIAN

WARSAW

$$\boxed{\alpha_s(M_Z^2) = 0.118 \pm 0.003}$$

SCHMELLING

EVENT SHAPES

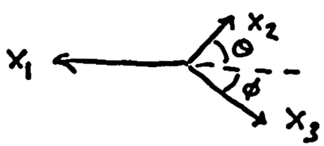
– global observables characterising structure of hadronic event

e.g. THRUST $$T = \max \frac{\sum |\underline{p}_a \cdot \underset{\uparrow \text{Thrust axis}}{\underline{n}}|}{\sum |\underline{p}_a|}$$

- for 2 particle (jet) events, thrust axis lies along particle direction

$$\longleftrightarrow \quad \Rightarrow \quad T = 1$$

- for 3 particle events, thrust axis coincides with most energetic particle



$$T = \frac{x_1 + x_2\cos\theta + x_3\cos\phi}{x_1 + x_2 + x_3} < 1$$

- for spherical events

$$T = \frac{1}{2}$$

e.g TOTAL JET BROADENING, B_T

$$B_T = \frac{\sum |\underline{p}_a \times \underline{n}|}{2 \sum |\underline{p}_a|}$$

THRUST AXIS

- for 2 particle events $\quad B_T = 0$

  3 particle events

$$B_T = \frac{x_2 \sin\theta + x_3 \sin\phi}{2(x_1 + x_2 + x_3)} > 0$$

$\longrightarrow$ FIG.

INFRARED SAFENESS

— FOR INFRARED POLES TO CANCEL FOR A PARTICULAR OBSERVABLE, IT MUST BE INFRARED SAFE

i.e. OBSERVABLE MUST BE INVARIANT UNDER

$$p_i \longrightarrow p_j + p_k$$

when $\quad p_j \parallel p_k \quad$ or $\quad |p_j| \to 0$

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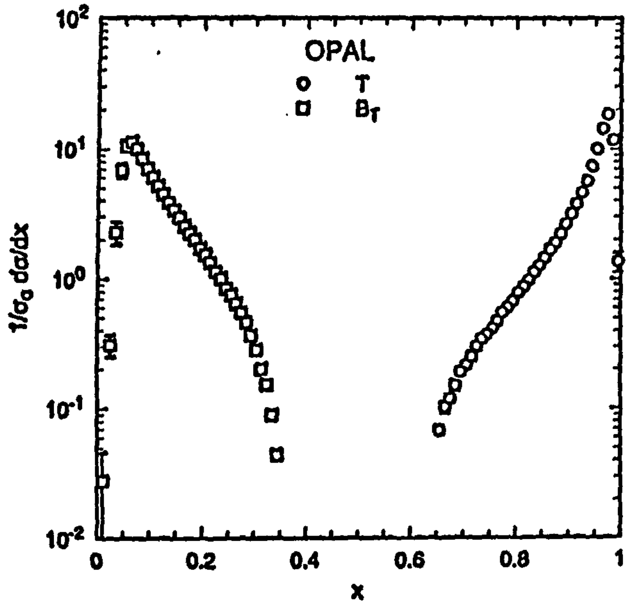
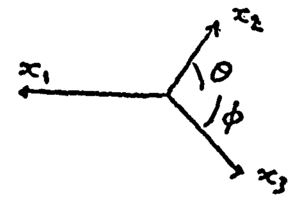


As $T \to 1$ or $B_T \to 0$ $\quad \frac{d\sigma}{dx} \to 0$

$O(\alpha_s)$ DISTRIBUTION

- $e^+e^- \to q\bar{q}g$



$x_1 > x_2, x_3$

$$\frac{2p_1\cdot p_2}{s} = \frac{2E_1E_2(1+\cos\theta)}{s}$$

$$= \frac{x_1x_2}{2}(1+\cos\theta) = \frac{(p_\gamma - p_3)^2}{s}$$

$$= 1 - \frac{2E_3}{\sqrt{s}} = 1 - x_3$$

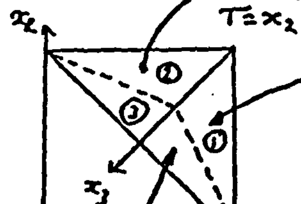
ie. $\quad x_2\cos\theta = \frac{2(1-x_3)}{x_1} - x_2$

$$x_3\cos\phi = \frac{2(1-x_2)}{x_1} - x_3 = x_1 - x_2\cos\theta$$

$$\Rightarrow \quad T = \frac{x_1 + x_2\cos\theta + x_3\cos\phi}{x_1+x_2+x_3}$$

$$\boxed{T = x_1} \qquad \text{if } x_1 > x_2, x_3$$

$x_2 > x_1, x_3$
$T = x_2$



$x_1 > x_2, x_3$
$T = x_1$

When $x_1 = x_2 = x_3$
$T = 2/3$
Minimum for 3 particle event

$x_3 > x_1, x_2$
$T = x_3$

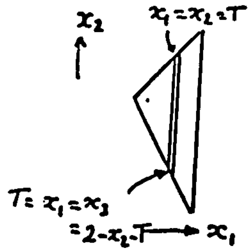
Starting from

$$\frac{1}{\sigma^0_{q\bar q}}\frac{d\sigma}{dx_1 dx_2} = \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

obtain $\frac{1}{\sigma^0}\frac{d\sigma}{dT}$ by integrating over the 3 regions

1) $x_1 > x_2, x_3 \Rightarrow x_1 = T$

ie. $2(1-T) < x_2 < T$

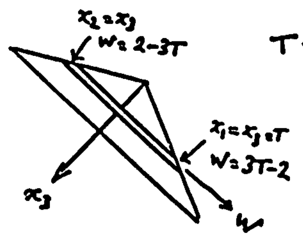


$$\left.\frac{1}{\sigma^0}\frac{d\sigma}{dT}\right)_1 \propto \int_{2(1-T)}^{T}\frac{T^2+x_2^2}{(1-T)(1-x_2)}\,dx_2$$

$$= \frac{\alpha_s}{2\pi}\,\frac{N^2-1}{2N}\left[\frac{(1+T^2)}{(1-T)}\ln\left(\frac{2T-1}{1-T}\right)+\frac{1}{1-T}\left(4-7T+\frac{3T^2}{2}\right)\right]$$

2) $x_2 > x_1, x_3$ — get same contribution as 1)

3) $x_3 > x_1, x_2$



rewrite variables

$T = x_3 = 2-x_1-x_2$

$\omega = x_1 - x_2$

$dT\,d\omega = 2\,dx_1 dx_2$

$x_1 = 1+\frac{W-T}{2}$ $\quad x_2 = 1-\frac{(W+T)}{2}$

$1-x_1 = \frac{T-W}{2}$ $\quad 1-x_2 = \frac{W+T}{2}$

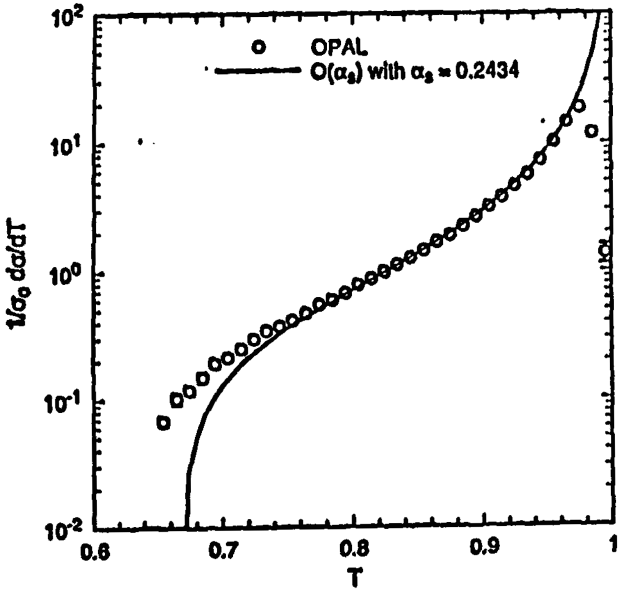
$$\text{ie}\ \left.\frac{1}{\sigma^0}\frac{d\sigma}{dT}\right)_3 \propto \frac{1}{2}\int_{2-3T}^{3T-2}\frac{(2-T+w)^2+(2-T-w)^2}{(T+w)(T-w)}\,dw$$

$$= \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\left[\frac{4-4T+2T^2}{T}\ln\left(\frac{2T-1}{1-T}\right)-6T+4\right]$$

$\Rightarrow$ adding up contributions from 3 regions

$$\boxed{\frac{1}{\sigma^0}\frac{d\sigma}{dT} = \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\left[2\,\frac{(3T^2-3T+2)}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right) - 3\,\frac{(3T-2)(2-T)}{1-T}\right]}$$

- $T > 2/3$
- As $T \to 1$ $\quad \frac{1}{\sigma_0}\frac{d\sigma}{dT} \sim \frac{1}{1-T}\log\left(\frac{1}{1-T}\right)$
-  virtual gluon contribution occurs at $T=1$
- EXPECT LARGE HADRONISATION CORRECTIONS AS $T \to 1$



- DEFICIENCY AT SMALL T DUE TO KINEMATIC BOUND
- SHAPE GOOD $0.75 < T < 0.95$

SPIN OF THE GLUON

if the gluon is a SCALAR

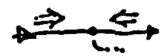
$$-\!-\!-\!<\ -ig_s T^a_{ij} \qquad \mathcal{L}_{INT} \sim \underbrace{g_s}_{\text{COUPLING}}\ \overbrace{\bar{\psi}_i\, T^a_{ij}\, \underbrace{\phi^a}_{\text{SCALAR GLUON FIELD}}\, \psi_j}^{\text{QUARKS}}$$

$$\Rightarrow \quad \frac{1}{\sigma^0}\frac{d\sigma}{dx_1 dx_2} = \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\frac{x_3^2}{2(1-x_1)(1-x_2)}$$

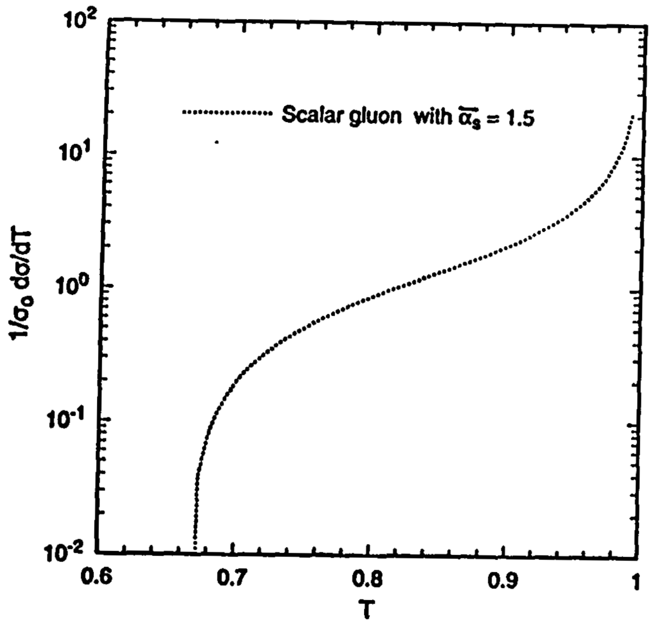
$$\therefore \quad \frac{1}{\sigma^0}\frac{d\sigma}{dT} = \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\frac{1}{2}\left[2\ln\left(\frac{2T-1}{1-T}\right) + \frac{(3T-2)(4-3T)}{(1-T)}\right]$$

<u>NOTE</u> as $E_g \sim x_3 \to 0 \quad |M|^2 \to 0$

- emission of gluon (scalar) changes helicity



- helicity violating

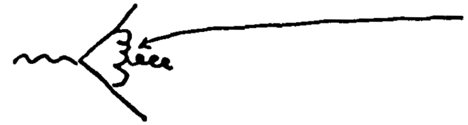


- SHAPE FOR SCALAR GLUON VERY DIFFERENT $\Rightarrow$ EVIDENCE THAT $J_g = 1$
- $\overline{\alpha_s}$ MUST BE LARGE

$O(\alpha_s^2)$ Thrust Distribution

At NLO, get contributions from double bremstrahlung

and virtual graphs



ggg VERTEX

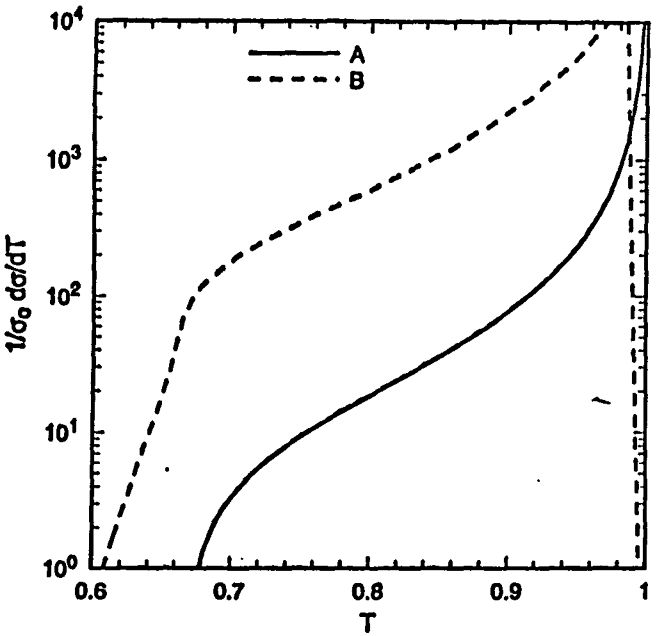
Same structure of NLO corrections as before

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \frac{\alpha_s(\mu)}{2\pi} A(T) \quad \ldots\ldots \text{ what we just calculated!}$$

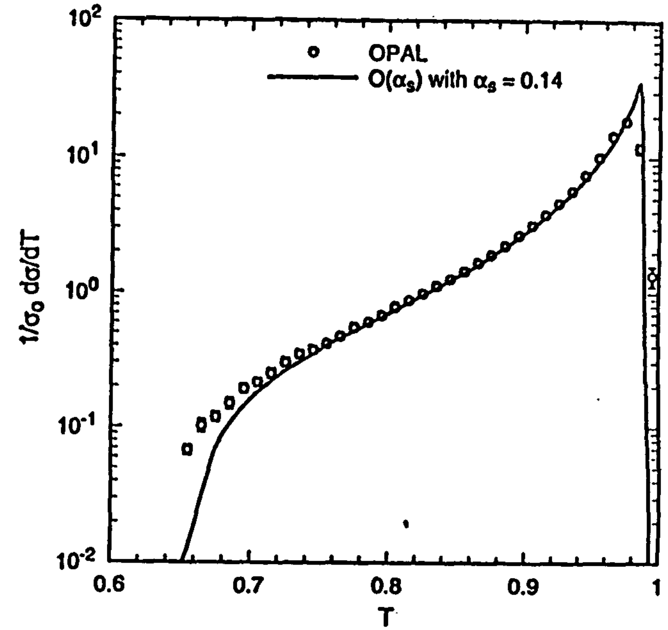
$$+ \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \Big[\underbrace{2\pi A(T) b_0 \log\frac{\mu^2}{s}}_{\substack{\text{renormalisation} \\ \text{term } b_0 = \frac{11N - 2n_f}{12\pi}}} + B(T) \Big]$$

genuine NLO

- too hard to do analytically
- numerically using EVENT program by Kunszt + Nason



- AS $T \to 1$ A diverges $\uparrow$
  B diverges $\downarrow$
- $\frac{1}{\sqrt{3}} < T$ ALLOWED at NLO



- AGREEMENT WITH DATA OVER WIDER RANGE OF T
- SMALLER VALUE OF α_S
- PREDICTION $\to -\infty$ AS $T \to 1$

RESUMMING LARGE LOGS

as $T \to 1$

$$\frac{1}{\sigma^0}\frac{d\sigma}{dT} \sim \frac{\alpha_s}{2\pi}\left(\underbrace{\frac{N^2-1}{2N}}_{C_F}\right)\left\{\frac{4}{1-T}\log\left(\frac{1}{1-T}\right)-\frac{3}{1-T}\right\}$$

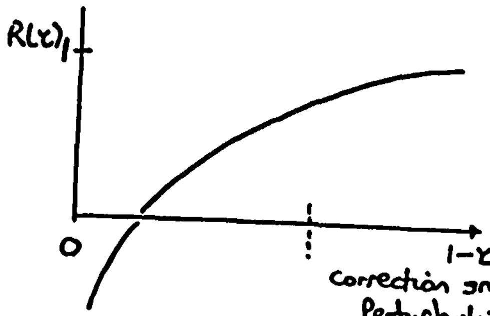
⇒ define cross section $T > \tau$

$$R(\tau) = \int_\tau^1 dT \, \frac{1}{\sigma^0}\frac{d\sigma}{dT}$$

singularity at $T=1$ cancelled by $\langle\,\rangle$

~ fraction of events with $T > \tau$.

$$R(\tau) \sim 1 - \underbrace{\frac{C_F\alpha_s}{\pi}\log^2(1-\tau)}_{\text{not small unless } \alpha_s\log^2(1-\tau) \text{ small}}$$



correction small
Perturbation theory believable

very large correction
≡ don't believe fixed order PT.

in fact

$$R(\tau) = 1 - \frac{C_F\alpha_s}{\pi}\log^2(1-\tau) + \frac{1}{2}\left(\frac{C_F\alpha_s}{\pi}\right)^2\log^4(1-\tau) \ldots$$

keeping only leading logs. – CAN RESUM for few variables e.g. T, B_T, EEC, jet crosssections in k_T alg.

$$\boxed{R(\tau) \sim \exp -\frac{C_F\alpha_s}{\pi}\log^2(1-\tau)}$$

so that as $\tau \to 1$ $\log^2(1-\tau) \to \infty$

$$\underline{R(\tau) \to 0}$$

- This is SUDAKOV form factor effect
  - for event to have high thrust, must have radiated very few gluons
  - very improbable

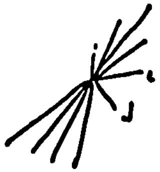
cf data, very improbable to have only 2 particle event.

- can also resum next-to-leading logs (Webber, Dokshitzer, Catani et al)

⇒ calculations believable when $\alpha_s \log(1-\tau)$ small.

WARNING: NOT ALL OBSERVABLES ARE EQUAL

e.g. JETS DEFINED THROUGH CLUSTERING



- FIND min d_{ij}
- IF $d_{ij} < d_{cut}$ COMBINE ij
- REPEAT UNTIL NO MORE CLUSTERING
  - ALL $d_{ij} > d_{cut}$

$\boxed{\text{JADE}}$ $\quad d_{ij} = 2E_iE_j(1-\cos\theta_{ij})$

$\boxed{K_T\ \text{ALGORITHM}}$ $\quad d_{ij} = 2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})$

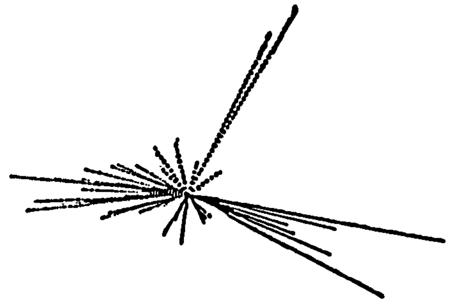
- BOTH INFRARED SAFE

  BUT LOGS CAN BE RESUMMED FOR K_T ALGORITHM

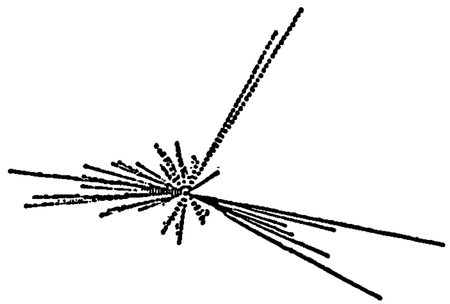
- MORE OBVIOUS ASSIGNMENT OF PARTICLES IN JETS
- HADRONISATION CORRECTIONS SMALLER

→ FIG

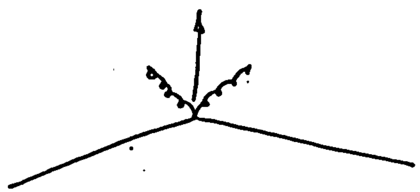
3 JET EVENTS.



K_T ALGORITHM



JADE



JADE

ADDS TWO SOFT GLUONS TOGETHER TO MAKE JET

MEASURING QCD GROUP PARAMETERS

Probabilities for parton splitting

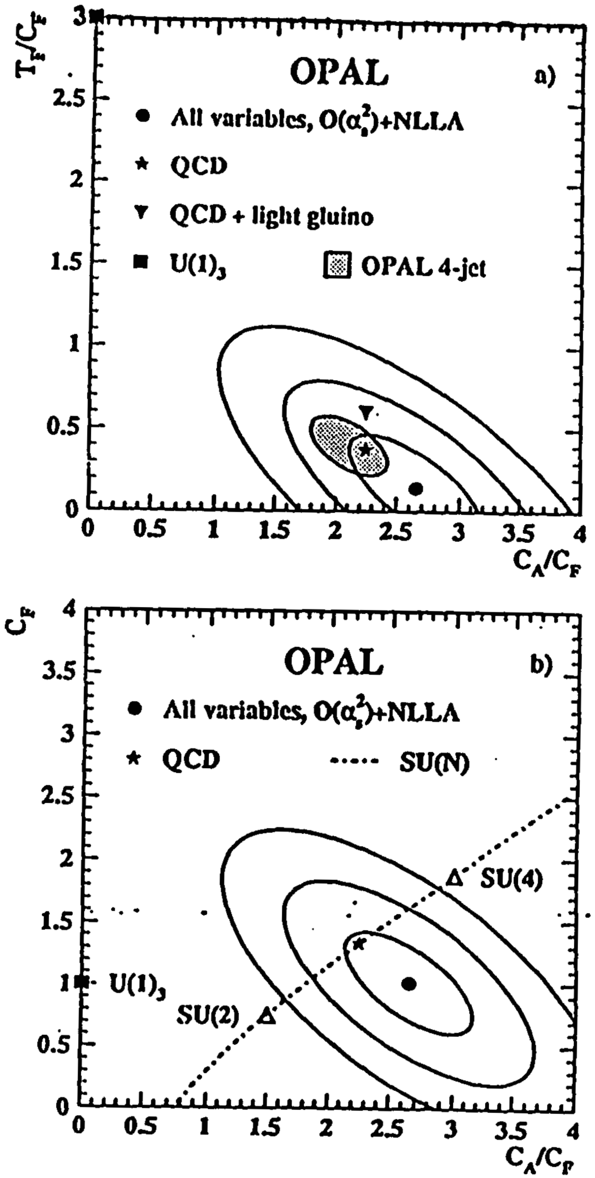
			QCD
$q \to qg$	$\left\vert \right.$  $\left. \right\vert^2$	$\propto \; C_F \alpha_s$	$\left(\frac{N^2-1}{2N}\right)\alpha_s$
$g \to gg$	$\left\vert \right.$  $\left. \right\vert^2$	$\propto \; C_A \alpha_s$	$N \alpha_s$
$g \to q\bar{q}$	$\left\vert \right.$  $\left. \right\vert^2$	$\propto \; T_F \alpha_s$	$\frac{1}{2} \alpha_s$

All present in $O(\alpha_s^2)$ EVENT shapes and 4jetrate

e.g. $B(T) = C_F \left[C_F B_{C_F}(T) + C_A B_{C_A}(T) + T_F B_{T_F}(T) \right]$

FIT TO DATA ON THRUST, B_T etc.

→ agreement with QCD.



