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Witten's cubic vertex in the comma theory (II)

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ABSTRACT

The proof that the Witten's 3-vertex solves the comma overlap equations is completed. The ghost sector is discussed in detail, using the fermionic formulation of the ghosts. The BRST operators in the comma theory, Q^L, Q^R , are constructed and the identity $Q^R + Q^L = Q$ is discussed.

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1 Introduction

In this paper we complete the proof of equivalence of the comma theory and Witten's string field theory (WSFT) at the level of vertices that we have started in [1, 2]. In [1, 2] we concentrated on the matter sector of the theory where we demonstrated that the Witten's interaction vertex, V_N^W (N=1,2,3,4) is indeed a solution to the comma overlap equations. We also discussed in some detail the invariance and potential anomalies of the theory due to the K_n generators (i.e., invariance under reparametrizations generated by K_n). However, we failed to address issues related to the $BRST$ invariance ($Q\,invariance$). Here we give a complete construction of the comma ghosts in the fermionic representation[2]. The ghosts were considered in [1, 2] in the bosonized form as presented by Witten [3]. However, to address the $BRST$ invariance, the bosonizied form is cumbersome as compared to that in the fermionic representation.

First let us review briefly the full string ghost sector. Recall that the ghost coordinates of the open bosonic string are the anticommuting fields

$$c_{\pm}(\sigma) = \sum_{n=-\infty}^{\infty} c_n e^{\pm in\sigma} = c(\sigma) \pm i\pi_b(\sigma), \tag{1.1}$$

$$b_{\pm}(\sigma) = \sum_{n=-\infty}^{\infty} b_n e^{\pm in\sigma} = \pi_c(\sigma) \pm ib(\sigma). \tag{1.2}$$

The c_+ (c_-) are the ghosts for reparameterization of $z = \tau + i\sigma$ ($\overline{z} = \tau - i\sigma$) respectively and the $b_\pm$ are the corresponding antighosts. Theses obey

[2]The ghost sector of the comma theory was also constructed in [4]. However, the ambiguity related to the mid-point was not settled there. One was not sure how to view the midpoint, since it was common to both the comma theory and WSFT. This lead to the modified definition of the comma introduced in [1, 2]. In the modified definition of the comma, the mid-point is excluded and is used to constrain the comma degrees of freedom. For more details see ref. [1]. Here also the comma definition is slightly modified along the lines of refs. [1, 2].

the anticomutation relations

$$\{c_n, c_m\} = \{b_n, b_m\} = 0, \tag{1.3}$$

$$\{c_n, b_m\} = \delta_{n+m,0}. \tag{1.4}$$

The fermionic Fock space is constructed in terms of a vacuum state annihilated by $c_n, b_n;\ n \geq 1$. Since the zero modes c_0 and b_0 anticommute, such a ground state is a doublet of states, $|\pm>$, with ghost number $\pm 1/2$. The overlap equations for $N - strings$ are

$$c^r(\sigma) + c^{r-1}(\pi - \sigma) = 0, \tag{1.5}$$

$$\pi_c^r(\sigma) - \pi_c^{r-1}(\pi - \sigma) = 0, \tag{1.6}$$

where $r = 1, 2, .., N$ (and $r - 1 = 0 \equiv N$). They are similar equations for $b(\sigma)$ and $\pi_b(\sigma)$, with the role of coordinates and momenta exchanged. The ghost Fock space vector ($N = 1$) satisfying the above overlaps (ignoring midpoint insertions, $b_+(\pi/2)b_-(\pi/2)$) is

$$|I_0^{ghost}>= e^{\sum_{n=1}^{\infty}(-)^n c_{-n} b_{-n}}|\Omega, c_0 = 0> . \tag{1.7}$$

The Witten's vertex expressing the coupling of three string ($N = 3$), satisfying the overlaps, takes the form (see ref. [5, 6, 7, 8, 9]

$$|V_3^{ghost}>= e^{\sum_{r,s=1}^{3}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} b_{-n}^r \tilde{N}_{nm}^{rs} c_{-m}^s}|\Omega> . \tag{1.8}$$