JET PRODUCTION

THREE DISTINCT STAGES

1) ON MOMENTUM SCALE $\boxed{t \sim Q^2}$ HARD SCATTERING INVOLVING SMALL NUMBER OF PRIMARY PARTONS

e.g $e^+e^- \to q\bar{q}$

2) OVER A PERIOD $\boxed{Q_0^2 < t < Q^2}$

PRIMARY PARTONS CASCADE / SHOWER BY MULTIPLE BREMSTRAHLUNG. CASCADES TEND TO FOLLOW DIRECTION OF PRIMARY PARTONS DUE TO SOFT/COLLINEAR BEHAVIOUR OF $|M|^2$

- COHERENCE (ANGULAR ORDERING) AND SUDAKOV BRANCHING — LEADING LOGS RESUMMED

3) $\boxed{\Lambda^2 < t < Q_0^2}$ NON-PERTURBATIVE

(LONG DISTANCE + LONG TIMES) WHERE PARTONS BLANCH AND FORM COLOURLESS HADRONS

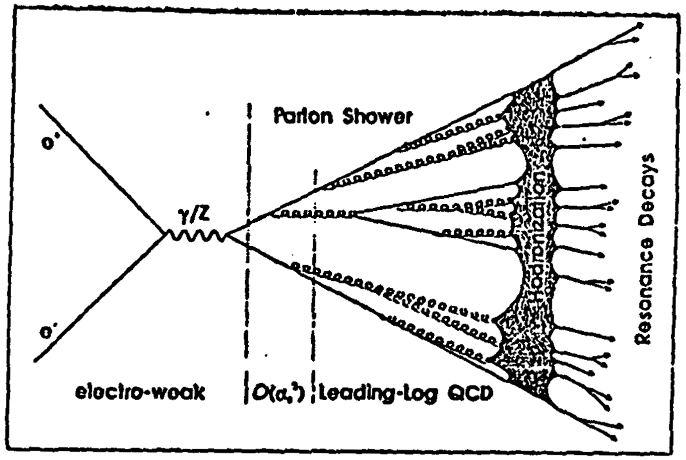


Figure 3: Schematic representation of a parton shower.

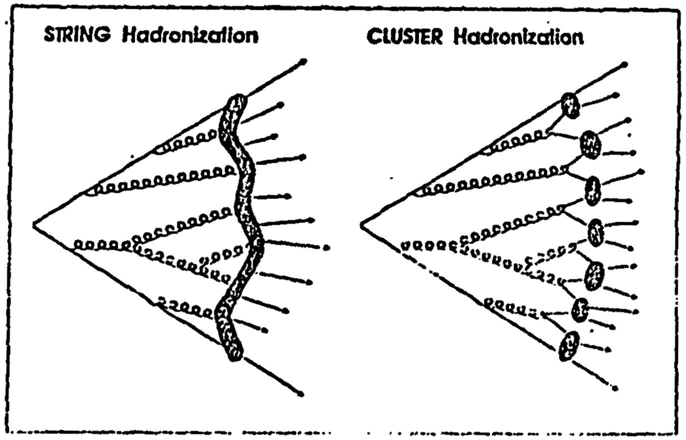


Figure 4: Pictorial presentation of the string and the cluster hadronization model.

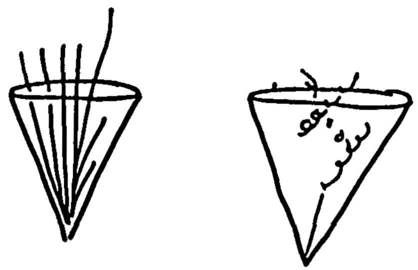
LOCAL PARTON HADRON DUALITY

- FLOW OF MOMENTUM + QUANTUM NUMBERS AT HADRON LEVEL TENDS TO FOLLOW FLOW ESTABLISHED AT PARTON LEVEL

e.g. The flavour of quark initiating jet should be found in hadron near jet axis

The extent the hadron flow deviates from parton flow reflects smearing due to hadronisation – of order λ.

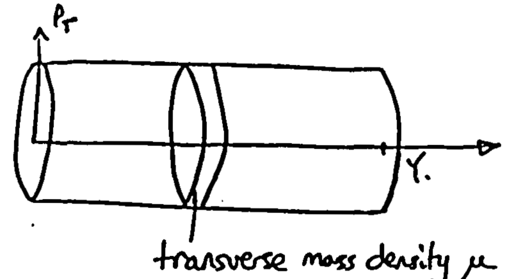
e.g



JET energy + direction determined at parton level $\approx$ that at hadron level.

SIMPLE HADRONISATION MODEL.

(Feynman).



pseudorapidity $= \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right)$

Parton produces a tube in (y, p_T) space of light hadrons wrt. initial parton direction

$$E_{jet} = \mu \int_0^Y \cosh y \, dy = \mu \sinh Y$$

$$p_{jet} = \mu \int_0^Y \sinh y \, dy = \mu(\cosh Y - 1)$$

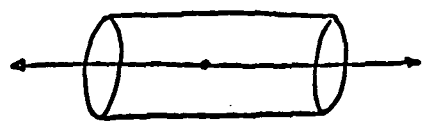
$$\Rightarrow \quad m_{jet}^2 = E_{jet}^2 - p_{jet}^2 = 2\mu^2(\cosh Y - 1)$$
$$= 2\mu \, p_{jet}$$

$$\boxed{E_{jet} = p_{jet} + \mu}$$

$\mu \sim 0.5 - 1$ GeV from expt.

HADRONISATION AND THRUST

$$T = \max \frac{\sum |\underline{p}_i \cdot \underline{n}|}{Q}$$

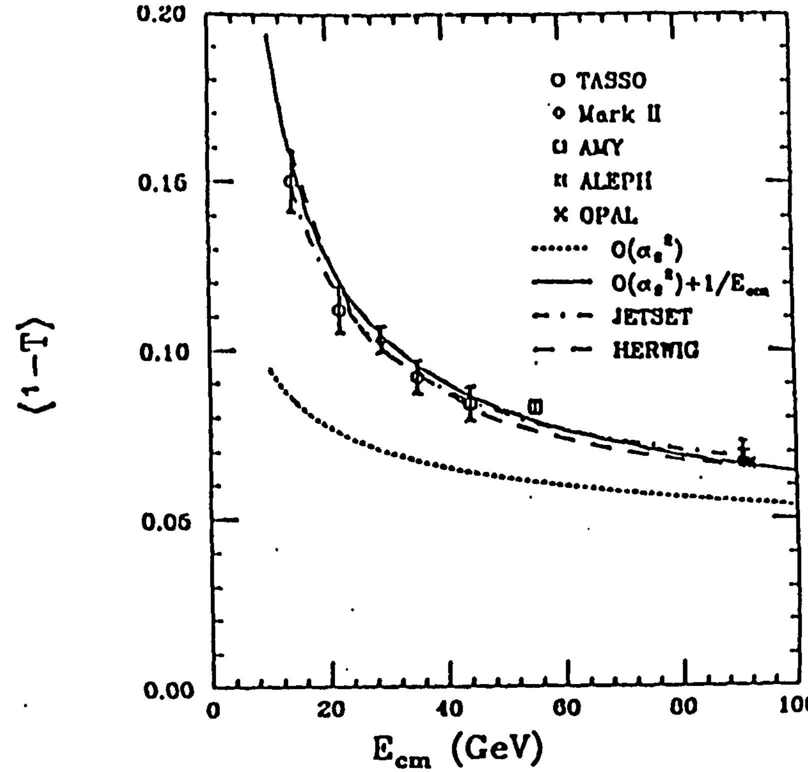


2 jet event $\quad T_{parton} = 1$

$$T_{hadron} = \frac{2P_{jet}}{Q} = \frac{2E_{jet} - 2\mu}{Q}$$

$$= 1 - \frac{2\mu}{Q}$$

i.e. $\boxed{\delta T = -\frac{2\mu}{Q}}$

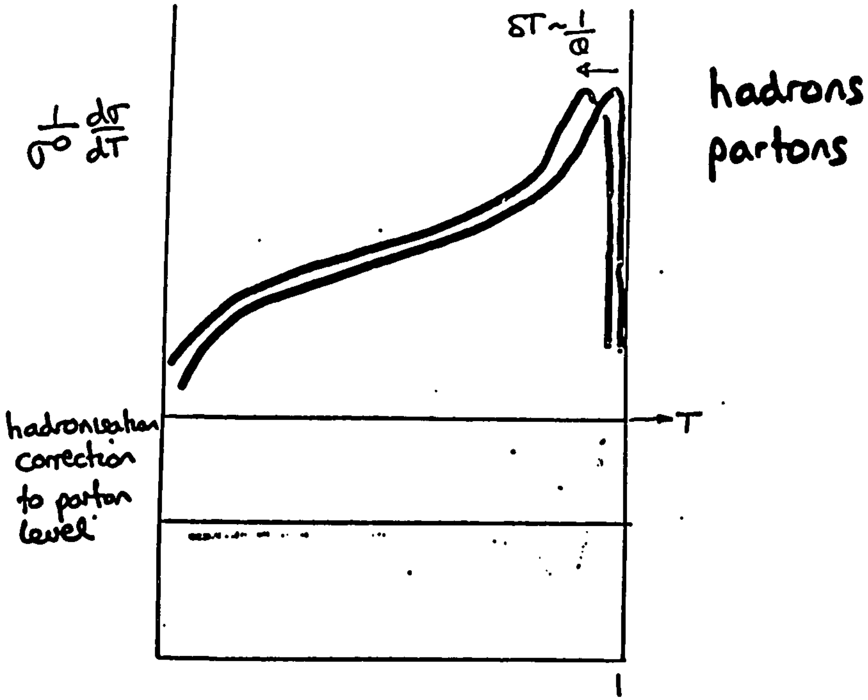


MEAN VALUE OF THRUST

$$\langle 1-T \rangle = \int (1-T) \frac{1}{\sigma_0} \frac{d\sigma}{dT} dT$$

$$= 0.334\,\alpha_s + 1.02\,\alpha_s^2 + \frac{1\,\text{GeV}}{Q}$$

HADRONISATION CORRECTION TO THRUST DISTRIBUTION



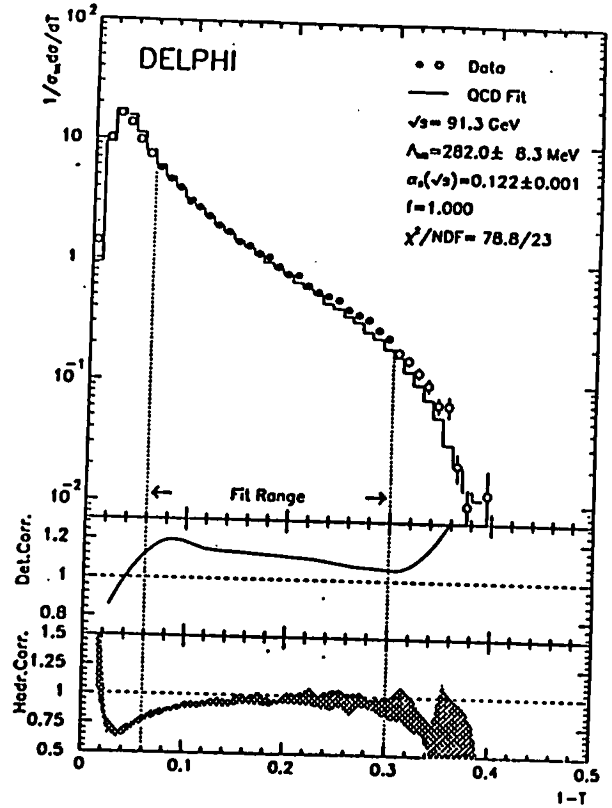


Figure 11: Measurement of the strong coupling constant from the Thrust distribution. The data points used in the fit are indicated by the full dots. Detector and hadronization corrections are indicated below. The theoretical prediction is the second order matrix element plus resummation of leading and next-to-leading logarithms.

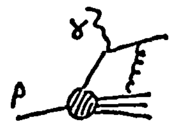
4. Deep Inelastic Scattering and QCD

- QCD improved Parton Model
- Mass Factorisation
- Evolution equations
- Callan Gross Relation
- Global fits and PDF's

QCD + PARTON MODEL
– DIS REVISITED

- EXPLAIN LOGARITHMIC SCALING VIOLATIONS SEEN IN DATA
- WILL IGNORE HIGHER TWIST TERMS

$$F(x,Q^2) = \underbrace{\boxed{F^{(2)}(x,Q^2)}}_{\text{LEADING TWIST}} + \underbrace{\frac{1}{Q^2} F^{(4)}(x,Q^2) + \ldots.}_{\text{HIGHER TWIST}}$$



e.g final state interactions

Milsztajn + Virchaux

- EMPIRICAL FITS AT low Q^2 (SLAC) + high Q^2 (BCDMS)

$\Rightarrow$ negligible for $Q^2 > 5\,GeV^2$, $W^2 = \frac{Q^2(1-x)}{x} \geq 10\,GeV^2$

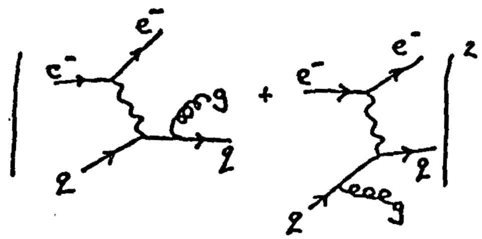
LEADING ORDER

$$F_2(x) = \sum_q e_q^2\, x\, f_q(x)$$

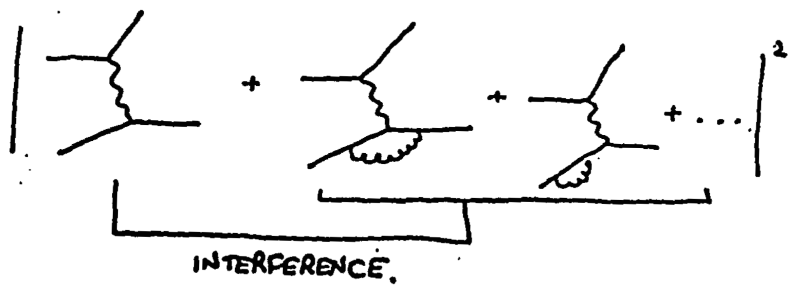
QCD IMPROVED PARTON MODEL

$O(\alpha_s)$ CORRECTIONS

1) BREMSSTRAHLUNG



2) VIRTUAL

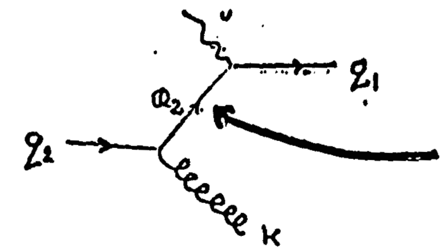


— SAME $|M|^2$ AS $e^+e^- \to q\bar{q}g$ and $e^+e^- \to q\bar{q}$ WITH e^+ CROSSED TO FINAL STATE AND $\bar{q}$ CROSSED TO INITIAL STATE

e.g. $e^-(p_2) + q(q_2) \longrightarrow e^-(p_1) + q(q_1) + g(k)$

$$\boxed{\frac{1}{4}|M|^2 = 2e^4 e_q^2 \left(\frac{N^2-1}{2}\right) g_s^2 \times \left[\frac{(q_1\cdot p_1)^2 + (q_1\cdot p_2)^2 + (q_2\cdot p_1)^2 + (q_2\cdot p_2)^2}{q_1\cdot k \;\; k\cdot q_2 \;\; p_1\cdot p_2}\right]}$$

from before



SINGULAR WHEN $k\cdot q_2 \to 0$
i.e. INITIAL STATE COLLINEAR SINGULARITY

$\Rightarrow$ IN <u>COLLINEAR LIMIT</u>

$$Q_2 = z q_2 \qquad k = (1-z) q_2 = \frac{(1-z)}{z} Q_2$$

so

$$q_1\cdot k = \frac{1-z}{z}\, q_1\cdot Q_2 = \frac{1-z}{z}\, p_1\cdot p_2$$

$$q_2\cdot p_1 = \frac{1}{z}\, Q_2\cdot p_1 = \frac{1}{z}\, q_1\cdot p_2$$

$$q_2\cdot p_2 = \frac{1}{z}\, Q_2\cdot p_2 = \frac{1}{z}\, q_1\cdot p_1$$

$$\frac{1}{4}\sum|M|^2 \longrightarrow 2e^4 e_q^2 \left(\frac{N^2-1}{2}\right) g_s^2 \left[\frac{(q_1\cdot p_1)^2 + (q_1\cdot p_2)^2}{(p_1\cdot p_2)^2}\right] \times \frac{1}{q_2\cdot k}\left[\frac{1+z^2}{z(1-z)}\right]$$

$$= \frac{g_s^2}{2 q_2\cdot k}\left(\frac{N^2-1}{2N}\right) \frac{2(1+z^2)}{z(1-z)}\; \frac{1}{4}\sum|M|^2_{eq}$$

$$\text{FLUX} = \frac{1}{2 q_2\cdot p_2} \longrightarrow \frac{z}{2 Q_2\cdot p_2}$$

Phase Space

$$d\Phi(p_1,q_1,k) \longrightarrow d\Phi(p_1,q_1)\,\frac{1}{16\pi^2}\,\overset{2k\cdot q_1}{dt}\,\frac{dz}{z}$$

Putting pieces together

$$d\sigma(eq\to eqg) = \frac{1}{\text{Flux}}\,\frac{1}{4}\sum|M|^2\, d\Phi$$

$$\longrightarrow \frac{\alpha_s}{2\pi}\,\frac{N^2-1}{2N}\,\frac{1+z^2}{1-z}\,\frac{dt}{t}\,\frac{dz}{z}\,d\sigma(eq\to eq)$$

$$= \frac{\alpha_s}{2\pi}\,\hat{P}_{qq}(z)\,\log\left(\frac{Q^2}{m^2}\right)\frac{dz}{z}\,d\sigma(eq\to eq)$$

SMALL SCALE TO CUT OFF INTEGRATION

Altogether, have now found $O(\alpha_s)$ CORRECTIONS TO F_2 $\left[z = x/y \Rightarrow dz/z = -dy/y \quad \int_{x/y}^{1} \to -\int_x^1 \right]$

$$\boxed{F_2(x,Q^2) = x\sum_q e_q^2 \int_x^1 \frac{dy}{y}\, f_q(y)\left[\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_s}{2\pi}\left\{P_{qq}\left(\frac{x}{y}\right)\log\frac{Q^2}{m^2} + R\left(\frac{x}{y}\right)\right\}\right]}$$

VIRTUAL GRAPHS CONTRIBUTE

$$P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)_+ + 2\,\delta(1-x)$$

— $R\left(\frac{x}{y}\right)$ is CALCULABLE FINITE CORRECTION

- As $t\to 0$, the collinear quark propagates for longer times
  - — eventually hadronisation occurs!

DIVIDE LONG DISTANCE / SHORT DISTANCE EFFECTS WITH SCALE μ

FINITE $\quad Q^2 > t > \mu^2 \quad$ SHORT DISTANCE — Perturbative

$\mu^2 > t > m^2 \quad$ LONG DISTANCE — NON-PERT.

— "RENORMALIZE" INTO PARTON DISTRIBUTIONS

MASS FACTORISATION

$$\boxed{f_q(x,\mu^2) = f_q(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y}\, f_q(y)\left\{P_{qq}\left(\frac{x}{y}\right)\log\frac{\mu^2}{m^2} + R_2\left(\frac{x}{y}\right)\right\}}$$

"bare" parton density function

divergence as $m\to 0$

scheme dependent

$$\boxed{F_2(x,Q^2) = x\sum e_q^2\int_x^1\frac{dy}{y}\, f_q(y,\mu^2)\left\{\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_s}{2\pi}\left\{P_{qq}\left(\frac{x}{y}\right)\log\frac{Q^2}{\mu^2} + R_1(x/y)\right\}\right\}}$$

PROCESS INDEPENDENT

$\Rightarrow$ CAN BE USED ELSEWHERE

$R - R_2$

$\Updownarrow$

Factorisation scheme dep.

Just as $\alpha_s(M_Z^2)$ is not directly calculable, neither is $f_q(x,\mu^2)$

– BUT VARIATION WITH μ^2 IS

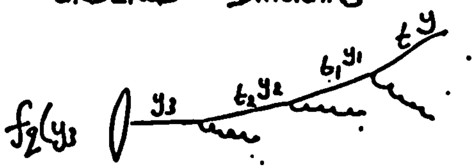
$$\boxed{\mu^2 \frac{\partial f_q(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi}\int_x^1 \frac{dy}{y}\, f_q(y,\mu^2)\, P_{qq}(x/y)}$$

– this is DGLAP equation and predicts scaling violations

– expt $\longrightarrow f_q(x,Q_0^2)$	– expt $\Rightarrow \alpha_s(M_Z)$
– running given by QCD	– running α_s
– μ is FACTORISATION scale	– μ is RENORMALISATION SCALE

<u>NOTE</u> – Physical cross sections are independent of μ – FIXED ORDER CALCULATIONS ARE <u>NOT</u>

DGLAP corresponds to resumming STRONGLY ORDERED DIAGRAMS



$t \gg t_1 \gg t_2 \gg t_3$

$y < y_1 < y_2 < y_3$

DGLAP

At $O(\alpha_s)$ ALSO GET CONTRIBUTION FROM GLUON PARTON DENSITY FUNCTION

– ALSO SPLITTINGS

$$\mu^2 \frac{\partial f_q(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi}\int_x^1 \frac{dy}{y}\left[P_{qq}\left(\frac{x}{y}\right) f_q(y,\mu^2) + P_{qg}\left(\frac{x}{y}\right) f_g(y,\mu^2)\right]$$

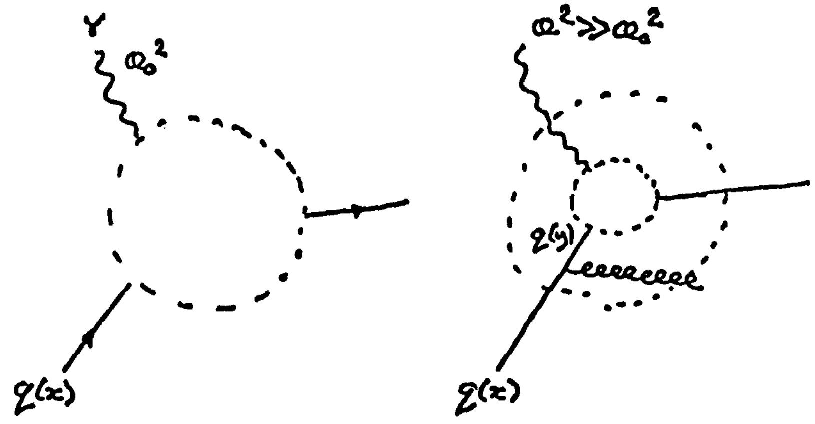
$$\mu^2 \frac{\partial f_g(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi}\int_x^1 \frac{dy}{y}\left[P_{gg}\left(\frac{x}{y}\right) f_g(y,\mu^2) + \sum_q P_{gq}\left(\frac{x}{y}\right) f_q(y,\mu^2)\right]$$

$\frac{\alpha_s}{2\pi} P_{ab}\left(\frac{x}{y}\right)$ IS PROBABILITY OF FINDING a WITH MOMENTUM FRACTION x INSIDE b WITH FRACTION y

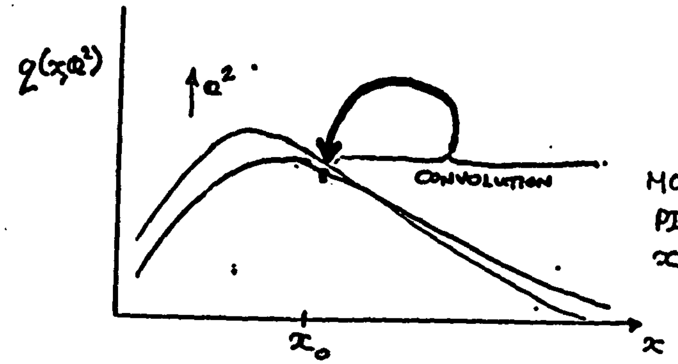
– for some input $f_i(x,\mu_0^2)$ can solve to find $f_i(x,\mu^2)$