The explicit values of the Neumann coefficients have been constructed in in refs. [5, 6, 7, 8, 9]. There are no ghost insertions at the midpoint, since the vacuum state in (1.8) has ghost number 3/2 which is the correct ghost number. We also recall that the Virasoro generators of reparametrization, L_n^{ghost}, is given by

$$L_n^{ghost} = \sum_m (n + m) b_{n-m} c_m, \tag{1.9}$$

and the $BRST$ charge is

$$Q = \sum_{n=1}^{\infty} [c_{-n}\mathcal{L}_n + \mathcal{L}_{-n}c_n] + c_0 [\mathcal{L}_0 - 1] \tag{1.10}$$

where

$$\mathcal{L}_n \equiv L_n^x + \frac{1}{2}L_n^{ghost}. \tag{1.11}$$

Now we are ready to address our problem. Let us denote by $c_{\pm}^{r}(\sigma) = c^{r}(\sigma) \pm i\pi_b^r$ the comma ghost coordinate analogous to $c_{\pm}(\sigma) = c(\sigma) \pm i\pi_b$ where the comma ghost coordinates are defined through the relation[3]

$$c^{r}(\sigma) = \begin{cases} c(\sigma) \ , \, if \ r = 1 \, , \\ c(\pi - \sigma) \ , \, if \ r = 2 \ , \\ \sigma \in [0, \frac{\pi}{2}) \, . \end{cases} \tag{1.12}$$

and likewise for π_b^r. Theses ghost coordinates are subjected to the constraint,

$$\lim_{\sigma \to \frac{\pi}{2}} c^{L,R}(\sigma) = c(\pi/2) \tag{1.13}$$

and likewise for π_b^r. If one expands the comma ghost, (1.12) in a Fourier series then they can be related to the full string ghosts. The comma boundary conditions are dictated by the boundary conditions of the full string and the comma definition. Choosing an even extension to the interval $(\pi/2, \pi]$ for $c^r(\sigma)$ with $c^r(\sigma) = c^r(-\sigma)$, only the even modes in the Fourier expansions of $c^r(\sigma)$ survive. Hence,

$$c^{r}(\sigma) = c_0^r + \sqrt{2} \sum_{n=1}^{\infty} g_{2n}^r \cos 2n\sigma, \quad \sigma \in [0, \pi/2) \, , \tag{1.14}$$

where

$$c_0^r = c_0 + (-)^r \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n}{2n-1} \left(c_{2n-1} + c_{-2n+1} \right) \, , \tag{1.15}$$

$$g_{2n}^r = \frac{1}{\sqrt{2}} \left(c_{2n} + c_{-2n} \right) + \sqrt{2} (-)^r \sum_{m=1}^{\infty} B_{2n\, 2m-1} \left(c_{2m-1} + c_{-2m+1} \right) \, , \tag{1.16}$$

and $r = 1, 2$ refers to the left (L) and right (R) parts of the string[4] respectively. The change of representation matrix (B) is given by

$$B_{nm} = \frac{(-)^{\frac{n+m+1}{2}}}{\pi} \left(\frac{1}{n+m} - \frac{1}{n-m} \right) \, . \tag{1.17}$$

[3] note that the only difference between this new definition of the comma and that in ref. [4] is the exclusion of the midpoint $c(\pi/2)$.

[4] Throughout the paper we will refer to the left and right parts of the string by 1 and 2 respectively; however to make things more transparent and to avoid confusion, sometimes we may refer to the left and right parts of the string by the letters L and R respectively. When dealing with more than one string the indices may become confusing; therefore indices referring to the parts of the string will always be written as superscripts while those labeling the string will be written as subscripts whenever possible.

Equations (1.16) and (1.17) can be inverted with the help of the identities

$$\sum_{n=1}^{\infty} B_{2n\,2k-1} B_{2n\,2m-1} = \frac{1}{4}\delta_{km}\,, \tag{1.18}$$

$$\sum_{k=1}^{\infty} \frac{2m}{2k-1} B_{2n\,2k-1} B_{2k-1\,2m} = -\frac{1}{4}\delta_{nm}\,. \tag{1.19}$$

Thus one gets

$$c_0 = \frac{1}{2}\sum_{r=1}^{2} c_0^r\,, \tag{1.20}$$

$$\frac{1}{\sqrt{2}}\left(c_{2n} + c_{-2n}\right) = \frac{1}{2}\sum_{r=1}^{2} g_{2n}^r\,, \quad n \geq 1, \tag{1.21}$$

$$\frac{1}{\sqrt{2}}\left(c_{2n-1} + c_{-2n+1}\right) = \sum_{r=1}^{2}(-)^{r+1}\sum_{m=1}^{\infty}\frac{2m}{2n-1} B_{2n-1\,2m}\, g_{2m}^r\,, \quad n \geq 1\,. \tag{1.22}$$