– EVOLUTION known to $O(\alpha_s^2)$

SIMPLE PICTURE

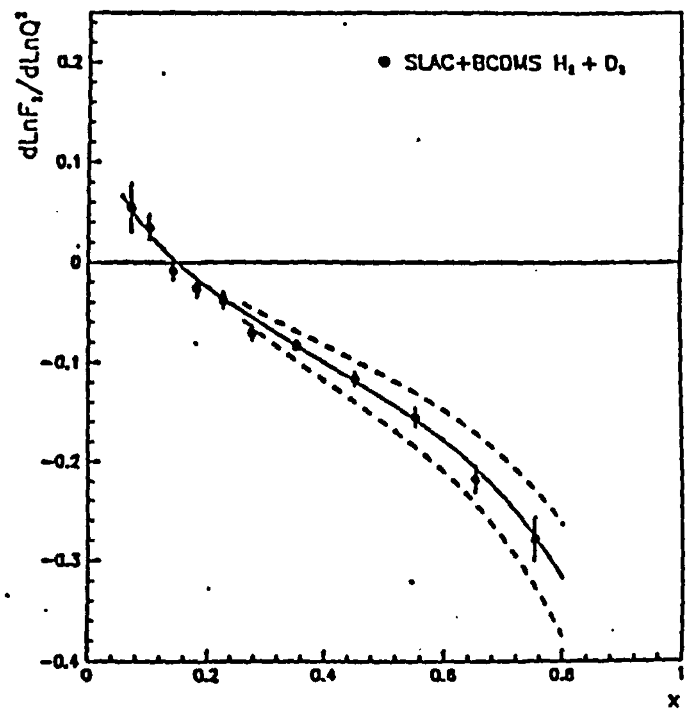


- AS $Q^2 \uparrow$ PHOTON RESOLVES QUARKS WITH LESS MOMENTA DUE TO RADIATION



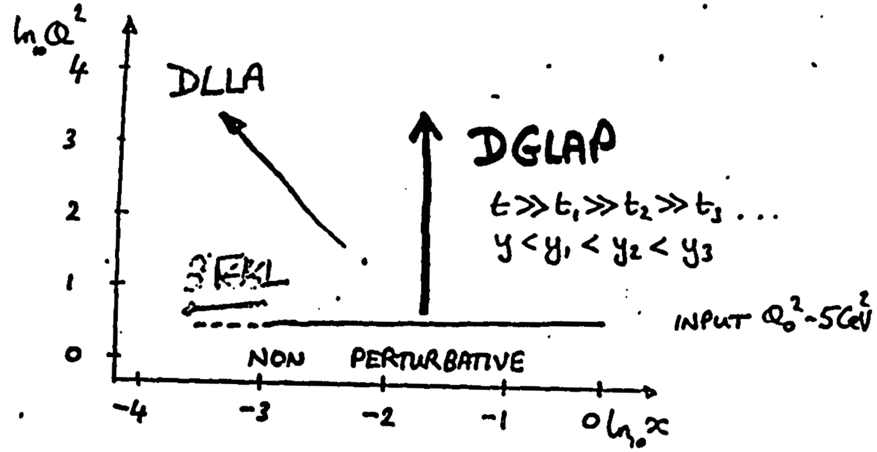
MOVEMENT OF PDF TO LOWER x — DICTATED BY $q(x)$ for $x > x_0$

$$q(x_0, Q^2+\Delta Q^2) = q(x_0, Q^2) + \frac{\alpha_s}{2\pi}\int_{x_0}^{1} \frac{dy}{y}\, P\left(\frac{x_0}{y}\right) q(y, Q^2)\, \ln\frac{Q^2+\Delta Q^2}{Q^2}$$



- different values of $\alpha_s(M_Z)$

EVOLUTION IN x, Q^2 SPACE



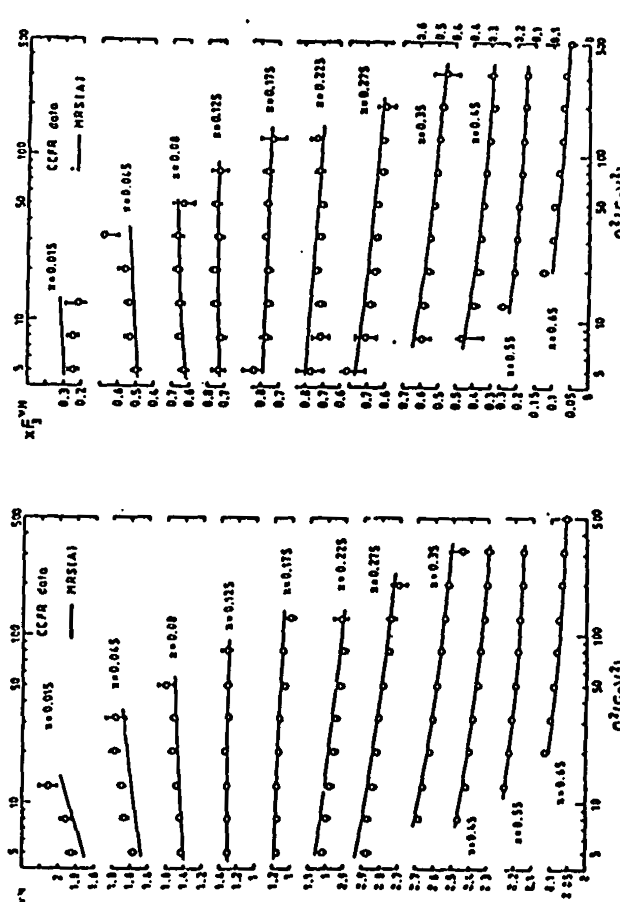


Fig. 8

DLLA – strong ordering in t and y

$$t \gg t_1 \gg t_2 \gg t_3$$
$$y \ll y_1 \ll y_2 \ll y_3$$

$\Rightarrow$ RESUM $\left[\alpha_s \log\left(\frac{1}{x}\right) \log\left(\frac{Q^2}{Q_0^2}\right)\right]^n$

BFKL – strong ordered ONLY in y

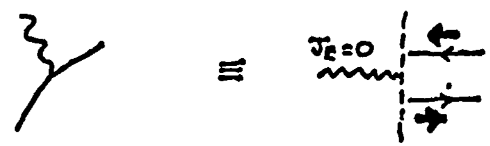
$$y \ll y_1 \ll y_2 \ll y_3$$

$\Rightarrow$ RESUM $\left[\alpha_s \log\left(\frac{1}{x}\right)\right]^n$

– BOTH PREDICT GROWTH AT SMALL x $[F_2]$

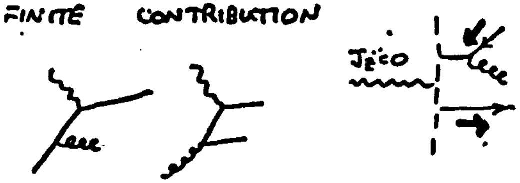
CALLAN - GROSS RELATION / F_L

- AT LOWEST ORDER $\boxed{F_2 - 2xF_1 = 0}$
  - This combination sensitive to helicity zero component of γ^* (LONGITUDINAL)
- $\Rightarrow$ Define $\boxed{F_L = F_2 - 2xF_1}$
- IN BREIT / BRICK WALL FRAME



DIRECTION OF QUARKS REVERSED

- helicity conserved by QED, but FORBIDDEN
- AT NLO, RADIATION OF GLUON GIVES FINITE CONTRIBUTION



$F_L \neq 0$

$$\boxed{F_2(x,Q^2) = \frac{\alpha_s}{\pi}\int_x^1 \frac{dy}{y}\left(\frac{x}{y}\right)^2\left[F_2(y,Q^2) + 2\sum_q e_q^2\left(1-\frac{x}{y}\right) y f_g(y,Q^2)\right]}$$

known — direct measure

GLOBAL FITS $\longrightarrow$ PDF

- PARAMETERISE $f_q(x,Q_0^2)$, $f_g(x,Q_0^2)$
  + QCD EVOLUTION + COMPARE WITH WIDE RANGE OF DATA (FIT) — MRS, CTEQ, GRV.

e.g.

$$x u_v = A_u x^{\eta_1}(1-x)^{\eta_2}(1+\varepsilon_u\sqrt{x}+\gamma_u x)$$
$$x d_v = A_d x^{\eta_3}(1-x)^{\eta_4}(1+\varepsilon_d\sqrt{x}+\gamma_d x)$$
$$x S = A_s x^{-\lambda}(1-x)^{\eta_s}(1+\varepsilon_s\sqrt{x}+\gamma_s x)$$
$$x g = A_g x^{-\lambda}(1-x)^{\eta_g}(1+\gamma_g x)$$

$$2\bar{u} = 0.4\,S - \Delta$$
$$2\bar{d} = 0.4\,S + \Delta$$
$$2\bar{s} = 0.2\,S$$

$x\Delta = A x^{\eta_\Delta}(1-x)^{\eta_S}$

GOTTFRIED SUM RULE

$\rightarrow$ MANY PARAMETER FITS

$\rightarrow$ AS DATA IMPROVE, FITS CHANGE

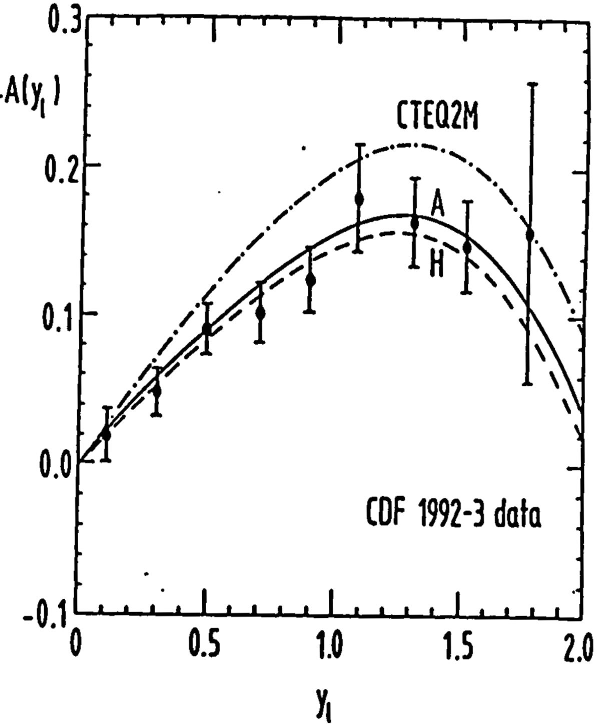
$\rightarrow$ CAN USE IN $p\bar{p}$, pp COLLISIONS TO PREDICT CROSS SECTIONS

MRS input

Process/ Experiment	Leading order subprocess	Parton determination
DIS ($\mu N \to \mu X$) BCDMS, NMC $F_2^{\mu p}, F_2^{\mu n}$	$\gamma^* q \to q$	Four structure functions $\to$ $u+\bar{u}$, $d+\bar{d}$, $\bar{u}+\bar{d}$, s (assumed $= \bar{s}$), but only $\int xg(x)dx \simeq 0.5$ [$\bar{u}-\bar{d}$ is not determined]
DIS ($\nu N \to \mu X$) CCFR (CDHSW) $F_2^{\nu N}, xF_3^{\nu N}$	$W^* q \to q'$	
$\nu N \to \mu^+\mu^- X$ CCFR	$\nu s \to \mu^- c$, $c \to \mu^+$	$s \approx \frac{1}{2}\bar{u}$ (or $\frac{1}{2}\bar{d}$) $\longrightarrow$ FIG
DIS (HERA) F_2^{ep} (H1,ZEUS)	$\gamma^* q \to q$	λ ($x\bar{q} \sim xg \sim x^{-\lambda}$, via $g \to q\bar{q}$)
$pp \to \gamma X$ WA70 (UA6)	$qg \to \gamma q$	$g(x \approx 0.4)$
$pN \to \mu^+\mu^- X$ E605	$q\bar{q} \to \gamma^*$	$\bar{q} = ...(1-x)^{\eta_s}$
$pp, pn \to \mu^+\mu^- X$ NA51	$u\bar{u}, d\bar{d} \to \gamma^*$; $u\bar{d}, d\bar{u} \to \gamma^*$	$(\bar{u}-\bar{d})$ at $x = 0.18$
$p\bar{p} \to WX(ZX)$ UA2, CDF, D0	$ud \to W$	u, d at $x_1 x_2 s \simeq M_W^2 \to$ $x \approx 0.13$ CERN, $x \approx 0.05$ FNAL
$\to W^{\pm}$ asym CDF		slope of u/d at $x \approx 0.05$ $\longrightarrow$ FIG

RISE OF F_2 AT LOW x

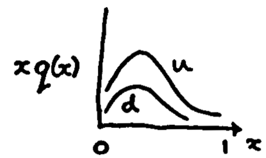
$$\frac{\sigma^+(y) - \sigma^-(+y)}{\sigma^+(y) + \sigma^-(+y)} = A(y_l)$$

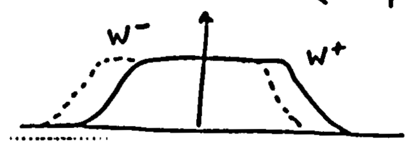


$u \longrightarrow W^+ \longleftarrow \bar{d}$

$d \longrightarrow W^- \longleftarrow \bar{u}$ Fig.3

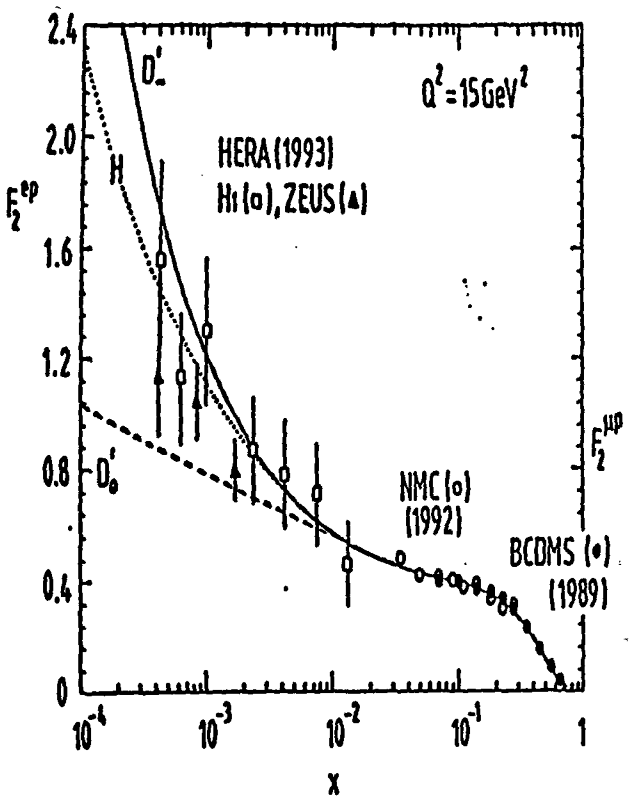
$p \Rightarrow$ $\Leftarrow \bar{p}$





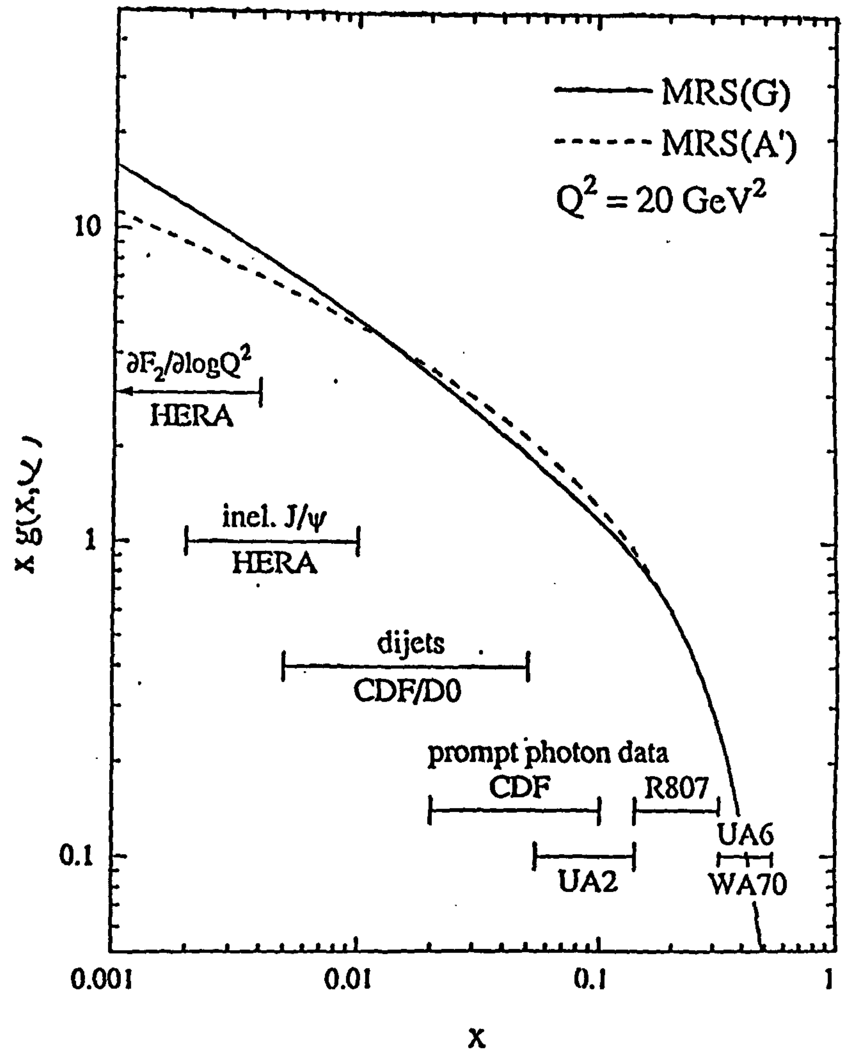
SENSITIVE TO u/d

W ASYMMETRY MAINTAINED AFTER $W^{\pm} \to \ell^{\pm}\nu$



2.4
2.0
1.6
1.2
0.8
0.4
0
F_2^{ep}
$F_2^{\mu p}$
D'_-
H
D'_0
$Q^2 = 15\,GeV^2$
HERA (1993)
H1 (o), ZEUS (▲)
NMC (o)
(1992)
BCDMS (•)
(1989)
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
x

Fig.1



MRS(G)
MRS(A')
$Q^2 = 20\ GeV^2$
$x\,g(x, Q^2)$
10
1
0.1
$\partial F_2/\partial \log Q^2$
HERA
inel. J/ψ
HERA
dijets
CDF/D0
prompt photon data
CDF
R807
UA6
UA2
WA70
0.001
0.01
0.1
1
x

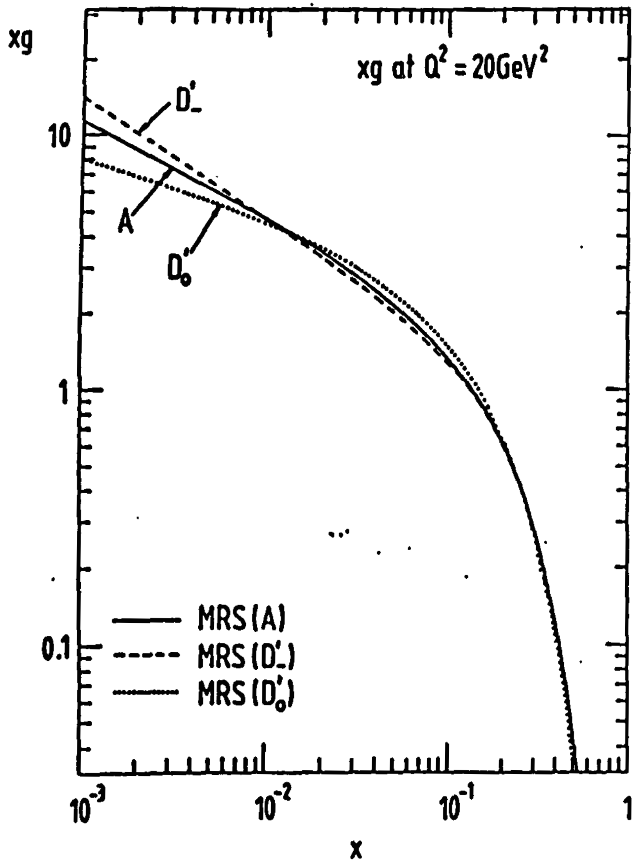

xg
xg at $Q^2 = 20\,GeV^2$
D'_-
A
D'_0
10
1
0.1
MRS (A)
MRS (D'_-)
MRS (D'_0)
10^{-3}
10^{-2}
10^{-1}
1
x

Fig. 15

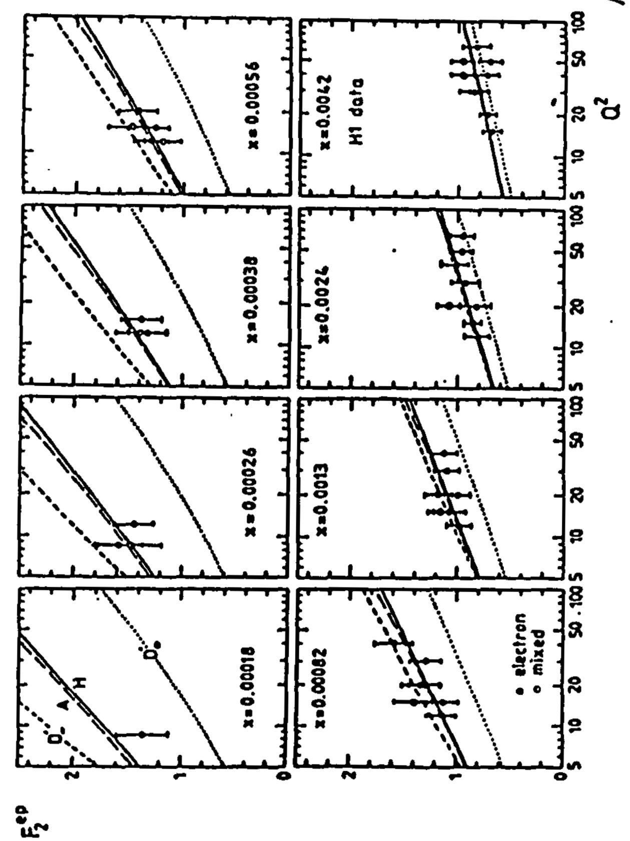
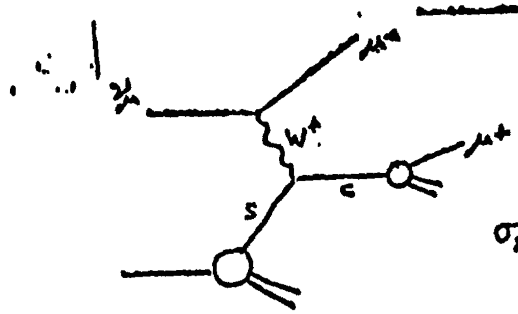
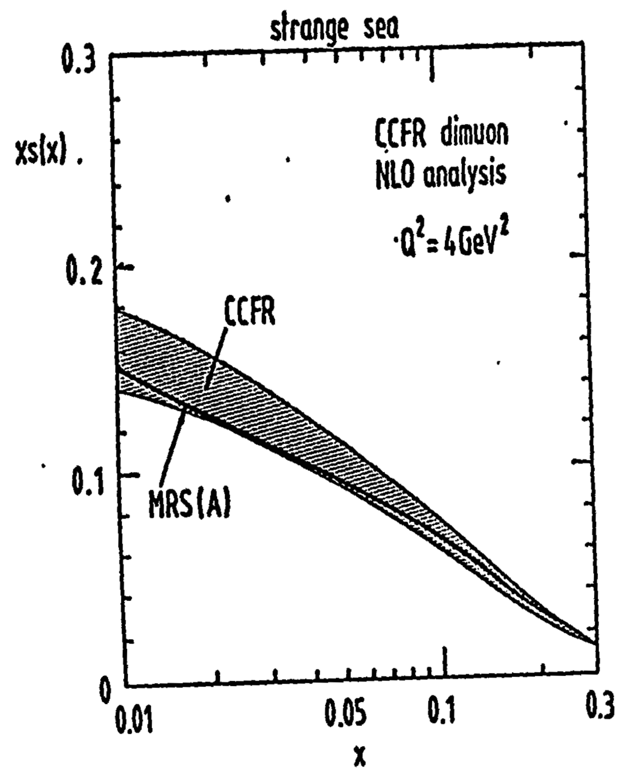

F_2^{ep}
x=0.00018
x=0.00026
x=0.00038
x=0.00056
x=0.00082
x=0.0013
x=0.0024
x=0.0042
H1 data
electron
mixed
Q^2

Fig. 11



$\sigma_{\nu N} \sim |V_{cs}|^2 \, s(x) \, D^{c \to \mu^+}$



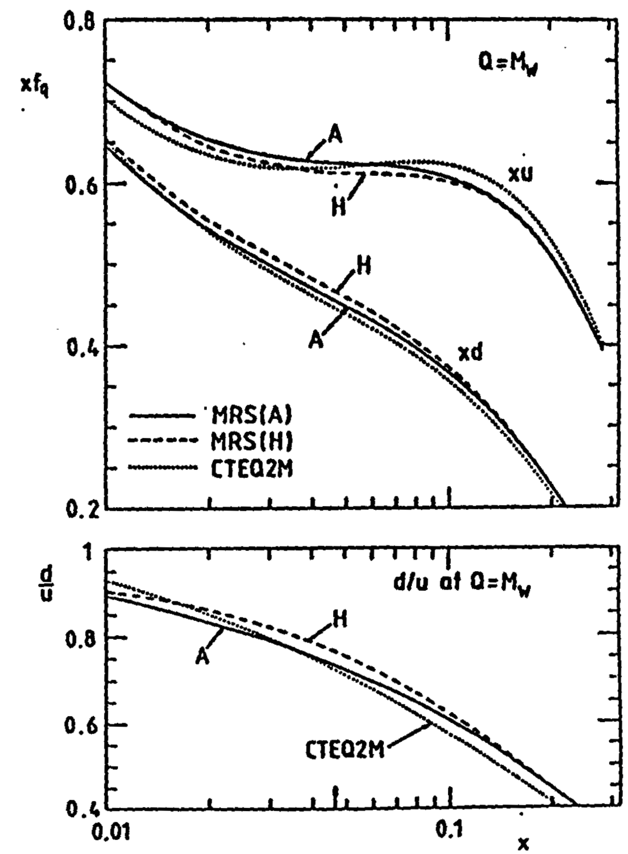


Fig.4

5. Precision Electroweak Physics at LEP

- **Parameters and Observables**
- **Radiative corrections from Vacuum Polarisation**
- **ρ parameter**
- **LEP Physics**
  - **Lineshape: M_Z, Γ_Z and peak cross section**
  - **Bhabha scattering and Luminosity**
  - **Forward Backward Asymmetry**
- **Radiative Corrections**
- **R_b**
- **Bounds on m_H and m_t**

ELECTROWEAK PHYSICS

BASIC IDEA

$$\mathcal{L}(\underbrace{g, g', g_s}_{\text{gauge couplings}}, \underbrace{\lambda, \mu^2}_{\text{Higgs}}, g_f)$$

g_f: fermion masses

$\Rightarrow$ all observables can be computed in terms of $g, g', g_s, \lambda, \mu^2, g_f$

USUALLY choose 6 PARAMETERS THAT CAN BE DETERMINED DIRECTLY

$$e^2,\ M_Z,\ G_F,\ M_H,\ m_f,\ \alpha_s(M_Z)$$

$$\updownarrow$$

$$M_W$$

$\Rightarrow$ compute observable O as

$$O = O_0 \, [\underbrace{1 + \alpha c_1 + \alpha^2 c_2 \ldots}_{\text{PERTURBATION SERIES}}]$$

O_0: Lowest order

O_0, c_1, c_2 DEPEND ON PARAMETERS

e.g. QED + $(g-2)_e$ parameters m_e, α

$$\left(\frac{g-2}{2}\right)_e = 0.001\ 159\ 652\ 188\ (4) \quad \text{expt}$$
$$0.001\ 159\ 652\ 133\ (29) \quad \text{theory.}$$

- here m_e, α input.
- if, for example, m_e <u>not</u> known, could use this to determine m_e

ELECTROWEAK PHYSICS

inputs

$$\alpha^{-1}(m_e^2) = 137.035\,9895\ (61)$$
Josephson

$$G_F = 1.16637\ (2) \times 10^{-5}\ \text{GeV}^{-2}$$
μ decay

$$M_Z = 91.1863 \pm 0.0020\ \text{GeV}$$
LEP

$$\alpha_s(M_Z) = 0.117 \pm 0.006$$
WORLD.

NOT KNOWN

m_t , m_H

- $\sin^2\theta_W$ DERIVED FROM INPUTS

e.g. $\sin^2\theta_W = 1 - \dfrac{M_W^2}{M_Z^2}$ USEFUL QUANTITY.

MAIN (VACUUM POLARISATION) EFFECTS DUE TO:

① 

light fermions

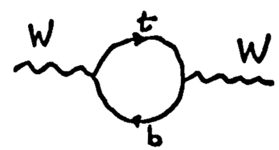
RUNNING α

$$\alpha^{-1}(M_Z) = 128.89 \pm 0.09$$
$$= \alpha^{-1} - \frac{\Delta\alpha}{\alpha}$$

② 

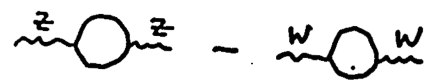
top, W, H

running Z MASS

③ 

running W mass

- actually since M_Z known only difference



appears

④ 

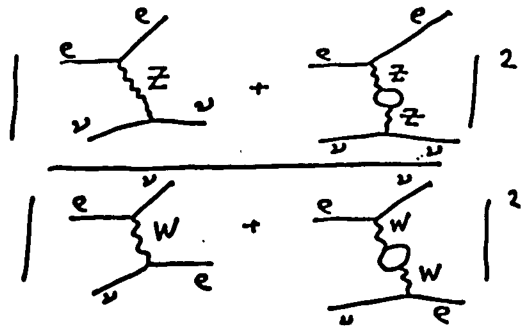
top, W, H

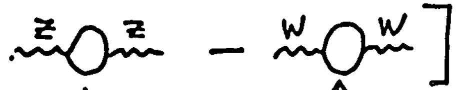
Z γ MIXING

- other diagrams will contribute but main m_t effects from these.

ρ PARAMETER

– Defined as RATIO OF NEUTRAL AND CHARGED CURRENT INTERACTIONS

$\rho \sim$ 

$\approx \rho_0 \Big[1 +$ 

$\Big]$

ρ_0: 1 IN SM; TOP; TOP VELTMAN

$$\approx 1 + \frac{3 G_F m_t^2}{8\pi^2\sqrt{2}} + \ldots$$

DATA $= 1.0044 \pm 0.0016$

$\Rightarrow \quad m_t = 118^{+20}_{-24}$ GeV

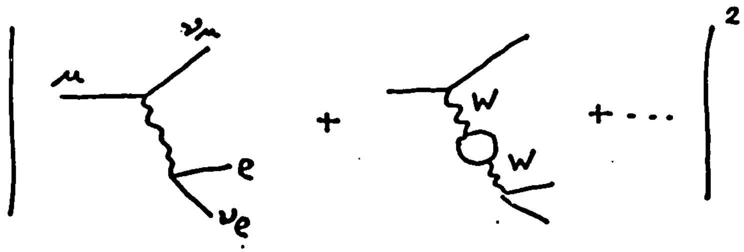
*  EVALUATED AT LOW q^2.

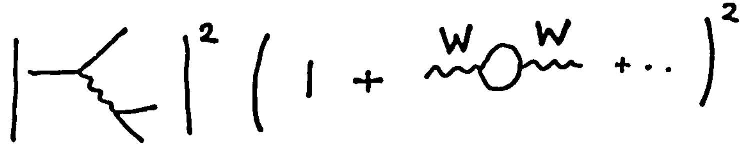
MUON DECAY CONSTANT

– constant of proportionality in μ decay

$$(\text{muon lifetime})^{-1} \equiv \frac{G_F^2 m_\mu^5}{192\pi^3} \Big[f(\alpha, m_e, m_\mu) \Big]$$

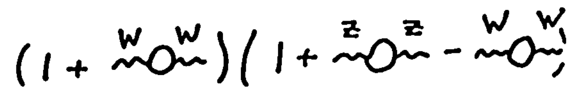
calculable in QED

$=$ 

$=$ 

- G_F "knows" about m_t

Useful combination

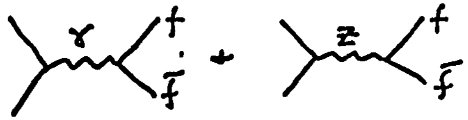
$G_F \rho \sim$ 

$\simeq 1 +$ 

LEP Physics

- BASIC PROCESS

$$e^+ + e^- \longrightarrow f + \bar{f}$$



$$\Rightarrow \quad \frac{d\sigma_0^q}{d\cos\theta} = N\frac{\pi\alpha^2}{2s}\left[(1+\cos^2\theta)\,A_q(s) + 2\cos\theta\, B_q(s)\right]$$

from before

$$A_q(s) = e_q^2 - 2\,e_q v_q v_e\, \chi_1(s) + (v_e^2+a_e^2)(v_q^2+a_q^2)\,\chi_2(s)$$

$$B_q(s) = -2\,e_q a_q a_e\, \chi_1(s) + 4 v_e a_e v_q a_q\, \chi_2(s)$$

$$\chi_1(s) = \frac{\sqrt{2}\,G_F M_Z^2}{16\pi\alpha}\quad \frac{s(s-M_Z^2)}{(s-M_Z^2)^2+\Gamma_Z^2 M_Z^2}$$

$$\chi_2(s) = \left(\frac{\sqrt{2}\,G_F M_Z^2}{16\pi\alpha}\right)^2 \quad \frac{s^2}{(s-M_Z^2)^2+\Gamma_Z^2 M_Z^2}$$

$$\Rightarrow \quad \boxed{\sigma_0^q(s) = \frac{4\pi\alpha^2}{3s}\cdot A_q(s)\cdot N}$$

$\longrightarrow$ LINESHAPE

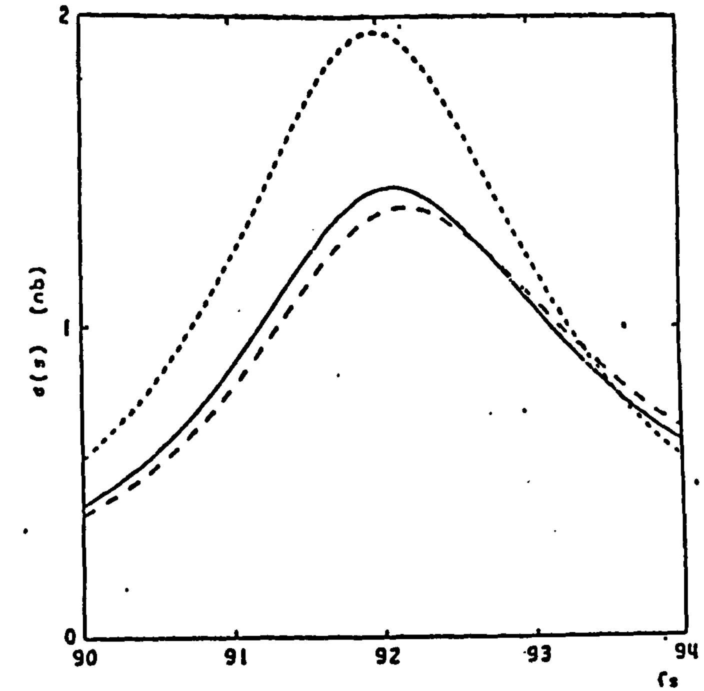


Fig. 2

Muon pair line shape curves including successive stages of corrections: only non-photonic corrections (fine dashed line), first order QED corrections applied to the previous one (dashed line) and second order exponentiated corrections applied to the first curve (solid line). The masses are $M_Z = 92$, $M_H = 100$ and $m_t = 60$ GeV and the minimum s' value is 1 GeV^2.

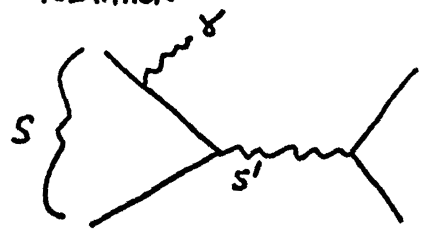
- SEE VERY LARGE QED CORRECTIONS

$$\sigma^f(s) = \int H(s,s')\, \boxed{\sigma_0^f(s')}\, ds'$$

$H(s,s')$: RADIATOR FUNCTION TO DESCRIBE QED EMISSION

$\sigma_0^f(s')$: CALCULATED ABOVE

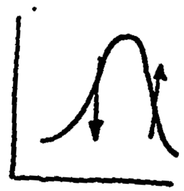
- LARGE EFFECT DUE TO QED INITIAL STATE RADIATION



Photon carries away energy so $s' < s$

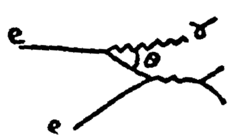
- if $s > M_Z^2$, CAN PUT Z ON SHELL $s' \sim M_Z^2$ $\quad \sigma \uparrow$

$s \sim M_Z^2$, CAN PUT Z OFF SHELL $\quad \sigma \downarrow$



OBSERVED
- PEAK SHIFTS TO HIGHER S $\sim +100\,\text{MeV}$
- needs to be corrected.

• LARGE EFFECT BECAUSE OF COLLINEAR RADIATION



$$\sim \alpha \log \frac{M_Z^2}{M_e^2} \sim 24\alpha$$

Z WIDTH

$$d\Gamma = \frac{1}{2M_Z} \frac{1}{3} |M|^2 \, d\phi$$

$\frac{1}{3}$: AVERAGE OVER Z POLARISATIONS

$$\Gamma_\ell = \frac{G_F M_Z^3}{24\pi\sqrt{2}} \cdot N \cdot \left[1 + (1 - 4|e_\ell| \sin^2\theta_W)^2\right]$$

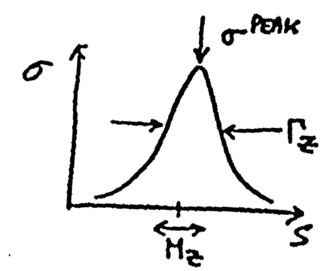
$$\Gamma_Z = \sum_\ell \Gamma_{lep} + \Gamma_{had} + \Gamma_{inv}$$

$$\Gamma_{inv} \to N_\nu \Gamma_\nu = N_\nu\, 165.8\,\text{MeV}$$

$\Rightarrow$ EXTRACTION OF M_Z, Γ_Z, σ_{PEAK} FROM LINESHAPE

IN RESONANCE REGION, IGNORE γ EXCHANGE

$$\sigma_0^f = \sigma_0^{PEAK} \frac{s\Gamma_Z^2}{(s - M_Z^2)^2 + \Gamma_Z^2 M_Z^2}$$



$\Rightarrow$ 3 PARAMETER FIT

$$\sigma_0^{PEAK} = \frac{12\pi \Gamma_e \Gamma_f}{M_Z^2 \Gamma_Z^2} = N_{events}/\mathcal{L}$$

Bhabha scattering $e^+e^- \to e^+e^-$

- MUST INCLUDE ALL GRAPHS
  - LARGE CONTRIBUTIONS FROM S-CHANNEL Z WHEN $s \sim M_Z^2$ AND t-CHANNEL γ EXCHANGE WHEN $\underbrace{t \sim 0}_{\text{i.e. } Q \sim 0}$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}\Big[(1+\cos^2\theta)A_2(s) + 2\cos\theta\, B_2(s) + 2\left(\frac{(1+\cos\theta)^2+4}{(1-\cos\theta)^2}\right) + \text{other terms}\Big]$$

$\hookrightarrow$ FIG

- USE LOW ANGLE TAGGER TO MEASURE QED Bhabha $\Rightarrow$ NORMALISATION OF σ ($\mathcal{L}$)

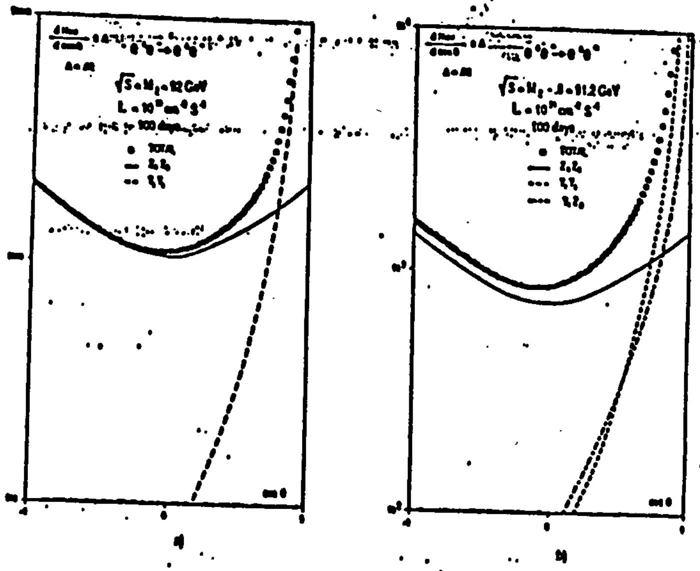


Figure 2 : The differential cross section $d\sigma/d\cos\theta$ for $e^+e^- \to e^+e^-$ at first order

a) at an energy $\sqrt{s} = M_z$

b) at an energy $\sqrt{s} = M_z - .8$

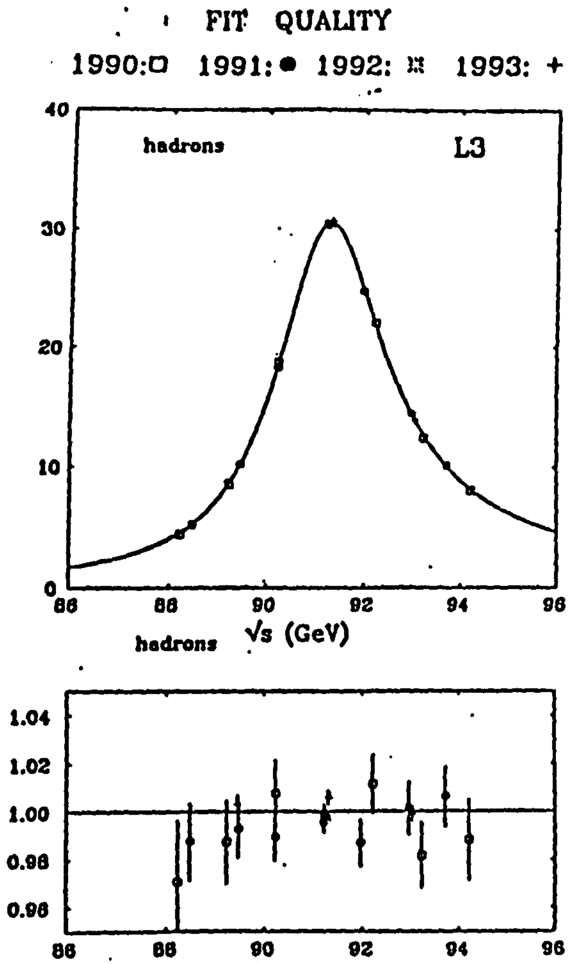
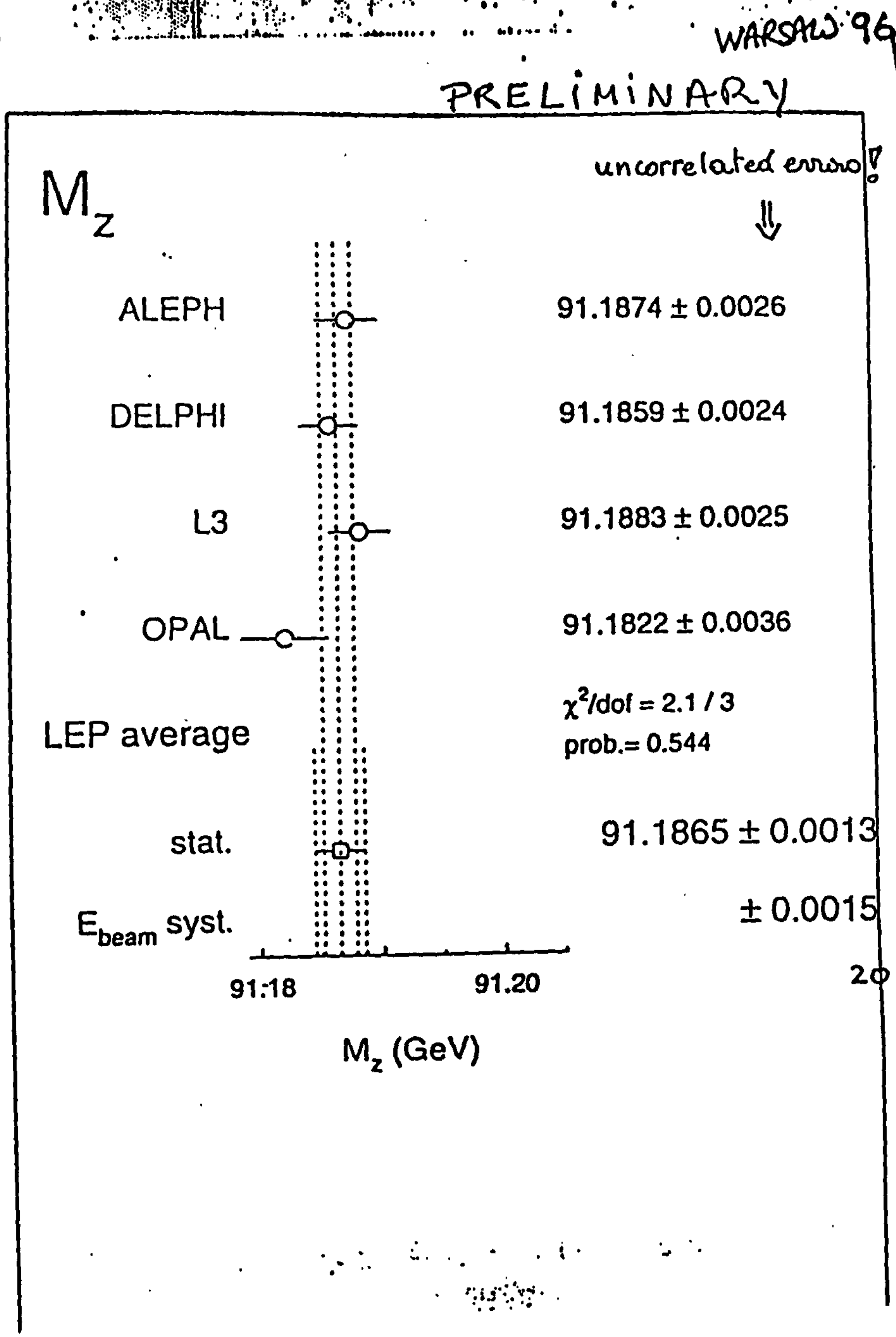
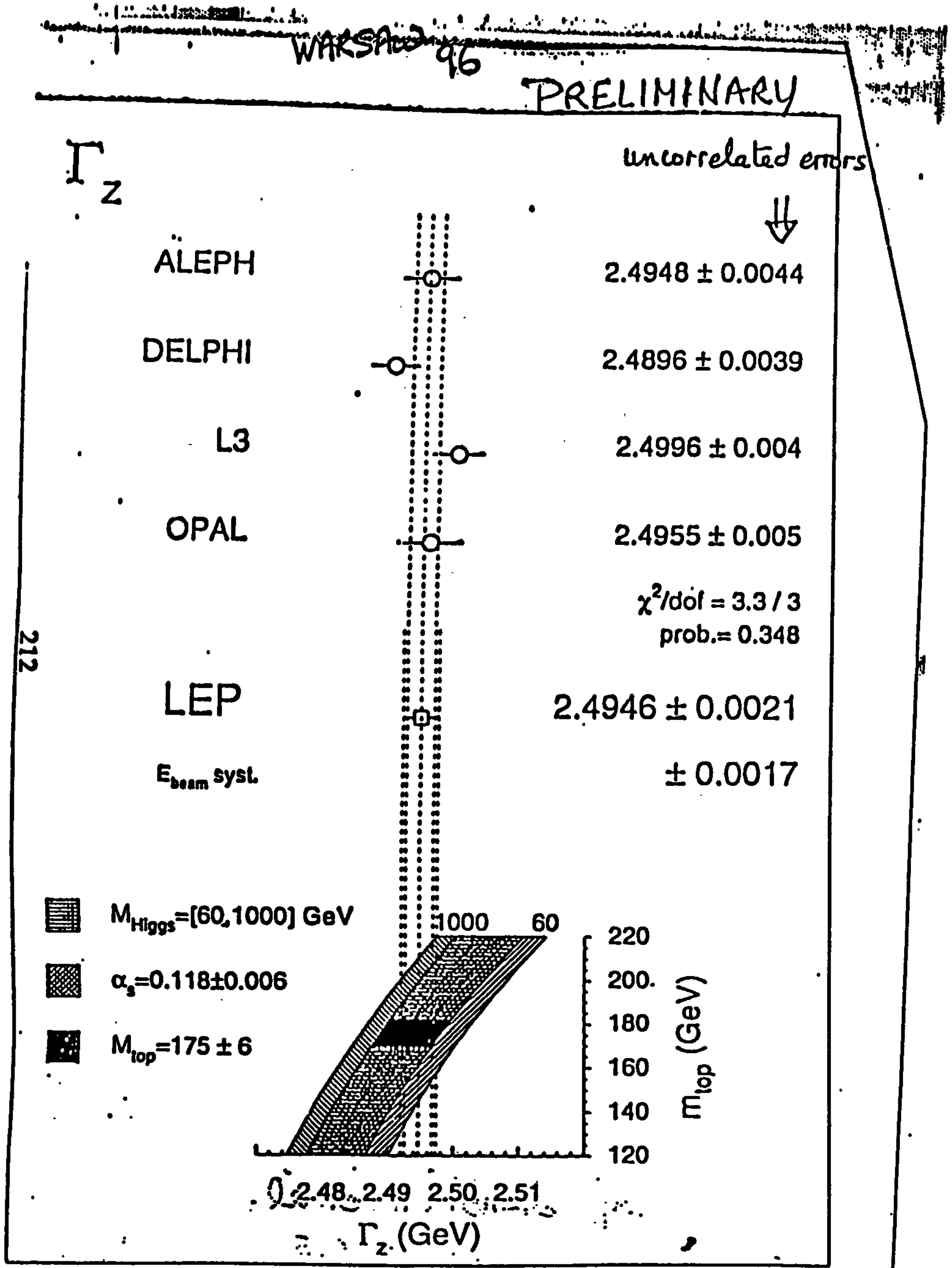