However, in (1.22) there are redundant degrees of freedom. Now, the constraint on the comma modes (1.13) can be explicitly solved and what results are the modes with no subsidiary condition. Hence one gets

$$\frac{1}{\sqrt{2}}\left(c_{2n-1} + c_{-2n+1}\right) = \frac{\sqrt{2}}{\pi}\frac{(-)^n}{2n-1}\sum_{r=1}^{2}(-)^r c_0^r + \sum_{r=1}^{2}(-)^r \sum_{m=1}^{\infty} B_{2m\,2n-1} g_{2m}^r\,, \quad n \geq 1\,, \tag{1.23}$$

for (1.22). Likewise for its conjugate momentum $\pi_c^r(\sigma)$ we obtain

$$\pi_c^r(\sigma) = b_0^r + \sqrt{2}\sum_{n=1}^{\infty} y_{2n}^r \cos 2n\sigma, \quad \sigma \in [0, \pi/2)\,, \tag{1.24}$$

where

$$b_0^r = b_0 + (-)^r \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{2n-1}\left(b_{2n-1} + b_{-2n+1}\right)\,, \tag{1.25}$$

$$y_{2n}^r = \frac{1}{\sqrt{2}}\left(b_{2n} + b_{-2n}\right) + \sqrt{2}(-)^r \sum_{m=1}^{\infty} B_{2n\,2m-1}\left(b_{2m-1} + b_{-2m+1}\right)\,. \tag{1.26}$$

The inverse of equations (1.25) and (1.26) reads[5]

$$b_0 = \frac{1}{2}\sum_{r=1}^{2} b_0^r\,, \tag{1.27}$$

[5] In arriving at these equations we used the midpoint constraint as before to remove the redundancy.

$$\frac{1}{\sqrt{2}}\left(b_{2n}+b_{-2n}\right)=\frac{1}{2}\sum_{r=1}^{2} y_{2n}^{r}\,, \quad n\geq 1\ . \tag{1.28}$$

$$\frac{1}{\sqrt{2}}\left(b_{2n-1}+b_{-2n+1}\right)=\frac{\sqrt{2}}{\pi}\frac{(-)^{n}}{2n-1}\sum_{r=1}^{2}(-)^{r}b_{0}^{r}+\sum_{r=1}^{2}(-)^{r}\sum_{m=1}^{\infty} B_{2m\;2n-1}y_{2m}^{r}\,, \quad n\geq 1\,, \tag{1.29}$$

The comma modes have been treated so far as classical objects. One way of quantizing them is to interpret them as $q-operators$ and demand that they satisfy the desired anticommutation relations, i.e., one imposes

$$\left\{O_{n}^{r}, \frac{\partial}{\partial O_{m}^{s}}\right\}=\delta^{rs}\delta_{nm} \tag{1.30}$$

for fermi operator, O_{n}^{r}. If we introduce the operators

$$\hat{c}_{0}^{r}=\frac{1}{\sqrt{2}}c_{0}^{r}\,, \qquad \hat{g}_{n}^{r}=\frac{1}{\sqrt{2}}g_{2n}^{r}\,, \tag{1.31}$$

$$\hat{b}_{0}^{r}=\frac{1}{\sqrt{2}}b_{0}^{r}\,, \qquad \hat{y}_{n}^{r}=\frac{1}{\sqrt{2}}y_{2n}^{r}\,, \tag{1.32}$$

we see that $\hat{c}_{0}^{r}$, $\hat{g}_{n}^{r}$ are conjugate to $\hat{b}_{0}^{r}$, $\hat{y}_{n}^{r}$ respectively as desired. These with their counter part obtained in the expansion of $b^{r}(\sigma)$ and $\pi_{b}^{r}(\sigma)$ will be used later to define the comma ghost modes γ_{n}^{r} and β_{n}^{r}. The construction of $b^{r}(\sigma)$ and its conjugate momentum $\pi_{b}^{r}(\sigma)$ are given in appendix A.

2 Construction of the three-string comma overlaps

The fermionic coordinates $c^{r}(\sigma)$ and $b^{r}(\sigma)$ are canonically conjugate. Thus only one of them is to be identified about the midpoint, the other one must be identified, as a momentum, with opposite sign. The correct prescription is that $b^{r}(\sigma)$, $b_{\pm}^{r}(\sigma)$ and $\pi_{c}^{r}(\sigma)$ are treated like coordinates and $c^{r}(\sigma)$, $c_{\pm}^{r}(\sigma)$ and $\pi_{b}^{r}(\sigma)$ are treated as momenta. The comma overlap equations in terms of

$$c^{r}(\sigma)=c_{0}^{r}+\sqrt{2}\sum_{n=1}^{\infty} g_{2n}^{r}\cos 2n\sigma\,, \tag{2.1}$$