FIT QUALITY
1990:□ 1991:● 1992: ⌘ 1993: +
hadrons
L3
σ (nb)
√s (GeV)
hadrons
σmeas/σfit
√s (GeV)

Fig. 1(a)


WARSAW 96
PRELIMINARY
M_Z
uncorrelated errors!
ALEPH
91.1874 ± 0.0026
DELPHI
91.1859 ± 0.0024
L3
91.1883 ± 0.0025
OPAL
91.1822 ± 0.0036
χ^2/dof = 2.1 / 3
prob.= 0.544
LEP average
stat.
91.1865 ± 0.0013
E_{beam} syst.
± 0.0015
91.18
91.20
M_z (GeV)



FORWARD - BACKWARD ASYMMETRY.

$$A^f_{FB} = \frac{\int_0^1 \frac{d\sigma}{d\cos\theta}\, d\cos\theta - \int_{-1}^{0} \frac{d\sigma}{d\cos\theta}\, d\cos\theta}{\int_0^1 \frac{d\sigma}{d\cos\theta}\, d\cos\theta + \int_{-1}^{0} \frac{d\sigma}{d\cos\theta}\, d\cos\theta}$$

$$= \frac{3}{4}\,\frac{B_f(s)}{A_f(s)}$$

$s \sim M_Z^2$

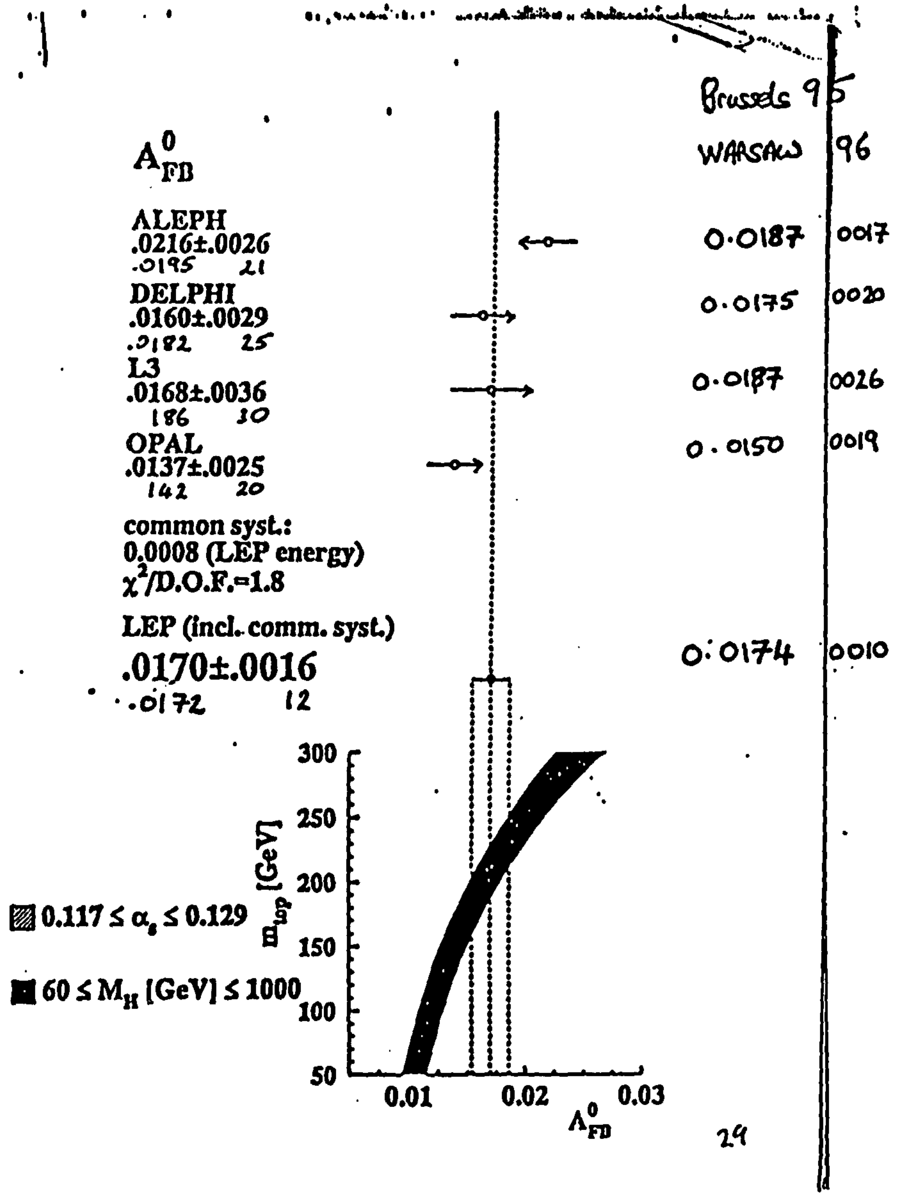
$$A^f_{FB} \approx \frac{3}{4} \cdot \frac{2 v_e a_e}{(v_e^2 + a_e^2)} \cdot \frac{2 v_f a_f}{(v_f^2 + a_f^2)}$$

so that for leptons

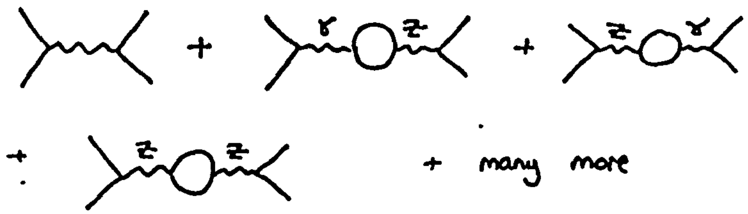
$$\boxed{A^{\ell}_{FB} = 3 \cdot \frac{v_e^2 a_e^2}{(v_e^2 + a_e^2)^2}}$$

cf leptonic width of Z

$$\boxed{\Gamma_\ell = \frac{G_F M_Z^3}{24\pi\sqrt{2}}\,(v_e^2 + a_e^2)}$$



ELECTROWEAK CORRECTIONS



— these universal self energy contributions can be absorbed into definitions

$$\Rightarrow \quad \boxed{\Gamma_\ell = \frac{G_F M_Z^3}{24\pi\sqrt{2}}\,\rho\,\left[1+(1-4\,k\sin^2\theta_W)^2\right]}$$

ρ parameter   contains 

$$\rho \sim 1 + \frac{3\,G_F m_t^2}{8\pi^2\sqrt{2}} = 1 + \Delta\rho$$

$$K \sim 1 + \frac{\cos^2\theta_W}{\sin^2\theta_W}\,\Delta\rho$$

$$\boxed{A_{FB}^{\ell} = 3\,\frac{\left[1-4K\sin^2\theta_W\right]^2}{\left[1+(1-4K\sin^2\theta_W)^2\right]^2}}$$

• ONLY K

Measurement	$\sin^2\theta_{eff}^{lept}$
LEP:	
$A_{FB}^{0,\ell}$	0.2311 ± 0.0009
$\mathcal{A}_\tau$	0.2320 ± 0.0013
$\mathcal{A}_e$	0.2330 ± 0.0014
$A_{FB}^{0,b}$	0.2327 ± 0.0007
$A_{FB}^{0,c}$	0.2310 ± 0.0021
$\langle Q_{FB}\rangle$	0.2320 ± 0.0016
Average LEP:	0.2321 ± 0.0004
SLC:	
A_{LR} (SLD)	0.2294 ± 0.0010
Average LEP+SLC:	0.2317 ± 0.0004

WARSAW 96

0.23209 ± 0.0003

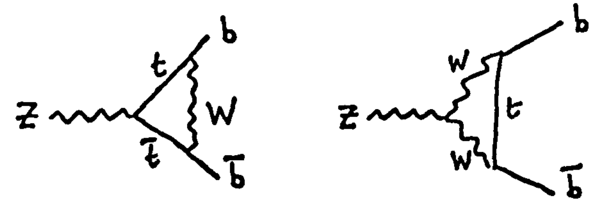
I USED $\sin^2\theta_W = 1 - \dfrac{M_W^2}{M_Z^2}$

OFTEN $\boxed{\sin^2\theta^{EFF} = K\sin^2\theta_W}$

So $A_{FB}^{\ell} = 3\dfrac{\left[1-4\sin^2\theta^{EFF}\right]^2}{\left[1+(1-4\sin^2\theta^{EFF})^2\right]^2}$

$$R_b = \Gamma_{b\bar{b}}/\Gamma_{had}$$

- ADDITIONAL m_t DEPENDENCE FROM VERTICES



$\Rightarrow$ $\Gamma_{b\bar{b}} = \dfrac{N G_F M_Z^3}{24\pi\sqrt{2}}\rho_b\left(1+\left(1-\frac{4}{3}K_b\sin^2\theta_W\right)^2\right)$

$\rho_b = 1 - \frac{1}{3}\Delta\rho$

$K_b = 1 + \left(\dfrac{\cos^2\theta_W}{\sin^2\theta_W} + \frac{2}{3}\right)\Delta\rho$

$\longrightarrow$ FIG.

- ALSO m_b mass effects — reduce R_b by ~10%

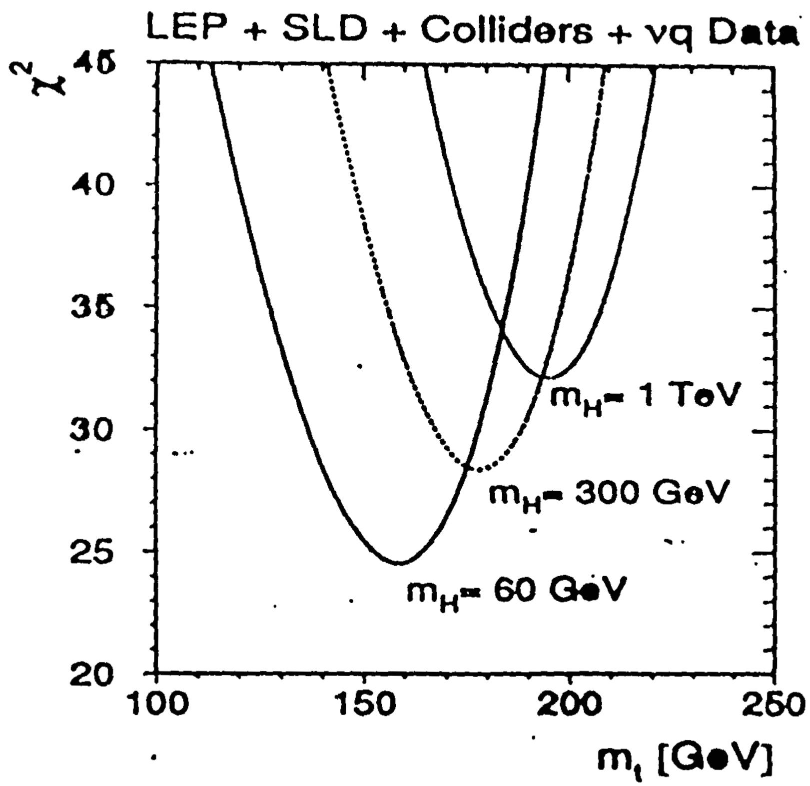
WARSAW 96

	Measurement with Total Error	Standard Model	Pull
a) LEP			
line-shape and lepton asymmetries:			
m_Z [GeV]	91.1863 ± 0.0020	91.1861	0.1
Γ_Z [GeV]	2.4946 ± 0.0027	2.4960	−0.5
σ_h^0 [nb]	41.508 ± 0.056	41.465	0.8
R_ℓ	20.778 ± 0.029	20.757	0.7
$A_{FB}^{0,\ell}$	0.0174 ± 0.0010	0.0159	1.4
+ correlation matrix Table 8			
τ polarization:			
$\mathcal{A}_\tau$	0.1401 ± 0.0067	0.1458	−0.9
$\mathcal{A}_e$	0.1382 ± 0.0076	0.1458	−1.0
b and c quark results:			
R_b [(a)]	0.2179 ± 0.0012	0.2158	1.8 *
R_c [(a)]	0.1715 ± 0.0056	0.1723	−0.1 *
$A_{FB}^{0,b}$ [(a)]	0.0979 ± 0.0023	0.1022	−1.8
$A_{FB}^{0,c}$ [(a)]	0.0733 ± 0.0049	0.0730	0.1
+ correlation matrix Table 13			
$q\bar{q}$ charge asymmetry:			
$\sin^2\theta_{eff}^{lept}$ ($\langle Q_{FB}\rangle$)	0.2320 ± 0.0010	0.23167	0.3
b) SLD			
$\sin^2\theta_{eff}^{lept}$ ($\mathcal{A}_{LR}$ [62,63])	0.23061 ± 0.00047	0.23167	−2.2
R_b [6][(a)]	0.2149 ± 0.0038	0.2158	−0.2
$\mathcal{A}_b$ [7]	0.863 ± 0.049	0.935	−1.4
$\mathcal{A}_c$ [7]	0.625 ± 0.084	0.667	−0.5
c) $p\bar{p}$ and νN			
m_W [GeV] ($p\bar{p}$ [73])	80.356 ± 0.125	80.353	0.3
$1 - m_W^2/m_Z^2$ (νN [8–10])	0.2244 ± 0.0042	0.2235	0.2
m_t [GeV] ($p\bar{p}$ [15–17])	175 ± 6	172	0.5

Table 20: Summary of measurements included in the combined analysis of Standard Model parameters. Section a) summarizes LEP averages, Section b) SLD results for $\sin^2\theta_{eff}^{lept}$ from the measurement of the left-right polarization asymmetry, for R_b and for $\mathcal{A}_b$ and $\mathcal{A}_c$ from polarized forward-backward asymmetries and Section c) electroweak precision measurements from $p\bar{p}$ colliders and νN scattering. The Standard Model results in column 2 and the pulls (difference between measurement and fit in units of the total measurement error) in column 3 are derived from the Standard Model fit including all data (Table 22, column 2) with the Higgs mass treated as a free parameter.

[(a)] For fits which combine LEP and SLD heavy flavor measurements we use as input the heavy flavor results given in Equation (11) and their correlation matrix in Table 14 in Section 4 of this note.

CORRECTIONS $\sim \; a\, m_t^2 - b\, M_Z^2 \log(M_H^2/M_Z^2)$

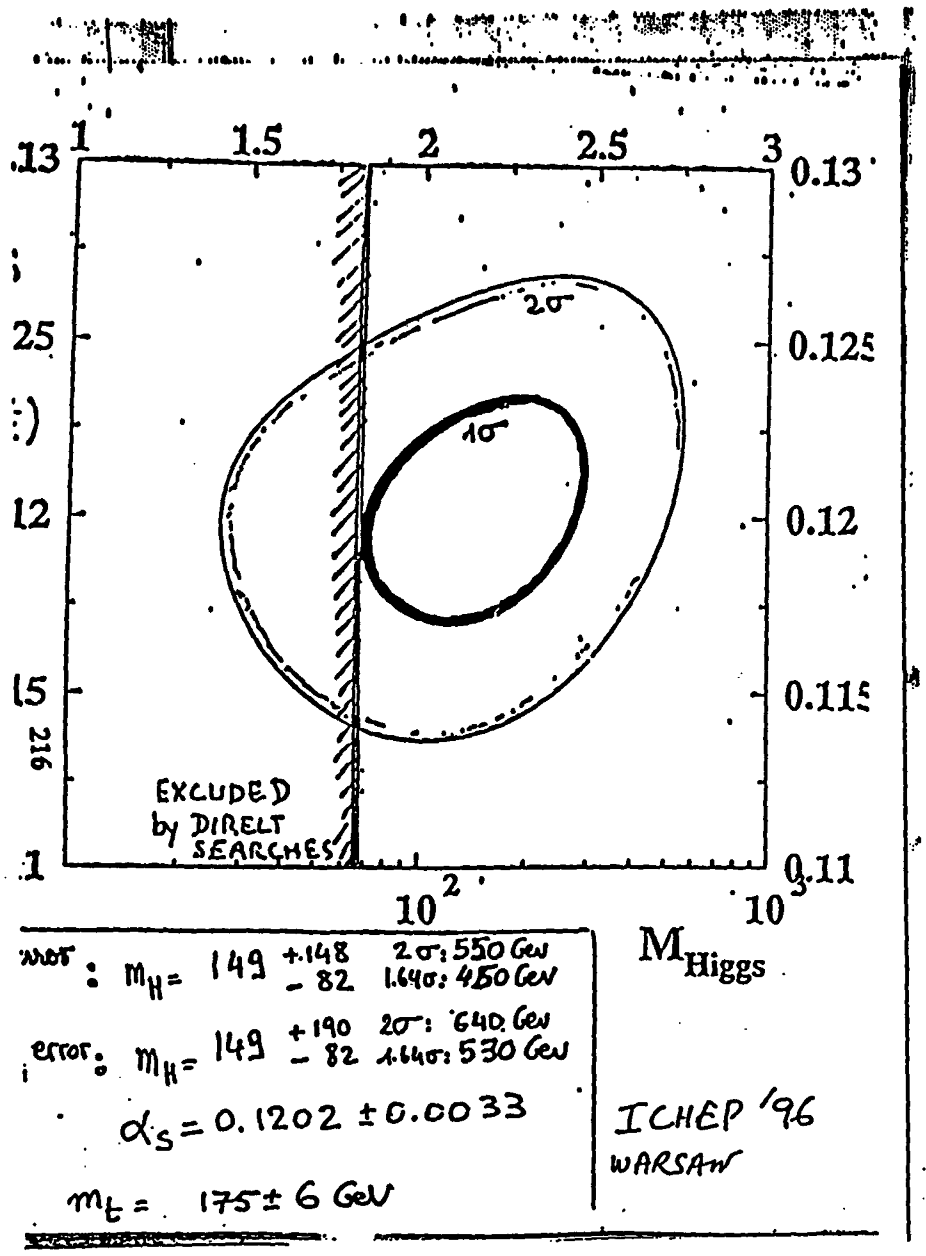


FITS TO ALL DATA

$$\boxed{\begin{aligned} m_t &= 171 \pm 11 \text{ GeV} \\ m_H &= 93 {}^{+189}_{-63} \text{ GeV} \\ \alpha_s(M_Z) &= 0.122 \pm 0.005 \end{aligned}}$$

POKORSKI + CHANKOWSKI

$\chi^2 = 25/14$



6. LEPII Physics

- **Standard Model Processes at LEPII**
- **Radiative return to the *Z* - a back of the envelope calculation**
- ***W* pair production**
- ***W* mass measurement**
- **Anomalous gauge boson couplings**

STANDARD MODEL PROCESSES AT LEP 2

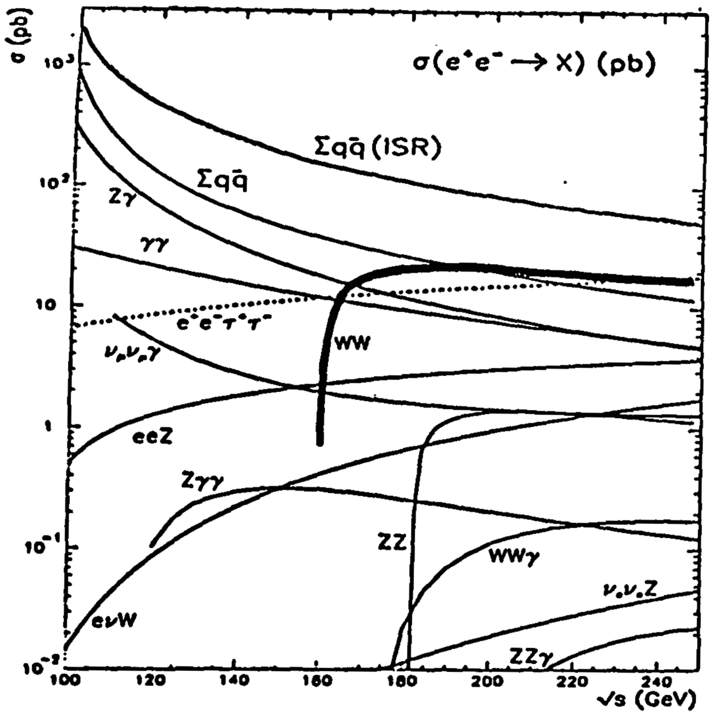
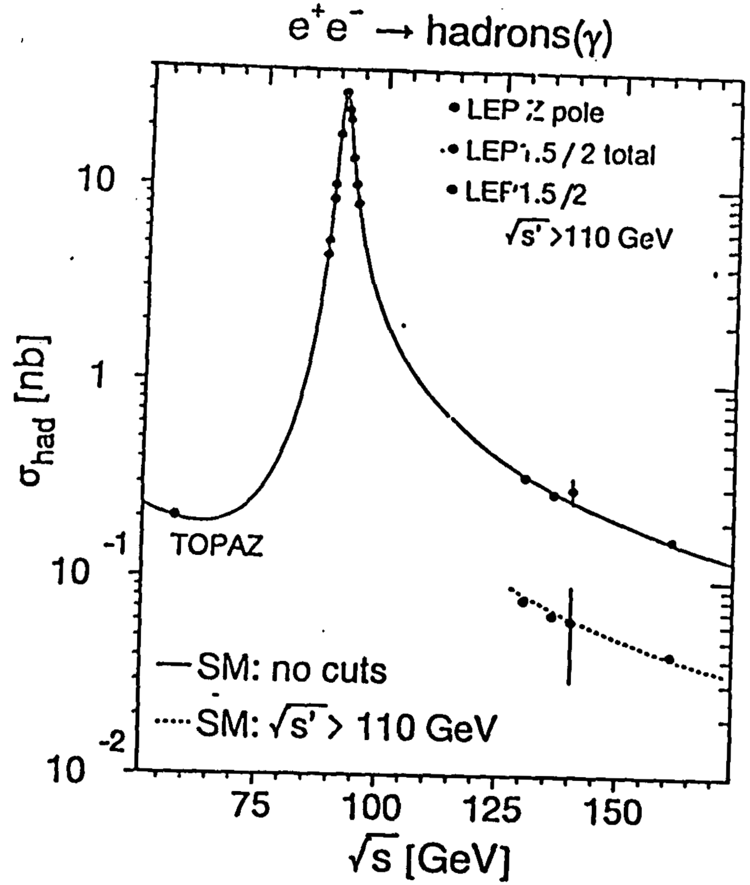
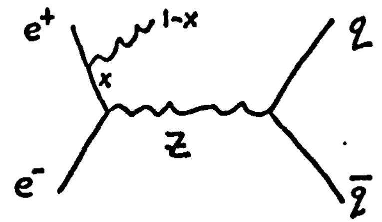


Figure 1: *Cross sections for some typical standard model processes.*
For $e^+e^- \rightarrow e^+e^-Z, e\nu_e W, \nu_e\bar{\nu}_e Z$ only the dominant t-channel contribution is shown. The photons in $Z\gamma$ and $\gamma\gamma$ are such that $|\cos_{e\gamma}| < 0.9$. For $\nu_\mu\bar{\nu}_\mu\gamma$ there is the additional cut $E_\gamma > 10GeV$. In $Z\gamma\gamma$, $W^+W^-\gamma$ and $ZZ\gamma$ the photon cut is $p_T^\gamma > 10GeV$ and all particles are separated with opening angles: $\widehat{eV} > 15^\circ, \widehat{VV'} > 10^\circ; V = W, Z, \gamma$.



RADIATIVE RETURN TO Z

$s \sim 4M_W^2$



$s' \sim M_Z^2$

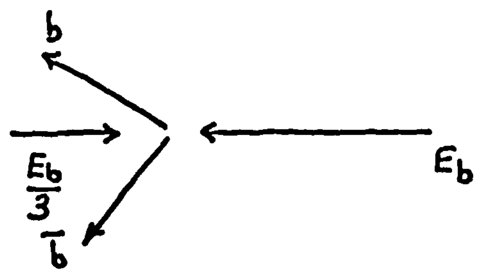
- single photon radiation at v. small angles

$$s = (p_{e^+} + p_{e^-})^2 = 2\,p_{e^+}\cdot p_{e^-}$$

$$s' = (x\,p_{e^+} + p_{e^-})^2 = 2x\,p_{e^+}\cdot p_{e^-} = xs$$

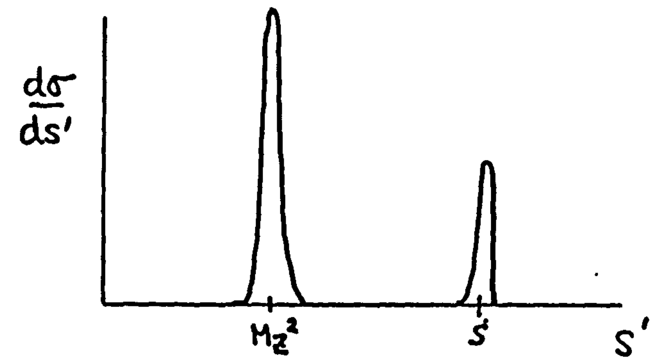
$$\Rightarrow \quad x \approx \frac{M_Z^2}{4M_W^2} \approx \frac{1}{3}$$

i.e. $E_\gamma \sim (1-x)\,E_{beam} \approx 50\ \text{GeV}$



i.e. LEP 2 + rad. return looks like asym B factory

but only 3k $b\bar{b}$ events/year



- get peak at $s' = s$ (no radiation)
  + $s' \sim M_Z^2$ (on-shell Z production)

FROM DATA: $\dfrac{\sigma(s' \sim M_Z^2)}{\sigma(s' \sim s)} \approx \boxed{}$

CAN WE ESTIMATE THIS RATIO?

⓪ What fraction of cross section at $s \sim 4M_W^2$ is due to Z exchange?

– photon exchange contribution doesn't do radiative return to Z!

$$\sigma_\gamma(s) = \frac{4\pi\alpha^2}{3s} N \sum e_q^2 = \boxed{}$$

$$\sigma_Z(s) \approx \sigma_{peak} \frac{s\,\Gamma_Z^2}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2} \nearrow 0 = \boxed{}$$

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so for $s \sim 4M_W^2$

$$\frac{\sigma_Z}{\sigma_\gamma} \approx \sigma_{peak} \cdot \frac{3s^2\Gamma_Z^2}{4\pi\alpha^2 N \sum e_q^2\, (s-M_Z^2)^2}$$

$$= \boxed{}$$

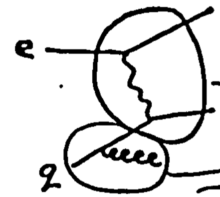
1 nb = $2.57\; 10^{-5}$ GeV^{-2}

i.e $\boxed{}$ % Z exchange

$$\Rightarrow \sigma_{\gamma+Z}(s) = \sigma_\gamma\left(1 + \frac{\sigma_Z}{\sigma_\gamma}\right) = \boxed{}$$

① Collinear behaviour of cross section

RECALL

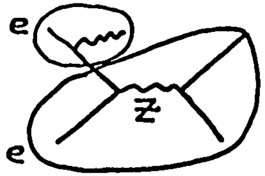


IN COLLINEAR LIMIT

$$d\sigma_{eq\to eqg} = \frac{\alpha_s}{2\pi}\,\frac{N^2-1}{2N}\,\frac{1+x^2}{1-x}\,\frac{dx}{x}\,\frac{dt}{t}\,\sigma_{eq\to eq}$$

USED FOR DIS

SAME COLLINEAR PHYSICS



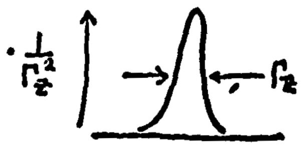
$$\underbrace{d\sigma^{Z}_{e^+e^-\to q\bar{q}\gamma}}_{\text{change in cross section due to }\gamma\text{ emission}} = 2\,\frac{\alpha}{2\pi}\,\frac{1+x^2}{1-x}\,\frac{dx}{x}\,\frac{dt}{t}\,\underbrace{\sigma_{e^+e^-\to q\bar{q}}(s')}_{\text{hadronic cross section at } s'=xs}$$

– photon could be radiated off either e^+ or e^-

② In the region of the Z

$$\sigma_{e^+e^-\to had}(s') \approx \sigma_{peak} \frac{s'\Gamma_Z^2}{(s'-M_Z^2)^2+\Gamma_Z^2 M_Z^2}$$

<u>but</u> for narrow resonances Breit-Wigner looks like δ fn



$$\frac{1}{(s'-M_Z^2)^2+\Gamma_Z^2 M_Z^2} \sim \overset{\text{FIXES AREA}}{\frac{\pi}{M_Z\Gamma_Z}}\ \delta(s'-M_Z^2)$$

WILL FIX $s' = xs = M_Z^2$

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SO

$$\sigma_{had}(s') \approx \sigma_{peak}\ \pi \frac{M_Z\Gamma_Z}{s}\ \delta\left(x-\frac{M_Z^2}{s}\right)$$

AND $\qquad$ σ_{peak} from previous page

$$\sigma_{had}(s') \approx \sigma_Z(s)\frac{(s-M_Z^2)^2}{s\Gamma_Z^2}\ \pi\frac{M_Z\Gamma_Z}{s}\ \delta\left(x-\frac{M_Z^2}{s}\right)$$

$$\doteq \sigma_Z(s)\ (1-x)^2\ \frac{\pi M_Z}{\Gamma_Z}\ \delta\left(x-\frac{M_Z^2}{s}\right)$$

③ Range of t integration

$$t = (p_\gamma - p_e)^2 = -2EE_\gamma(1-\cos\theta)$$

$$t_{min} = -m_e^2 \qquad t_{max} = -2E^2(1-x)(1-\cos\theta)$$

$$\Rightarrow \quad L = \int\frac{dt}{t} \simeq \log\frac{2E^2(1-x)(1-\cos\theta)}{m_e^2}$$

$E \sim M_W$, $x \sim 1/3$ $\qquad \cos\theta \sim \boxed{}$

$$\Rightarrow \quad L = \boxed{}$$

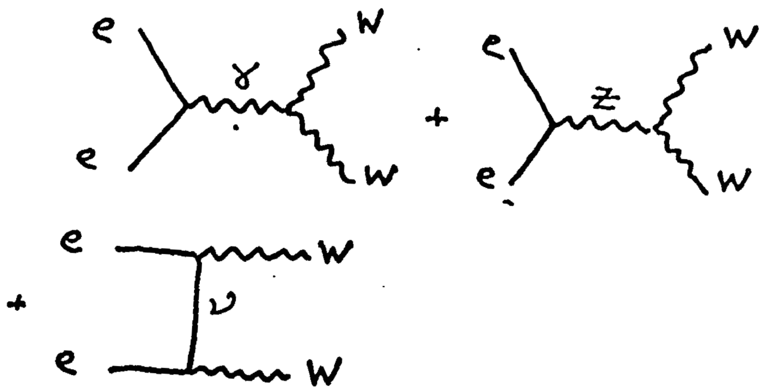
SO

$$\frac{\sigma^{\text{RAD RET}}}{\sigma_Z(s)} = 2\frac{\alpha}{2\pi}\frac{1+x^2}{1-x}\frac{(1-x)^2}{x}\frac{\pi M_Z}{\Gamma_Z}L$$

$$= \boxed{}$$

$$\Rightarrow \quad \frac{\sigma(s'\sim M_Z^2)}{\sigma(s'\sim s)} \approx \boxed{}$$

$e^+e^- \longrightarrow W^+W^-$



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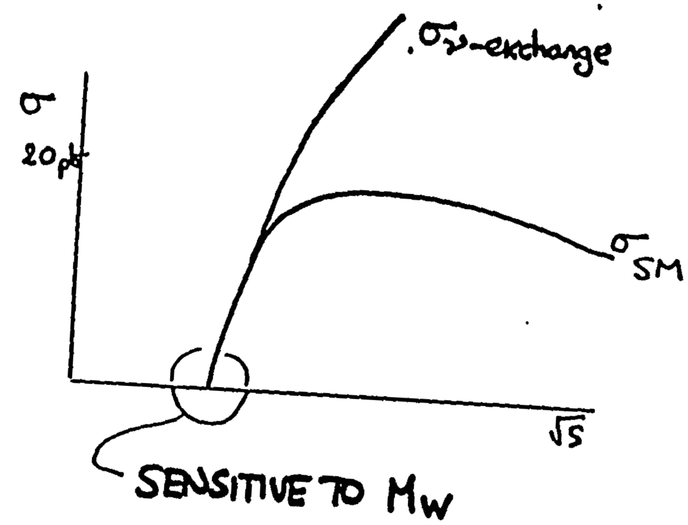
– very large cancellations between different terms

☆ $\sigma_{\nu\text{-exch}} \approx \dfrac{G_F^2 s}{48\pi}$ $\qquad s \gg M_W^2$

$\Rightarrow$ VIOLATES UNITARITY

☆ $\sigma_{SM} \approx \dfrac{G_F^2 M_W^4}{s\pi} \ln\left(\dfrac{s}{M_W^2}\right)$ $\qquad s \gg M_W^2$

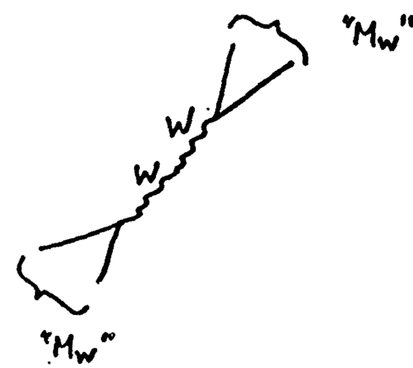
– because of v. large cancellations sensitive to TRIPLE GAUGE BOSON vertices



- Near threshold

☆ $\sigma_{SM} \sim \dfrac{2 G_F^2 M_W^4}{s\pi} \underbrace{\sqrt{1 - \dfrac{4M_W^2}{s}}}_{\text{velocity of W in CM}}$

- Away from threshold



can try to combine particles to reconstruct W mass

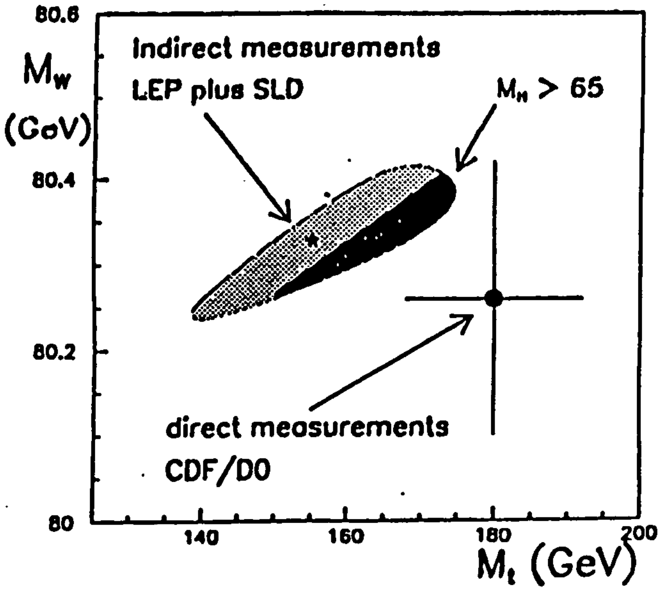
I. IMPORTANCE OF M_W

$+ G_F$. OVERCONSTRAINS SM

e.g. $\sin^2\theta_{EFF} = K \sin^2\theta_W$

$$\left(1 + \frac{\cos^2\theta_W}{\sin^2\theta_W}\Delta\rho\right)\left(1 - \frac{M_W^2}{M_Z^2}\right)$$

as $m_t \uparrow$ so $M_W \uparrow$

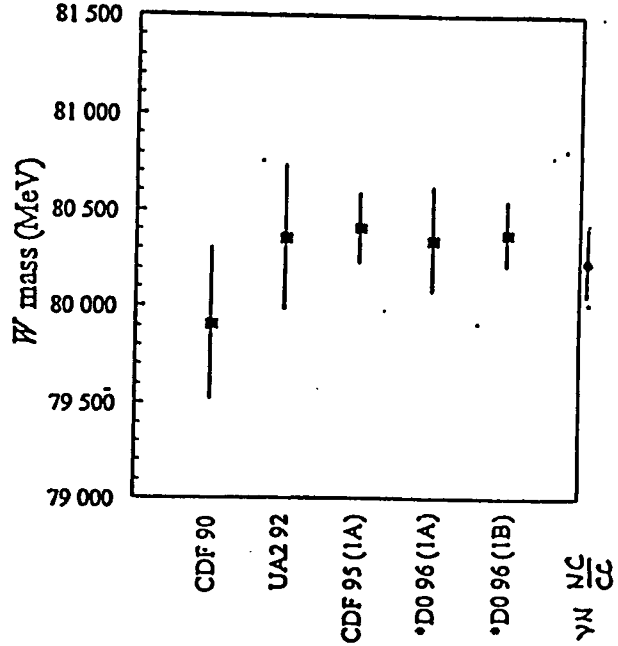


M_W STATUS

WARSAW

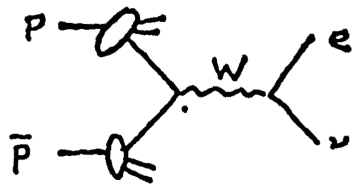
Other recent W mass measurements:

Experiment	m(W) (MeV)	stat	syst	scale	total error
CDF 90	79 910	292	227		390
UA2 92	80 360	220	297		370
CDF 95 (1A)	80 410	118	136		180
*D0 96 (1A)	80 350	140	160	160	270
*D0 96 (1B)	80 380	70	130	80	170
	*preliminary				
νN WA	80 220				210



also HERA verifies W propagator $\rightarrow 79 \pm 4$ GeV

W MASS IN $p\bar{p}$ COLLISIONS



single production

$$\sigma = \int dx_1\, dx_2\, f_1(x_1,\mu^2)\, f_2(x_2,\mu^2)\quad \hat{\sigma}_{12\to e\nu}$$

same parton density functions as DIS

parton level cross section

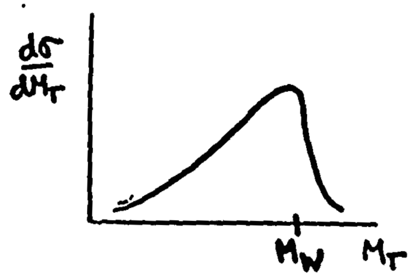
$u\bar{d} \to e^+\nu$
$d\bar{u} \to e^-\bar{\nu}$

Observable high p_T lepton + missing momentum

e^+

longitudinal info lost

ν_T $\Rightarrow M_T^2(e,\nu) = 2p_{Te}p_{T\nu}(1-\cos\phi)$



- sensitive to input PDF's

$\Delta M_W = 120\,\mathrm{MeV}$

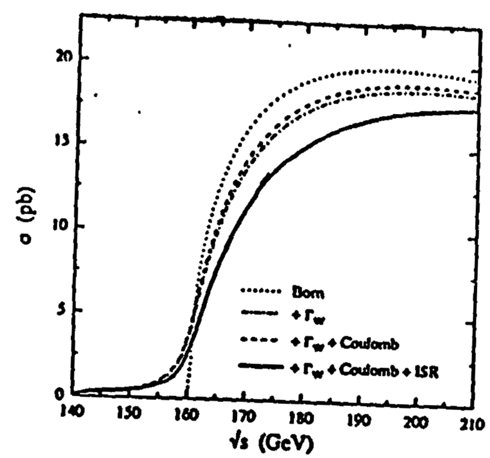
both expts run 1B

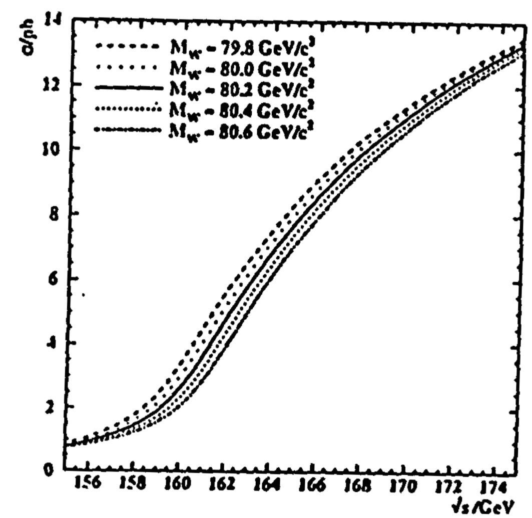
$\Delta M_W = 70\,\mathrm{MeV}$

main injector

W MASS FROM THRESHOLD

$\sqrt{s} = 161\,\mathrm{GeV}$





4 expts @ $25\,pb^{-1}$ $\longrightarrow$ $\Delta M_W = 144\,\mathrm{MeV}$

W MASS FROM RECONSTRUCTION

$\sigma_{WW} \sim 15\,pb$ $\qquad \sqrt{s} \sim 172\,GeV$

$\Gamma_W \sim 2\,GeV$ $\qquad \sigma_{q\bar{q}} \sim 30\,pb$

$Br(W \to hadrons) \sim 2/3$

$Br(W \to \ell\nu) \sim 1/9$

e.g

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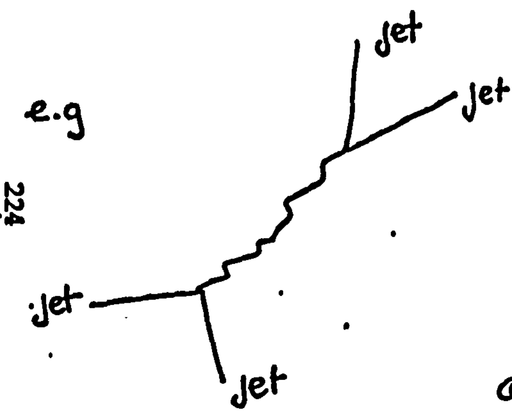


MUST ASSOCIATE RIGHT PARTICLES INTO RIGHT JET

$+ \; M_{jj} \approx M_W$

QCD background tends to have softer 4th jet

$\Rightarrow$ eliminate with cut

$q\bar{q}q\bar{q}$ $\quad$ 4 @ $500\,pb^{-1}$ @ $175\,GeV \Rightarrow \Delta M_W = 45\,MeV$

$q\bar{q}\ell\nu$ $\quad$ $44\,MeV$

$34\,MeV$

INTERCONNECTION

- usually assume W hadronic decay products don't interfere

EVENT PICTURE

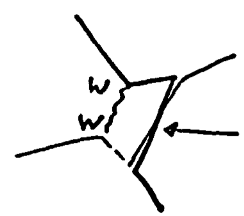
W's are colourless so each makes a separate colour antenna

- can gluon radiation from one antenna feel presence of other antenna

spatial separation $\sim 1/\Gamma_W$

$\Rightarrow$ any gluon $E_g > \Gamma_W$ $\quad \left(\lambda \sim 1/E_g < \frac{1}{\Gamma_W}\right)$
doesn't see other antenna

<u>but</u> $E_g < \Gamma_W$ DOES



May mix up assignment of hadrons to jets !

ANOMALOUS COUPLINGS

– introduce extra (non gauge invariant) interactions

$$\mathcal{L} = g_{WWV}\Big[\, g_1^V V^\mu \big(W^-_{\mu\nu} W^{+\nu} - W^+_{\mu\nu} W^{-\nu}\big) + k_V W^+_\mu W^-_\nu V^{\mu\nu} + \frac{\lambda_V}{M_W^2} V^{\mu\nu} W^{+\rho}_\nu W^-_{\rho\mu} + \text{terms with } \varepsilon_{\mu\nu\rho\sigma} \Big]$$

$g_{WW\gamma} = e$

$g_{WWZ} = e \cot\theta_W$

$V^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu \qquad V = \gamma \text{ or } Z$

in SM $\quad g_1^\gamma = g_1^Z = k_\gamma = k_Z = 1$

$\lambda_V = 0$

– NON SM VALUES CHANGE σ AND SHAPE OF EVENTS

FROM FNAL $W\gamma$ events $|\Delta k| \lesssim 1 \quad |\Delta\lambda| \lesssim 0.5$

LEP-II $\quad 4 @ 500\,pb^{-1} \longrightarrow$ error ~ 0.04

7. Higgs Physics

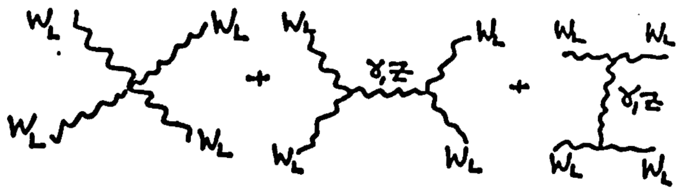
- General arguments
- Experimental bounds on m_H and m_t
- Theoretical bounds on m_H and m_t
- Higgs Decays
- Higgs search at LEP-I
- Higgs search at LEP-II
- Higgs search at LHC

HIGGS PHYSICS

– A "HIGGS" MUST EXIST !