and

$$\pi_{c}^{r}(\sigma)=b_{0}^{r}+\sqrt{2}\sum_{n=1}^{\infty} y_{2n}^{r}\cos 2n\sigma\,, \tag{2.2}$$

for the $N - string$ are

$$c_j^L(\sigma) = -c_{j-1}^R(\sigma), \tag{2.3}$$

$$\pi_{c_j}^{\ L}(\sigma) = \pi_{c_{j-1}}^{\ R}(\sigma), \tag{2.4}$$

where $j = 1, 2, ..., N$ and $j - 1 = 0 \equiv N$. For $b^r(\sigma)$ and $\pi_b^r(\sigma)$ we obtain similar equations (the same form with the relative sign exchanged). Now we are ready to consider the overlaps for the *cubic* vertex[6]. In terms of the complex coordinates

$$C_k^r(\sigma) = \frac{1}{\sqrt{3}}\sum_{l=1}^{3} c_l^r(\sigma)e^{2\pi i l k/3}, \ r = L\, R, \quad k = 1, 2, 3, \tag{2.5}$$

the overlaps (2.3) yields

$$\left[C^L(\sigma) + e^{2\pi i/3}C^R(\sigma)\right] |V_3^{comma} >= 0\,, \ \sigma \in [0, \pi/2), \tag{2.6}$$

$$\left[C_3^L(\sigma) + C_3^R(\sigma)\right] |V_3^{comma} >= 0\,, \ \sigma \in [0, \pi/2), \tag{2.7}$$

where $C^r(\sigma) \equiv C_1^r(\sigma) = \overline{C_2^r(\sigma)}$. Similarly , in the complex space of the commas, the overlaps specified by (2.4) are given by the same equations with $C^r(\sigma) \to \Pi_c^r(\sigma)$ and the relative sign flipped. The overlaps for $B^r(\sigma)$ and $\Pi_b^r(\sigma)$ are given in appendix A.

3 The Comma Overlaps in FS Oscillator Hilbert Space

The fact that the Witten's vertex solves the comma overlaps only prove that the Witten's vertex is a solution of the comma overlaps and not necessarily the only solution[7]. If this turn to be the case it will be interesting to see what other solutions are admitted by the comma formulation; certainly one of them will be the comma vertex itself if one can show that it is different from the Witten's vertex (i.e., possesses different properties from the Witten's vertex.). These questions will be addressed later in the paper. The proof that the operator form of the Witten's $3 - vertex$ solves the

[6]The overlaps for the $1 - vertex$ is given in appendix B.

[7]This statement is true, since no one has yet proven that Witten's interaction fixes the form of the vertex uniquely.

comma overlaps is not a trivial one; it involves double infinite sums (the second coming from integrating σ over the range $[0, \pi/2)$ in formulating the comma theory). The double infinite sums may not converge absolutely and the convergence may depend on the order of the sums. The case of the full string [5, 6, 7, 8, 9] is different, the expression for the vertices involve absolutely convergent sums. This ambiguity is not an accident, we have seen in [10, 4] that Witten's theory can be viewed as an infinite dimensional local matrix algebra where the star product " $*$ " becomes matrix multiplication over infinite dimensional matrices that does not conserve associativity. To establish that the Witten's $3-vertex$ solves the comma overlaps we to show that it solves the comma overlaps stated in the previous section. First we observe that (2.7) has the same form as the identity vertex[8] and therefore the proof follows from the form of the vertex. However, this is not the case for eq. (2.6) and therefore one must prove it explicitly. The overlap conditions on $C^r(\sigma)$ and $\Pi_c{}^r(\sigma)$ imply that their Fourier components satisfy

$$\left[C^L_{2n} + e^{2\pi i/3} C^R_{2n}\right] |V_3^{comma}>= 0\,, \quad n \geq 0 \tag{3.1}$$

$$\left[\Pi^L_{c2n} - e^{2\pi i/3} \Pi^R_{c2n}\right] |V_3^{comma}>= 0\,, \quad n \geq 0\,. \tag{3.2}$$

Now we have to show that the comma equations, (3.1) and (3.2), hold for the Witten's cubic vertex constructed in [5, 6, 7, 8, 9]. Let us start with (3.1). It is fairly easy to express the comma overlaps, (3.1), in the Hilbert space of the full string, by using the change of representation formulas derived earlier. After a little algebra one finds that (3.1), for the Witten's vertex, may be written as

$$\frac{1}{\sqrt{3}} \sum_{r=1}^{3} e^{2i\pi r/3} \left[c_0^r - i \frac{2\sqrt{3}}{\pi} \sum_{n=0}^{\infty} \frac{(-)^n}{2n+1} \left(c^r_{2n+1} + c^r_{-2n-1}\right)\right] |V_3^W>= 0 \tag{3.3}$$

for $n = 0$ and

$$\frac{1}{\sqrt{6}} e^{-i\pi/3} \sum_{r=1}^{3} e^{2i\pi r/3} \left[\left(c^r_{2n} + c^r_{-2n}\right) + 2i\sqrt{3} \sum_{m=0}^{\infty} B_{2n\,2m+1}\right.$$

[8] We have checked that the identity vertex in WSFT indeed satisfy the comma overlaps derived in appendix B. However, we will not produce the proof here since it is a straight forward proof.

$$\left(c^r_{2m+1} + c^r_{-2m-1}\right)\Big] |V_3^W >= 0\,, \tag{3.4}$$

for $n \geq 1$. Using the identity

$$c^r_n |V_3^W >= \tilde{N}^{rs}_{nm} m c^s_{-m} |V_3^W >\,, \tag{3.5}$$

eqs. (3.3) and (3.4) may be written in the form

$$\sum_{s=1}^{3} \sum_{k=0}^{\infty} \Omega^s_{2n\,k} c^s_{-k} |V_3^W >= 0; \quad n \geq 0\,. \tag{3.6}$$

The coefficients of c^s_{-k}, are given by

$$\Omega^s_{0\,k} = \sum_{r=1}^{3} e^{2i\pi r/3} \left[\tilde{N}^{rs}_{0\,k} k - 2i\frac{\sqrt{3}}{\pi} \sum_{l=0}^{\infty} \frac{(-)^l}{2l+1} \left(\tilde{N}^{rs}_{2l+1\,k} k + \delta^{rs} \delta_{2l+1\,k}\right)\right], \tag{3.7}$$

and

$$\Omega^s_{2n\,k} = \sum_{r=1}^{3} e^{2i\pi r/3} \Big[\tilde{N}^{rs}_{2n\,k} k + \delta^{rs}\delta_{2n\,k} + 2i\sqrt{3} \sum_{m=0}^{\infty} B_{2n\,2m+1}$$
$$\left(\tilde{N}^{rs}_{2m+1\,k} k + \delta^{rs}\delta_{2m+1\,k}\right)\Big]; \quad n \geq 1\,. \tag{3.8}$$

Now to prove (3.6) we must show that the identity

$$\Omega^s_{2n\,k} = 0 \tag{3.9}$$

is true separately for all $s = 1, 2, 3$ and all $k, n \geq 0$. The proof of this identity is highly nontrivial (apart from the case $n = k = 0$ which is obvious); it involves performing a large number of infinite sums over the change of basis matrix B and the Neumann coefficients $\tilde{N}^{rs}_{nm}$. The proof of the zero mode (i.e., $n = 0, k \geq 1$) of the identity is relatively easier than that for higher modes (i.e., $n \geq 1, k \geq 0$). Therefore we shall not give it here[9]. Before proceeding to prove (3.9) for the non zero modes let us give the values of Neumann coefficients $\tilde{N}^{rs}_{nm}$ (for $n, m \geq 0$) as derived in ref. [5]. For $m + n$ odd we have

$$\tilde{N}^{rs}_{mn} = -\frac{2}{3} \sin \frac{2}{3}\pi(r-s)\tilde{N}_{mn}\,, \tag{3.10}$$

with

$$\tilde{N}_{mn} = \frac{(-)^{\frac{m+n-1}{2}}}{2} \left[\frac{b_m a_n + b_n a_m}{m-n} + \frac{b_m a_n - b_n a_m}{m+n}\right], \tag{3.11}$$

[9] We have checked that the identity in (3.9) for the zero mode is indeed satisfied.

and for $(m+n)$ = even (excluding the case $m=n=0$),

$$\tilde{N}^{rs}_{mn} = \delta^{rs}\frac{(-)^{m+1}}{m}\delta_{mn} - \frac{2}{3}\cos\frac{2}{3}\pi(r-s)\tilde{N}_{mn}\,, \tag{3.12}$$

with

$$\tilde{N}_{mn} = \frac{(-)^{\frac{m+n}{2}}}{2}\left[\frac{b_m a_n - b_n a_m}{m-n} + \frac{b_m a_n + b_n a_m}{m+n}\right]\,,\quad m \neq n, \tag{3.13}$$

$$\tilde{N}_{mm} = -\frac{(-)^m}{m}\left(\sum_{k=0}^{m}(-)^k a_k^2 - \frac{1}{2}(-)^m a_m^2 + 1\right) + \frac{(-)^m}{m}a_m b_m. \tag{3.14}$$