– IF NOT UNITARITY VIOLATED

e.g. SCATTERING OF LONGITUDINALLY POLARISED W BOSONS

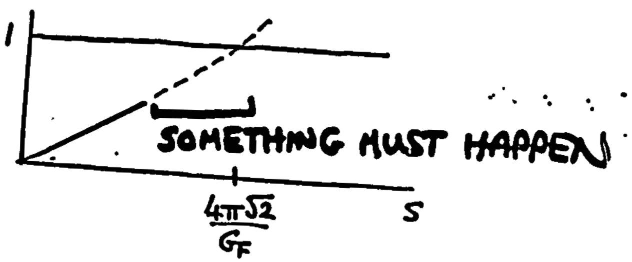


$$\underbrace{\qquad\qquad\qquad\qquad}_{\mathcal{M} \sim s}$$

ie $\sigma \sim \frac{1}{2s} |\mathcal{M}|^2 d\phi \sim s$

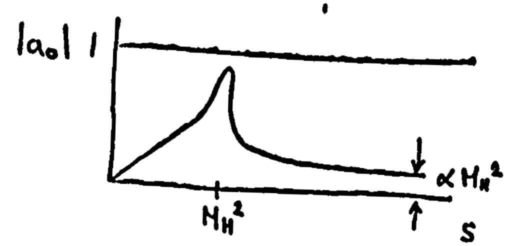
– this violates UNITARITY (PROBABILITY OF SCATTERING > 1 !)

J=0 partial wave $|a_0|$

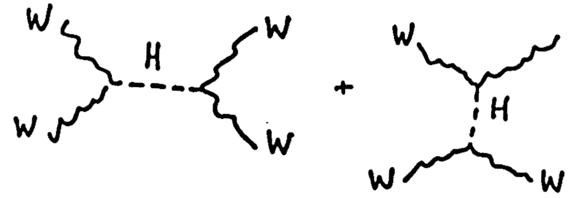


$s \sim \frac{4\pi\sqrt{2}}{G_F} = (1.2\,\text{TeV})^2$

$|a_0|$



Higgs exchange graphs prevent unitarity violation
$M_H^2 \lesssim \frac{8\pi\sqrt{2}}{3G_F} \sim (1\,\text{TeV})^2$.

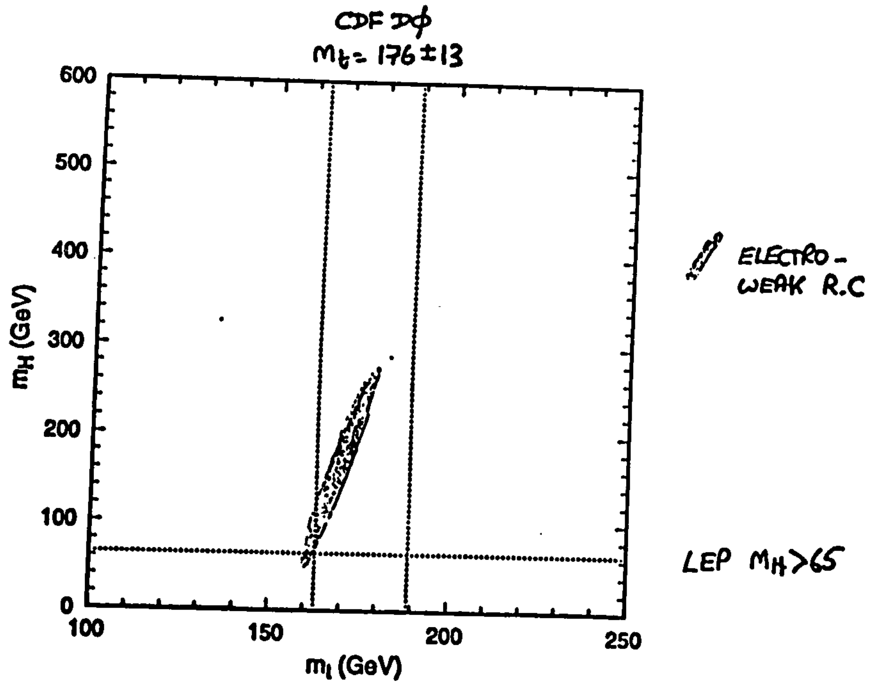


– also seen in $t\bar{t} \to W_L W_L$

$\Rightarrow$ "NEW PHYSICS (HIGGS) COUPLING TO MASSES MUST OCCUR ON A SCALE OF O(1TeV)"

BOUNDS ON $m_H - m_t$

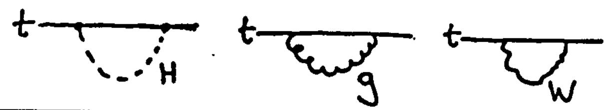
1) EXPERIMENTAL



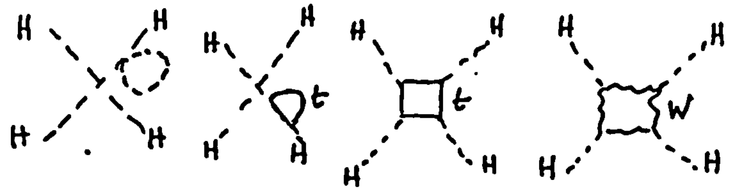
2) THEORETICAL

- JUST AS STRONG COUPLING CONSTANT AND BOSON MASSES

  THE TOP QUARK MASS AND HIGGS SELF COUPLING ALSO RUN



- $$\boxed{\frac{d m_t^2}{d \ln \mu^2} \sim a m_t^4 - b m_t^2 \alpha_s - c m_t^2 \alpha_W \ldots}$$



- $$\boxed{\frac{d\lambda}{d \ln \mu^2} \sim a' \lambda^2 + b' \lambda m_t^2 + c' m_t^4 + \ldots}$$

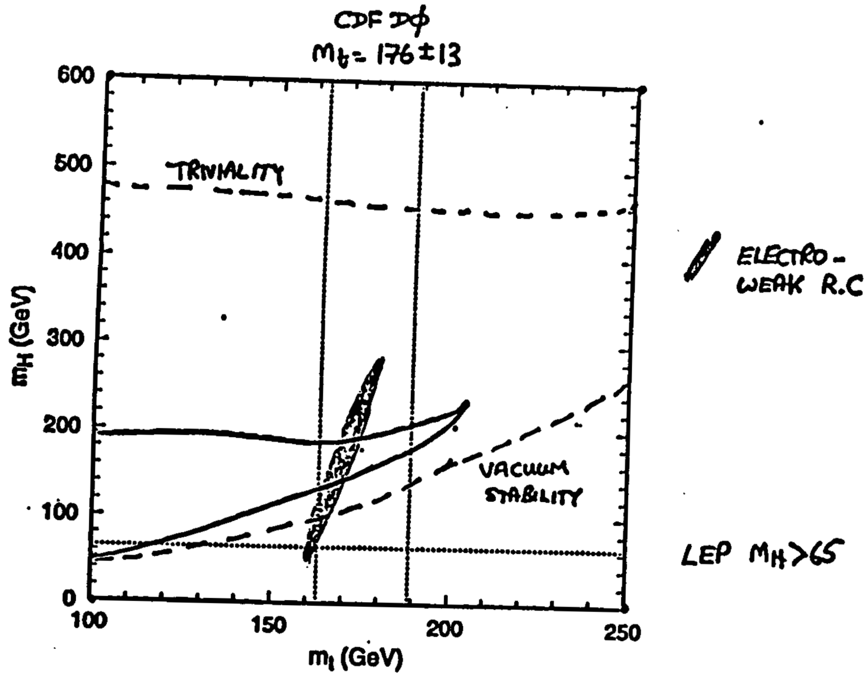
<u>BUT</u> 1) minimum of $V(\phi)$ must still be at $\frac{v}{\sqrt{2}}$ VACUUM STABILITY

2) $\lambda(\mu)$ MUST NOT DIVERGE $\boxed{\mu < \Lambda}$

- IF IT DOES THEN $\lambda(v) \to 0$
- NO SPONTANEOUS SYMMETRY BREAKING TRIVIALITY.

Bounds on $m_H - m_t$

1) EXPERIMENTAL



—— NO NEW PHYSICS SCALES $< 10^{19}$ GeV

- - - - NO NEW PHYSICS SCALES $< 10^{3}$ GeV

HIGGS DECAYS

1) DECAY TO FERMIONS

Vertex: $-\dfrac{ig\, m_f}{2M_W}$, outgoing p_1, p_2

$$\mathcal{M} = \bar{u}(p_1)\, v(p_2)\, \frac{g\, m_f}{2M_W}$$

$$\Rightarrow \quad \sum |M|^2 = \frac{g^2 m_f^2}{4M_W^2} \cdot \mathrm{Tr}\Big[\underbrace{(\not{p}_1 + m_f)(\not{p}_2 - m_f)}_{4p_1\cdot p_2 - 4m_f^2}\Big]$$

$$= 2(m_H^2 - 4m_f^2)$$

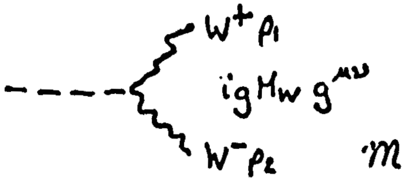
$$\Rightarrow \quad \Gamma_f = \frac{1}{2m_H} \sum |M|^2\, d\phi_2$$

$$\Gamma_f = \frac{g^2 m_f^2\, m_H}{32\pi M_W^2} \left(1 - \frac{4m_f^2}{m_H^2}\right)^{3/2}$$

- if f is quark, extra factor of N colours

- LARGER m_f, larger decay rate.

2) DECAY TO GAUGE BOSONS



$$\mathcal{M} = gM_W \, \varepsilon_\mu^+ \varepsilon_\nu^- g^{\mu\nu}$$

$$\Rightarrow \quad \sum |M|^2 = g^2 M_W^2 \underbrace{\sum \varepsilon_\mu^+ (\varepsilon_\nu^+)^*}_{-g_{\mu\nu} + \frac{p_{1\mu} p_{1\nu}}{M_W^2}} \sum \varepsilon_\mu^- (\varepsilon_\nu^-)^*$$

$$= g^2 M_W^2 \left[4 - \frac{p_1^2}{M_W^2} - \frac{p_2^2}{M_W^2} + \frac{(p_1 \cdot p_2)^2}{M_W^4} \right]$$

$$= \frac{g^2}{4M_W^2} \left[m_H^4 - 4m_H^2 m_W^2 + 3m_W^4 \right]$$

grows with M_H^2

$$\Rightarrow \quad \Gamma_W = \frac{1}{2m_H} \int \sum |M|^2 d\phi$$

$$= \frac{g^2}{64\pi} \frac{m_H^3}{m_W^2} \sqrt{1 - \frac{4m_W^2}{m_H^2}} \left(1 - \frac{4m_W^2}{m_H^2} + \frac{3m_W^4}{m_H^4} \right)$$

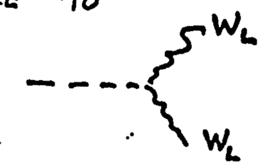
similarly

$$\Gamma_Z = \frac{g^2}{128\pi} \frac{m_H^3}{m_W^2} \sqrt{1 - \frac{4M_Z^2}{m_H^2}} \left(1 - \frac{4m_Z^2}{m_H^2} + \frac{3m_Z^4}{m_H^4} \right)$$

- FACTOR $\frac{1}{2}$ IN Γ_Z DUE TO ZZ IDENTICAL PARTICLES !

- RAPID GROWTH OF $\Gamma_W, \Gamma_Z \propto M_H^3$

DUE TO



"ORIGINALLY GOLDSTONE BOSON BEFORE SSB"

- **RULES OF THUMB**

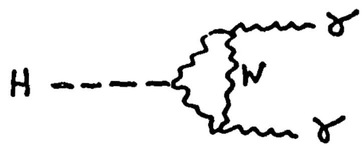
1) $\Gamma(H \to WW, ZZ) \simeq \frac{1}{2} \left(\frac{M_H}{1\,\text{TeV}} \right)^3$

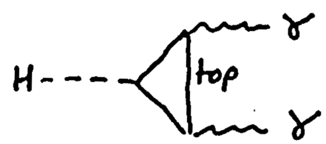
2) DECAY TO HEAVIEST ALLOWED PARTICLE DOMINATES

3) FOR $M_H > 2m_Z$

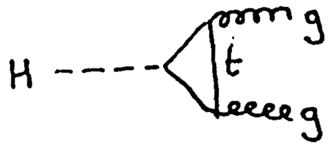
$$\Gamma_W \simeq 2\,\Gamma_Z$$

3) RARE DECAYS

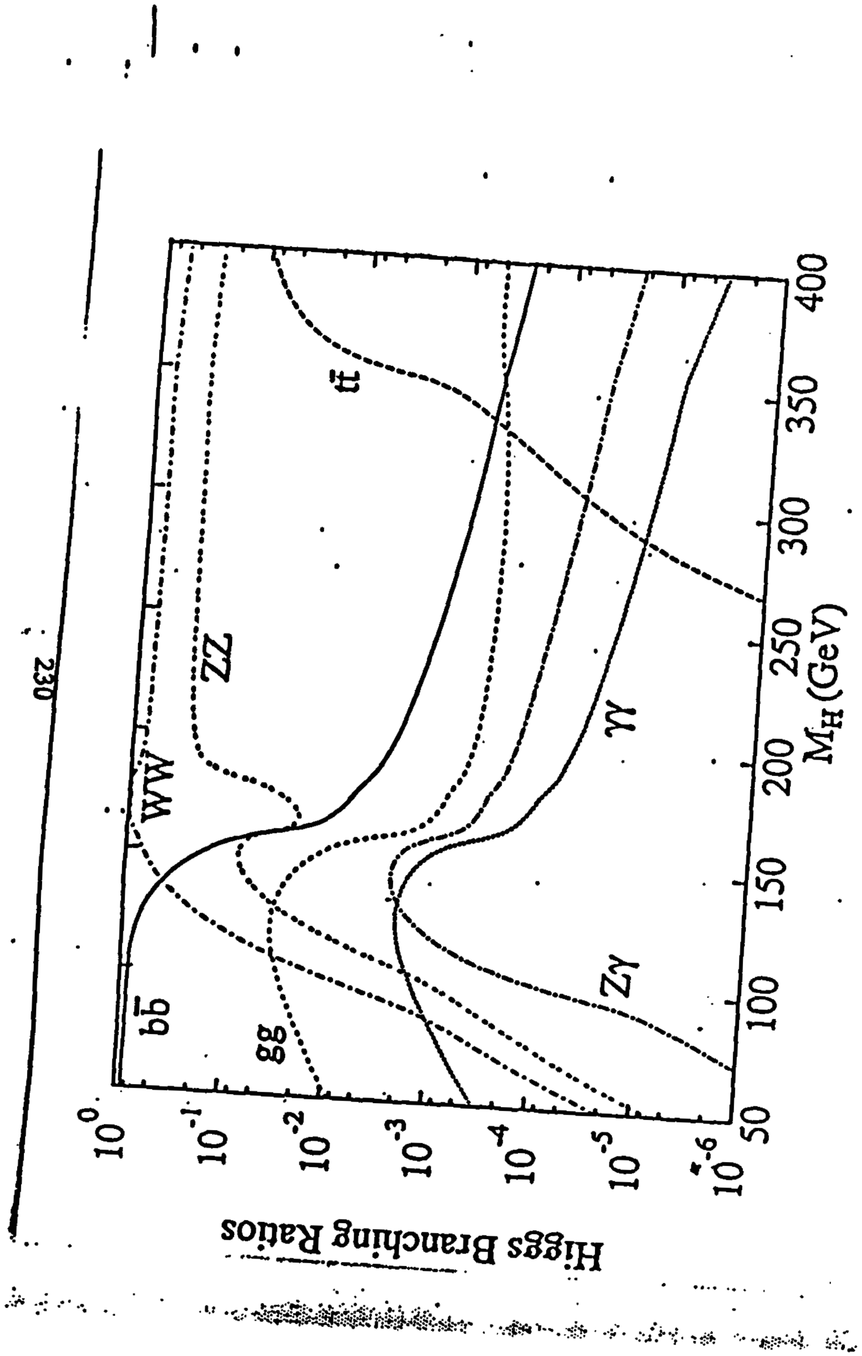




— VERY SMALL, BUT USEFUL SIGNAL.

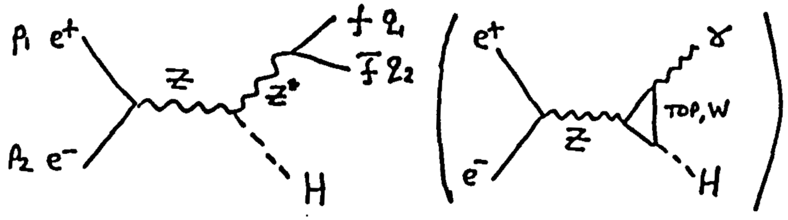


VITAL PRODUCTION MECHANISM AT LHC



HIGGS AT LEP I

— PRODUCED IN Z DECAY



$$\mathcal{M} = \bar v(p_1)\gamma_\mu(v-a\gamma_5)u(p_2)\,\frac{g^{\mu\alpha}}{s-M_Z^2+i\Gamma_Z M_Z}$$

$$\cdot\; g_{\alpha\beta}\,\frac{g^{\beta\nu}}{s'-M_Z^2+i\Gamma_Z M_Z}\;\bar u(q_1)\gamma_\nu(v_f-a_f\gamma_5)v(q_2)$$

$$\cdot\; \frac{g^3}{\cos^2\theta_W}\cdot\frac{M_Z^4}{M_W^2}$$

$$\Rightarrow\quad \sum|M|^2 = \frac{g^6 M_Z^8}{\cos^4\theta_W M_W^4}\;\frac{L^e_{\mu\nu}\,L_f^{\mu\nu}}{\left[(s-M_Z^2)^2+\Gamma_Z^2M_Z^2\right]\left[(s'-M_Z^2)^2+\Gamma_Z^2M_Z^2\right]}$$

- SIGNAL $\quad H\to b\bar b \quad + f\bar f \qquad m_{b\bar b}\sim m_H$ (narrow)

BACKGROUND

e.g.

NO PEAKING IN $m_{b\bar b}\sim m_H$

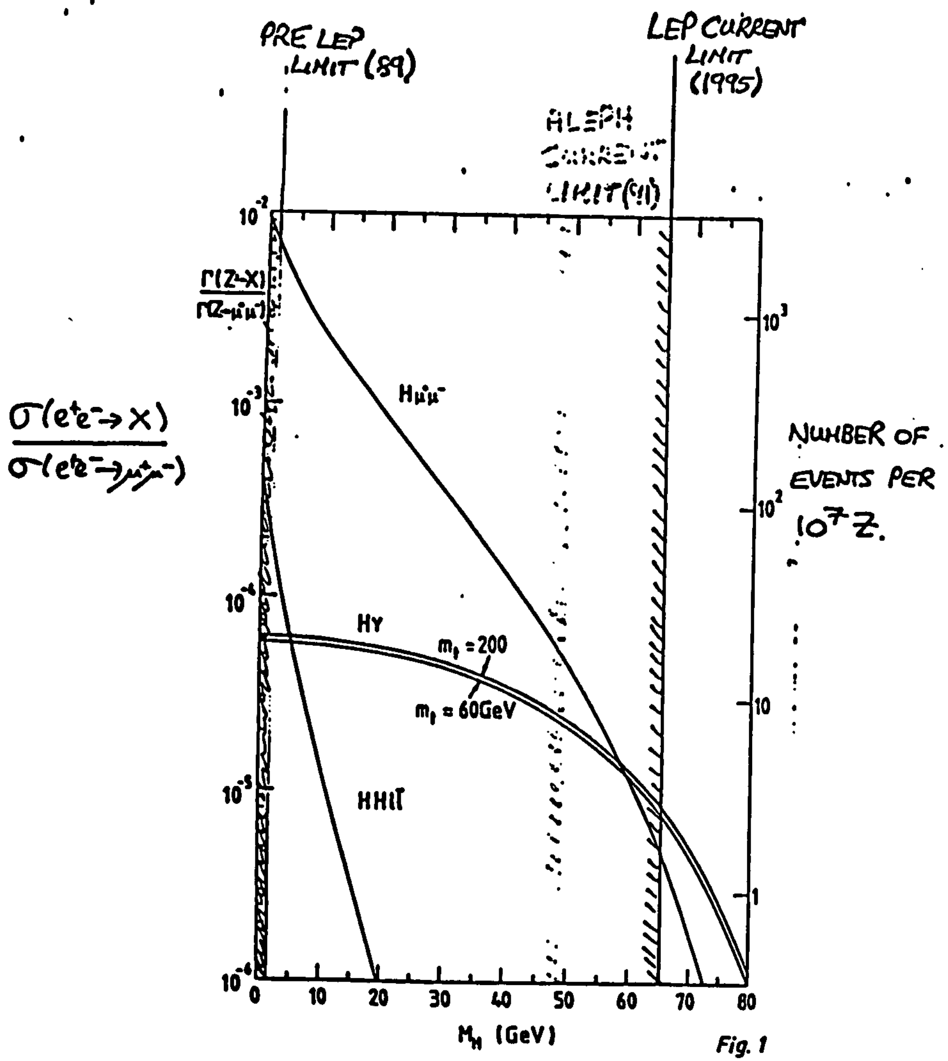


Fig. 1

Fig. 6

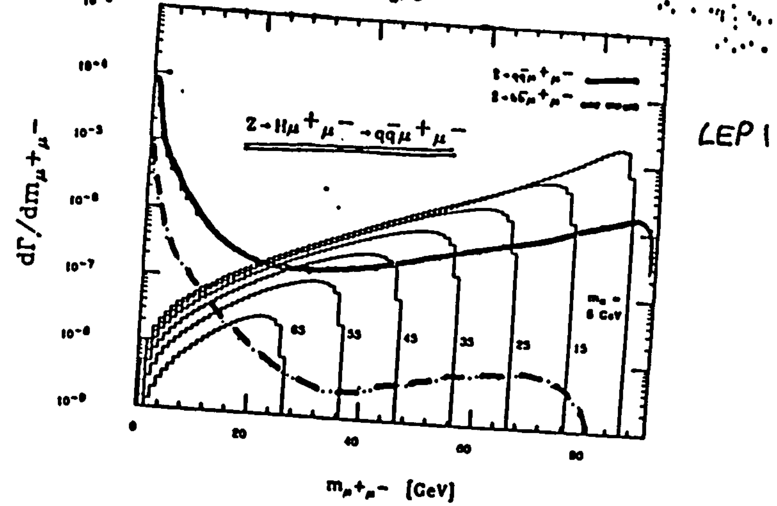
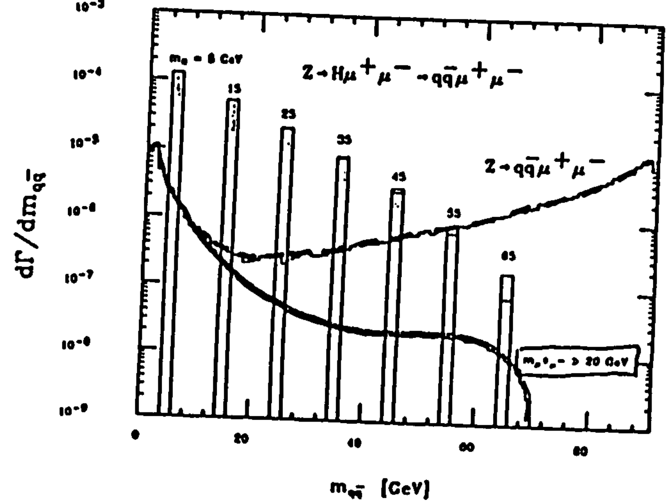
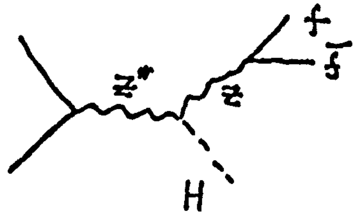


Fig. 7



HIGGS AT LEP II

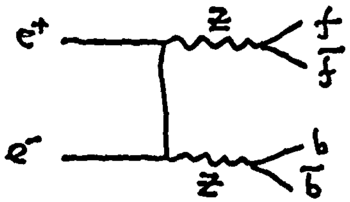
– SAME PRODUCTION PROCESS, BUT NOW INITIAL Z OFF SHELL



⇒ SAME MATRIX ELEMENTS

– KINEMATICS $\quad m_{f\bar{f}} \sim M_Z \,, \quad m_{b\bar{b}} \sim m_H$

• BACKGROUND

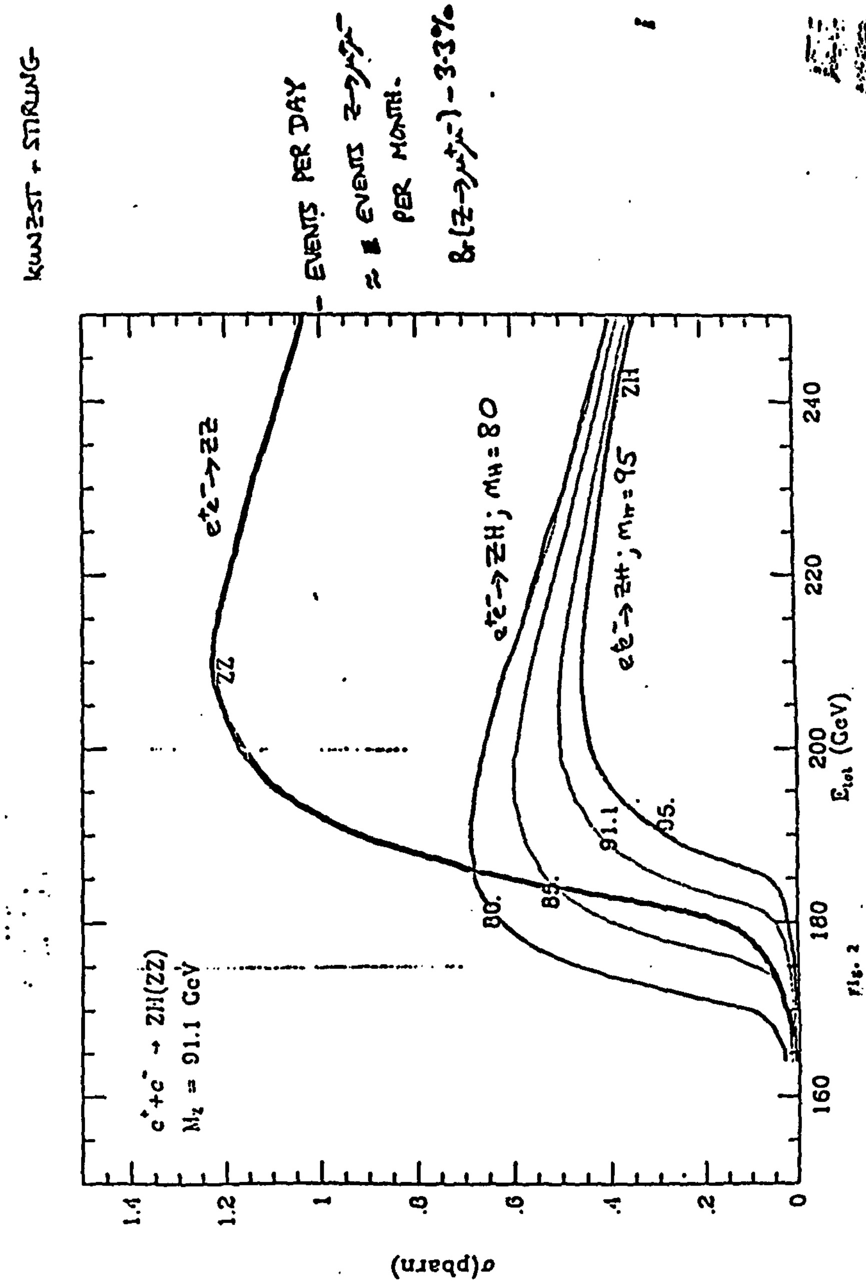


$m_{f\bar{f}} \sim m_Z$

$m_{b\bar{b}} \sim m_Z$

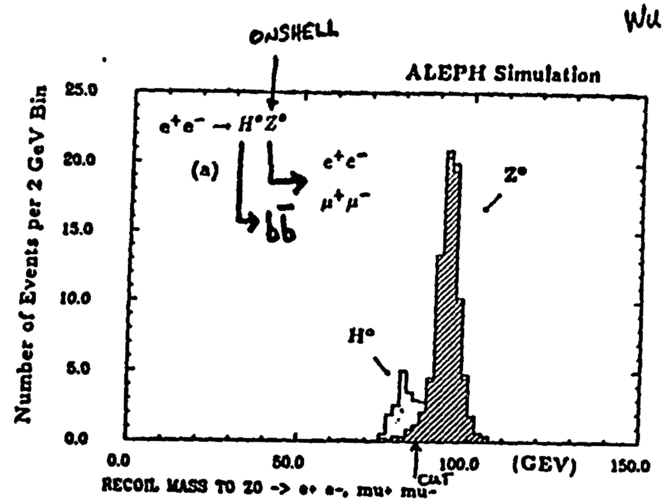
• HOPE TO SEE

$$\boxed{m_H \leq \sqrt{s} - 100 \text{ GeV}}$$



LEP II

WU ET AL.



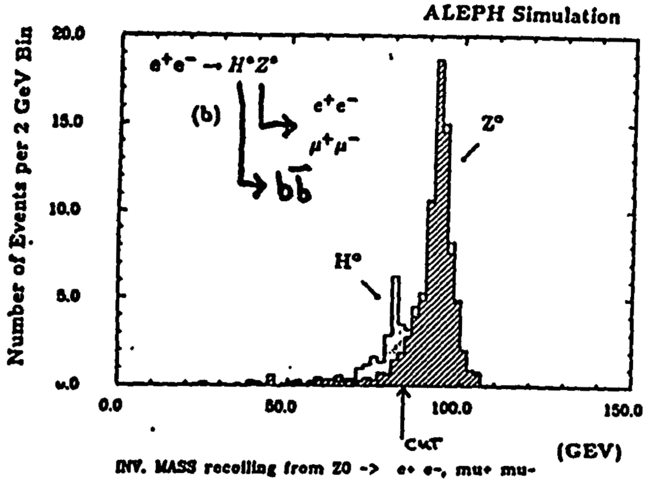
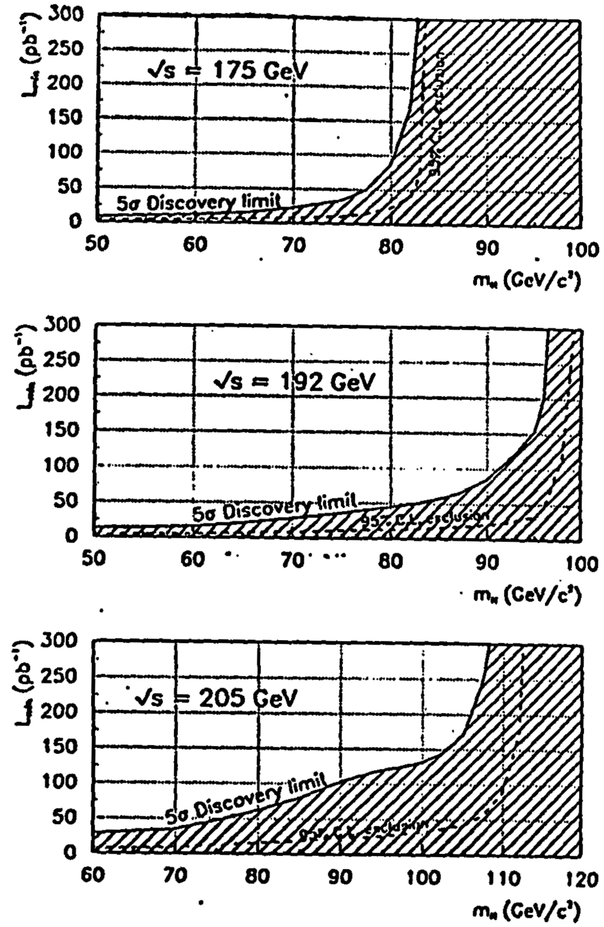


Fig. 17 (a)(b) Same as Fig. 15 for $\boxed{M_{H^0} = 80\ \text{GeV}}$



BACKGROUND

$e^+e^- \to ZZ \to e^+e^-, \mu^+\mu^-$

$\quad\to$ JETS

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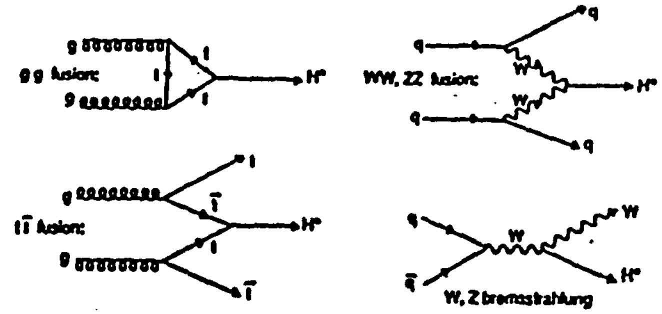
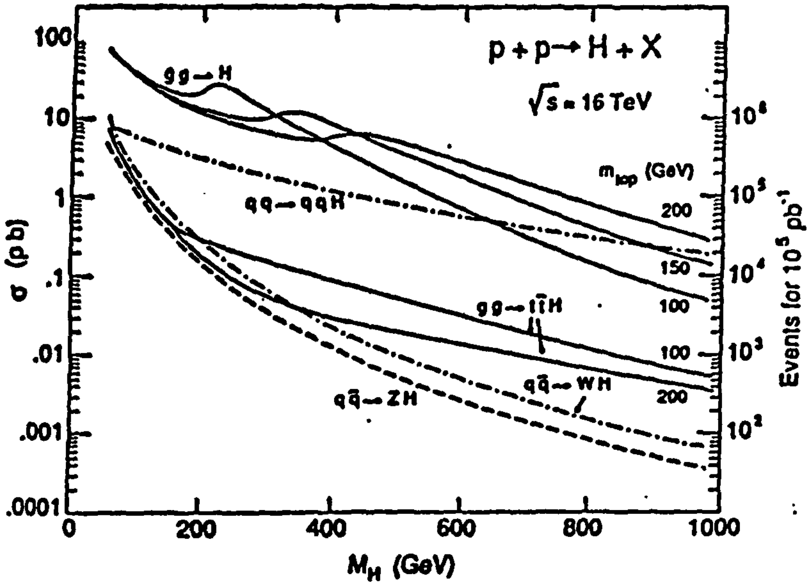
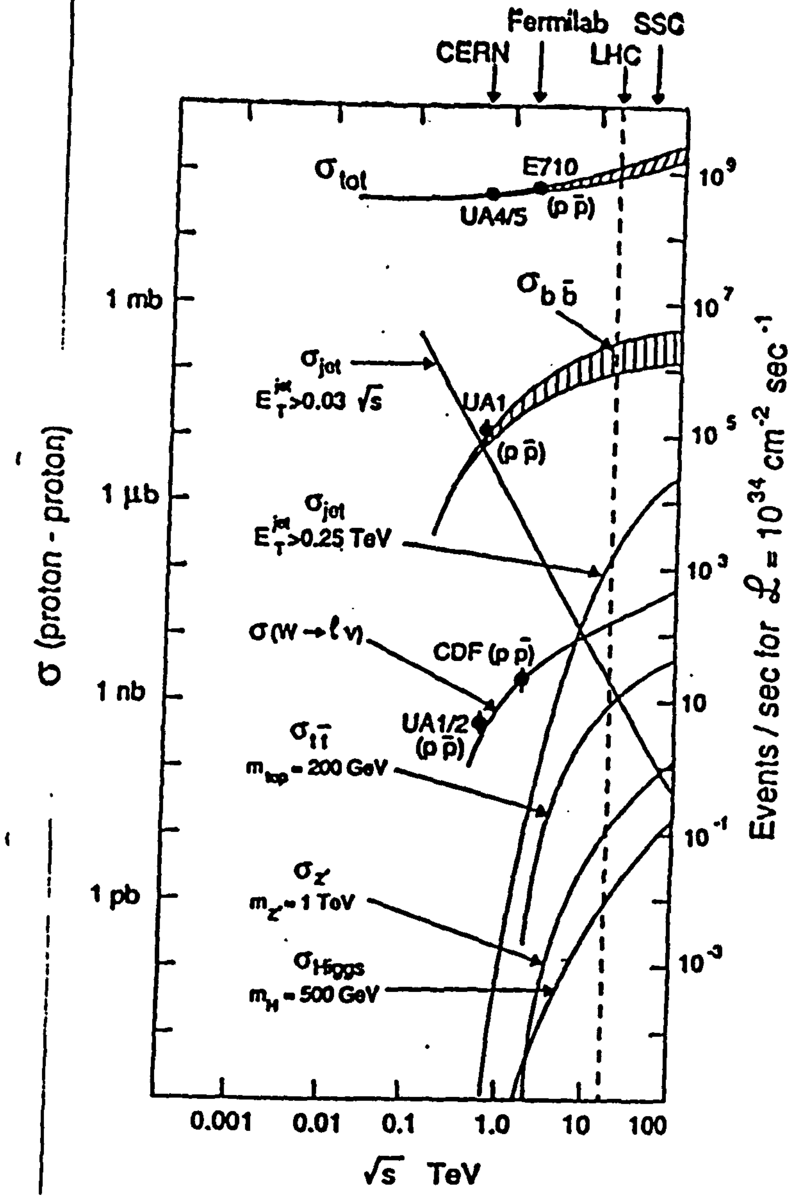
HIGGS AT LHC



gg fusion:
H^0
WW, ZZ fusion:
$t\bar{t}$ fusion:
W, Z bremsstrahlung

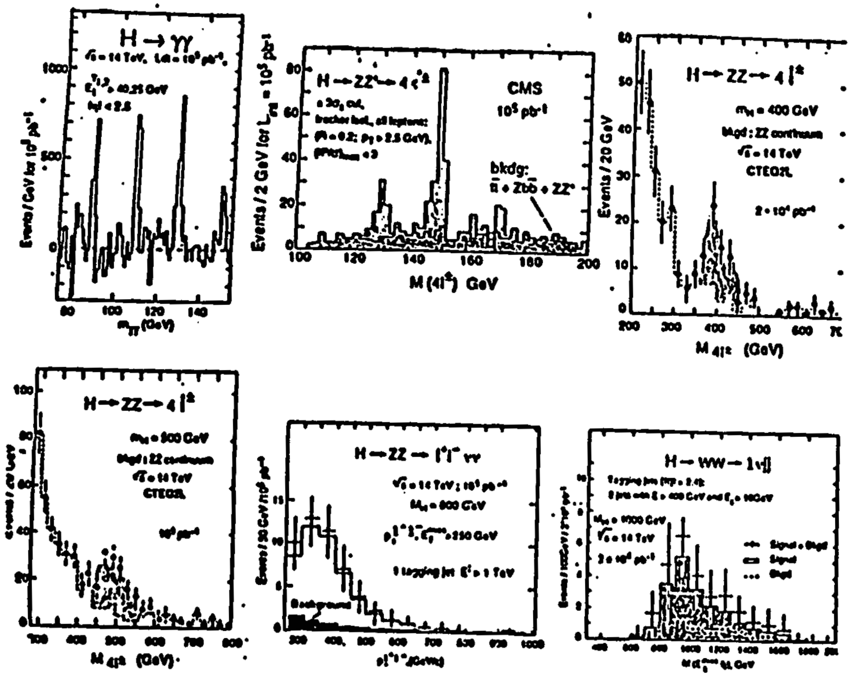


$p + p \to H + X$
$\sqrt{s} = 16$ TeV
$gg \to H$
$qq \to qqH$
$gg \to t\bar{t}H$
$q\bar{q} \to WH$
$q\bar{q} \to ZH$
m_{top} (GeV)
200
150
100
100
200
σ (pb)
Events for 10^5 pb^{-1}
M_H (GeV)



CERN
Fermilab
LHC
SSC
σ_{tot}
E710
UA4/5
$(p\bar{p})$
$\sigma_{b\bar{b}}$
σ_{jet} $E_T^{jet} > 0.03\sqrt{s}$
UA1
σ_{jet} $E_T^{jet} > 0.25$ TeV
$\sigma(W \to \ell\nu)$
CDF $(p\bar{p})$
UA1/2 $(p\bar{p})$
$\sigma_{t\bar{t}}$ $m_{top} = 200$ GeV
$\sigma_{Z'}$ $m_{Z'} = 1$ TeV
σ_{Higgs} $m_H = 500$ GeV
σ (proton - proton)
Events / sec for $\mathcal{L} = 10^{34}$ cm^{-2} sec^{-1}
$\sqrt{s}$ TeV

SM Higgs search in CMS

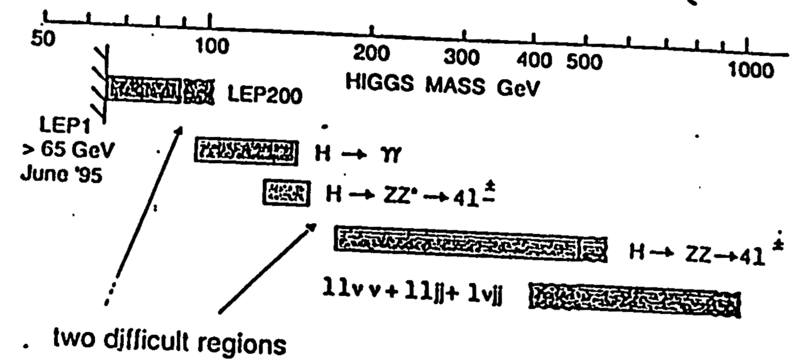


235

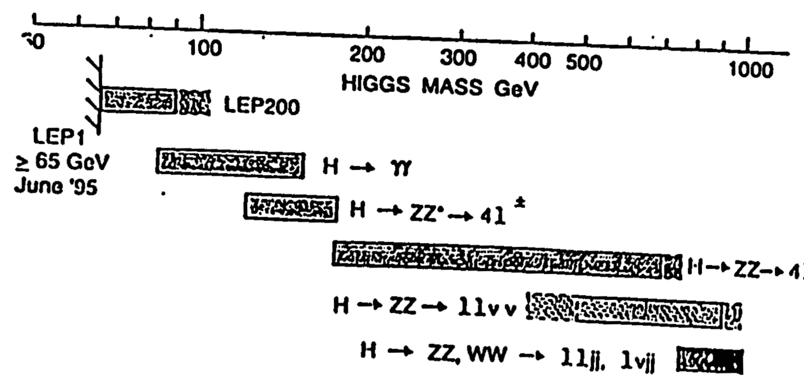
- INTERMEDIATE HIGGS $80 < M_H < 180$ GeV
  - LARGE PRODUCTION CROSS SECTION
  - DETECT SMALL B.R. $H \to \gamma\gamma$, $H \to \ell^+\ell^-\ell^+\ell^-$
- NORMAL HIGGS $180 < M_H < 600$ GeV
  - GOOD RATE, GOLD PLATED $H \to ZZ \to \ell^+\ell^-\ell^+\ell^-$
- WIDE HIGGS $600 < M_H$
  - RATE LOW, WIDE RESONANCE
  - ∴ USE BIGGER $ZZ \to \ell\ell\nu\nu$ DECAY.

SM Higgs search at LHC

i) mass range explorable at $\sqrt{s} = 14$ TeV with $3 \cdot 10^4$ pb^{-1} taken at 10^{33} cm^{-2}s^{-1} (~ 1 year at $\cdot 10^{33}$



ii) explorable mass range with 10^5 pb^{-1} taken at 10^{34} cm^{-2}s^{-1}



Questions

1. Show that
$$q^\mu L_{\mu\nu} = 0.$$

2. The Gottfried sum rule,
$$\int_0^1 \frac{dx}{x}\left(F_2^{ep}(x) - F_2^{en}(x)\right),$$
has been measured by the NMC Collaboration to be,
$$\int_{0.0014}^{0.8} \frac{dx}{x}\left(F_2^{ep}(x) - F_2^{en}(x)\right) = 0.236 \pm 0.008 \pm 0.0014.$$
Assuming that this can be extrapolated to cover the whole x range,
$$\int_0^1 \frac{dx}{x}\left(F_2^{ep}(x) - F_2^{en}(x)\right) \sim 0.258 \pm 0.017,$$
what can be said about,
$$\int_0^1 dx\left(u_{sea}(x) - d_{sea}(x)\right)?$$

3. Bearing in mind that $F_2^{ep}(x)$ is constant or increasing as $x \to 0$, what can be said about the number of up quarks in the proton,
$$\int_0^1 u(x)dx?$$

4. Data for the ratio $F_2^{en}(x)/F_2^{ep}(x)$ was shown in the lecture. Explain why this ratio tends to 1 as $x \to 0$. What can be said about d_v/u_v at large x where the sea quark density functions are negligible?

5. For the $\gamma^* \to q\bar{q}g$ process at centre-of-mass energy $\sqrt{s}$, x_1, x_2 and x_3 are the scaled energy fractions carried by the quark, antiquark and gluon respectively. Show that the invariant mass of the quark and antiquark pair is given by $2q_1.q_2 = s(1 - x_3)$.

6. Show that in the limit where the quark and gluon momenta are parallel,
$$\frac{1}{4}\sum|M|^2_{q\bar{q}g} \to \frac{g_s^2}{2k.q_1}\left(\frac{N^2-1}{2N}\right)\frac{2(1+(1-z)^2)}{z}\frac{1}{4}\sum|M|^2_{q\bar{q}}.$$

7. Show that in the limit where the gluon momentum is soft,
$$\frac{1}{4}\sum|M|^2_{q\bar{q}g} \to g_s^2\left(\frac{N^2-1}{2N}\right)\frac{2q_1.q_2}{q_1.kk.q_2}\frac{1}{4}\sum|M|^2_{q\bar{q}}.$$

8. Show that
$$\sigma_{q\bar{q}g} + \sigma_{q\bar{q}(g)} = \frac{\alpha_s}{\pi}\sigma^0_{q\bar{q}}.$$

9. Show that if $x_1 >> x_2, x_3$,
$$D_T = \frac{\sqrt{(1-x_1)(1-x_2)(1-x_3)}}{x_1}.$$
Show that the maximum value of D_T is $1/2\sqrt{3}$ for a three jet event.

10. Show that in the limit $x_3 \to 0$, $D_T \to 0$. Show that in the limit that the quark and gluon are parallel, $D_T \to 0$.

11. Starting from,
$$\frac{1}{\sigma^0}\frac{d\sigma}{dx_1 dx_2} = \frac{\bar{\alpha}_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\frac{x_3^2}{2(1-x_1)(1-x_2)},$$
show that the thrust distribution for $q\bar{q}g$ events involving a scalar gluon is,
$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \frac{\bar{\alpha}_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\left(\log\left(\frac{2T-1}{1-T}\right) + \frac{(3T-2)(4-3T)}{2(1-T)}\right).$$

12. Given that
$$\int_0^1 \frac{f(x)}{(1-x)_+}dx = \int_0^1 \frac{f(x)-f(1)}{(1-x)}dx,$$
show that,
$$\int_0^1 P_{qq}(x)dx = 0,$$
where,
$$P_{qq}(x) = \frac{4}{3}\left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right).$$
What is the significance of this result?

13. Draw Feynman diagrams to motivate why the $pp \to \gamma +$ jet process is sensitive to the gluon density function.

14. Considering only ud scattering, show that the W asymmetry,
$$A_W(y) = \frac{\frac{d\sigma(W^+)}{dy} - \frac{d\sigma(W^-)}{dy}}{\frac{d\sigma(W^+)}{dy} + \frac{d\sigma(W^-)}{dy}},$$
effectively determines the d/u ratio in $p\bar{p}$ collisions.