The coefficient $\tilde{N}_{00}$ is not needed in the proof. It is worth observing that (3.13) and (3.14) may be combined into a single expression using several results from ref. [2]. Hence (3.12) takes the form

$$\tilde{N}^{rs}_{nm} = \frac{(-)^{m+1}}{m}\delta^{rs}\delta_{nm} - \frac{2}{3}\cos\frac{2}{3}\pi(r-s)\left[\tilde{N}_{nm} - \frac{3(-)^m}{4}\left(a_m S^b_n + b_m S^a_n\right)\delta_{mn}\right], \tag{3.15}$$

where

$$S^{a(b)}_n = \sum_{n+m=even}\frac{a(b)_m}{n+m}\,. \tag{3.16}$$

It is clear that the above expression for the off diagonal elements reduces to (3.12) with $\tilde{N}_{nm}$ given by (3.13). For the diagonal elements one has to take the limit $n \to m$ in (3.15) to recover (3.12) with $\tilde{N}_{mm}$ given by (3.14). The limits needed here can easily be evaluated using results derived in [2]. Here we only give their values

$$\tilde{N}_{nn} = lim_{m\to n}\tilde{N}_{2n\,2m} = -\frac{\sqrt{3}}{2\pi}\left(a_n\tilde{\Sigma}^b_n - b_n\tilde{\Sigma}^a_n\right) + \frac{(-)^n}{2n}a_n b_n, \tag{3.17}$$

where

$$\tilde{\Sigma}^{a(b)}_n \equiv \sum_{n+m=2k}^{\infty}\frac{a(b)_n}{(n+m)^2}\,, \tag{3.18}$$

can be evaluated using the integral representation of the Taylor modes (see refs. [5, 11, 2]). Now we are ready to prove (3.9) for the non zero modes ($n \geq 1$). For $k=0$ it is clear that (3.9) is true for all values of $s = 1,2,3$. Next we consider $k = even \geq 2$. For this case (3.8) reduces to

$$\Omega^s_{2n\,2k} = \sum_{r=1}^{3} e^{2i\pi r/3}\left[2k\tilde{N}^{rs}_{2n\,2k} + \delta^{rs}\delta_{n\,k} + 2i\sqrt{3}\sum_{m=0}^{\infty} B_{2n\,2m+1}2k\tilde{N}^{rs}_{2m+1\,k}\right]. \tag{3.19}$$

To evaluate the sum in the above expression we substitute the explicit values of the Neumann Coefficients and make use of the various identities in ref. [2] to obtain

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\tilde{N}^{rs}_{2m+1\,2k} = -\frac{1}{3\sqrt{3}}\sin\frac{2}{3}\pi(r-s)\tilde{N}_{2m+1\,2k}\,, \tag{3.20}$$

for $k \neq n$. Similarly we obtain

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\tilde{N}^{rs}_{2m+1\,2n} = -\frac{1}{3\pi}\sin\frac{2}{3}\pi(r-s)$$

$$\left[\frac{\pi}{2\sqrt{3}}\frac{a_{2n}b_{2n}}{2n} - a_{2n}\left(\tilde{\Sigma}^b_{2n} + \tilde{\Sigma}^b_{-2n}\right) + b_{2n}\left(\tilde{\Sigma}^a_{2n} - \tilde{\Sigma}^a_{-2n}\right)\right], \tag{3.21}$$

for $k = n$. Substituting (3.20) into (3.19), we get after some algebra

$$\Omega^s_{2n\,2k}|_{k\neq n} = -\frac{2}{3}(2k)\tilde{N}^{rs}_{2n\,2k}e^{-2i\pi s/3}\sum_{r=1}^{3} e^{4i\pi r/3} = 0\,, \tag{3.22}$$

since $\sum_{r=1}^{3} e^{4i\pi r/3} = 0$. For the case $k = n$ we still need to simplify (3.21). Using the following identity (see ref. [2] for derivation)

$$\tilde{\Sigma}^{a(b)}_{-2n} = +(-)\frac{1}{2}\tilde{\Sigma}^{a(b)}_{2n} + \frac{\pi\sqrt{3}}{4}S^{a(b)}_{2n}\,, \tag{3.23}$$

eq. (3.21) reduces to

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\tilde{N}^{rs}_{2m+1\,2n} = -\frac{1}{3\sqrt{3}}\sin\frac{2}{3}\pi(r-s)$$

$$\left[\frac{a_{2n}b_{2n}}{2(2n)} - \frac{\sqrt{3}}{2\pi}\left(a_{2n}\tilde{\Sigma}^b_{2n} - b_{2n}\tilde{\Sigma}^a_{2n}\right) - \frac{3}{4}\left(2_{2n}S^b_{2n} + b_{2n}S^a_{2n}\right)\right]. \tag{3.24}$$

Recalling (3.17) the above expression reduces to

$$-\frac{1}{3\sqrt{3}}\sin\frac{2}{3}\pi(r-s)\left[\tilde{N}_{2n\,2n} - \frac{3}{4}\left(2_{2n}S^b_{2n} + b_{2n}S^a_{2n}\right)\right]. \tag{3.25}$$

Substituting this back into (3.19) we see that

$$\Omega^s_{2n\,2n} = -\frac{2}{3}\left(\tilde{N}_{2n\,2n} - \frac{3}{2(2n)}\right)\sum_{r=1}^{3} e^{\frac{2}{3}i\pi(2r-s)} = 0. \tag{3.26}$$

Combining (3.22) and (3.26) we see that

$$\Omega^s_{2n\,2k} = 0\,, \tag{3.27}$$

for all values of $s = 1, 2, 3$ and all $n, k \geq 1$. To complete the proof of (3.9) we still have the case $k = odd \geq 1$ to deal with. For $k = odd \geq 1$, eq. (3.8) reduces to

$$\Omega^s_{2n\,2k+1} = \sum_{r=1}^{3} e^{2i\pi r/3} \Big[(2k+1)\tilde{N}^{rs}_{2n\,2k+1}$$
$$+2i\sqrt{3}\sum_{m=0}^{\infty} B_{2n\,2m+1}\left((2k+1)\tilde{N}^{rs}_{2m+1\,2k+1} + \delta^{rs}\delta_{m\,k}\right)\Big]\,. \tag{3.28}$$

Here $\tilde{N}^{rs}_{2n\,2k+1}$ is given by (3.11) and $\tilde{N}^{rs}_{2m+1\,2k+1}$ is given by (3.15). The only difficulty in (3.28) is the sum over the Neumann coefficients since this sum is potentially divergent. Let us first consider the sum over $\tilde{N}_{2m+1\,2k+1}$ in $\tilde{N}^{rs}_{2m+1\,2k+1}$, i.e.,

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\tilde{N}_{2m+1\,2k+1}\,. \tag{3.29}$$

Recalling the explicit expression for $\tilde{N}_{2m+1\,2k+1}$, the above expression reduces to

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\tilde{N}_{2m+1\,2k+1} = \frac{(-)^{k+1}}{2}\sum_{m=0}^{\infty} B_{2n\,2m+1}$$
$$(-)^m \frac{(2m+1)b_{2m+1}a_{2k+1} - (2k+1)b_{2k+1}a_{2m+1}}{(2m+1)^2 - (2k+1)^2}\,. \tag{3.30}$$

This expression is very delicate since it is potentially divergent when m takes the value k. We recall that this problem of potentially divergent sums have been studied in detail in ref. [2] and various identities were derived there. Thus using the following identity (derived in ref. [2])

$$\sum_{m=0}^{\infty} \frac{a_{2k+1}b_{2n+1} + (-)b_{2k+1}a_{2n+1}}{(2k+1) + (-)(2n+1)} = \frac{2(-1)}{2k+1}\,, \tag{3.31}$$

and many other identities derived in [2], we see after a lengthy otherwise a straight forward calculation that eq. (3.30) reduces to

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\tilde{N}_{2m+1\,2k+1} = \frac{1}{2}$$
$$\left[\frac{\pi}{2\sqrt{3}}\left(a_{2k+1}b_{2n}B_{2k+1\,2n} - b_{2k+1}a_{2n}B_{2n\,2k+1}\right) + \frac{1}{2k+1}B_{2n\,2k+1}\right]. \tag{3.32}$$

Noting that the expression in the round bracket in (3.33) is proportional to $\tilde{N}_{2n\,2k+1}$ we obtain

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\tilde{N}_{2m+1\,2k+1} = \frac{1}{2}\left[-\frac{1}{\sqrt{3}}\tilde{N}_{2n\,2k+1} + \frac{1}{2k+1}B_{2n\,2k+1}\right]. \qquad (3.33)$$

Now using (3.33) and the explicit value of $\tilde{N}^{rs}_{2n\,2k+1}$ one finds, after a little manipulation , that (3.28) reduce to

$$\Omega^s_{2n\,2k+1} = \sum_{r=1}^{3} e^{2i\pi r/3}\left[\frac{2i}{3}(2k+1)\tilde{N}_{2n\,2k+1}e^{\frac{2}{3}i\pi(r-s)}\right.$$
$$\left. +4i\sqrt{3}B_{2n\,2k+1}\left(\delta_{rs} - \frac{2}{3}\cos\frac{2}{3}\pi(r-s)\right)\right], \qquad (3.34)$$

which is identically zero for all $s = 1,2,3$ as required, since

$$\sum_{r=1}^{3} e^{4i\pi r/3} = \sum_{r=1}^{3} e^{2i\pi r/3}\left(\delta_{rs} - \frac{2}{3}\cos\frac{2}{3}\pi(r-s)\right) = 0\,. \qquad (3.35)$$

This completes the proof of (3.9) and consequently (3.1) follows. It remains to show that (3.2) is also true in the full string Oscillator Hilbert space. The proof follow exactly the same lines and one only need to prove the identity given in (3.9) with Ω given by

$$\Omega^s_{2n\,k} = \sum_{r=1}^{3} e^{2i\pi r/3}\left[2n\tilde{N}^{rs}_{2n\,k} - \delta^{rs}\delta_{2n\,k} - \frac{2i}{\sqrt{3}}\sum_{m=0}^{\infty} B_{2n\,2m+1}\right.$$
$$\left.\left((2m+1)\tilde{N}^{rs}_{2m+1\,k} - \delta^{rs}\delta_{2m+1\,k}\right)\right]; \quad n > 0, k \geq 0\,. \qquad (3.36)$$

At this point it is worth mentioning that the sum over m in (3.36), is different from that in (3.8); it has a an extra factor $(2m+1)$ multiplying the neumann coefficient $\tilde{N}^{rs}_{2m+1\,k}$. This extra factor may easily cause the convergent series in (3.8) to diverge. Luckily this is not the case as we shall see bellow. To prove (3.2) we only need to show that

$$\Omega^s_{2n\,k} = 0\,, \qquad (3.37)$$

is true separately for all $s = 1,2,3$ and all $n > 0, k \geq 0$. Now there are three cases to consider $k = 0, even \geq 2, odd \geq 1$ for each value of n. The $k = 0$ case can easily be checked using the identity

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}(2m+1)\tilde{N}^{rs}_{2m+1\,0} = \frac{1}{\sqrt{3}}\sin\frac{2}{3}\pi(r-s)\,(-)^n b_{2n}\,. \qquad (3.38)$$

This identity can easily be verified using the fact

$$\Sigma^b_{-n} = \frac{1}{2}\Sigma^b_n = \frac{1}{2}\pi\left(\frac{1}{3}\right)^{1/2} b_n\,, n \geq 1\,. \tag{3.39}$$

The other two case are more involved and more identities are needed to carry out the proof. Let us start with $k = even \geq 2$. For this case (3.36) becomes

$$\Omega^s_{2n\,2k} = \sum_{r=1}^{3} e^{2i\pi r/3}\left[2n\tilde{N}^{rs}_{2n\,2k} - \delta^{rs}\delta_{n\,k} - \frac{2i}{\sqrt{3}}\sum_{m=0}^{\infty} B_{2n\,2m+1}\right.$$

$$\left.(2m+1)\tilde{N}^{rs}_{2m+1\,2k}\right]\,. \tag{3.40}$$

Let us consider the sum in the above expression, i.e.,

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}(2m+1)\tilde{N}^{rs}_{2m+1\,2k}\,. \tag{3.41}$$

This sum can be evaluated, by using several results from ref. [2] . For $k \neq n$, we get

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}(2m+1)\tilde{N}^{rs}_{2m+1\,2k}|_{k\neq n} = \frac{1}{\sqrt{3}}\sin\frac{2}{3}\pi(r-s)(2n)\tilde{N}_{2n\,2k}. \tag{3.42}$$

Now using this and the explicit value of $\tilde{N}^{rs}_{2n\,2k}$ we obtain

$$\Omega_{2n\,2k}|_{k\neq n} = -\frac{2}{3}(2n)\tilde{N}_{2n\,2k}e^{-2i\pi s/3}\sum_{r=1}^{3} e^{4i\pi r/3} = 0\,. \tag{3.43}$$

Similarly for $k = n$ the sum can be performed using the various identities of ref. [2]. Thus

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}(2m+1)\tilde{N}^{rs}_{2m+1\,2n} = \frac{1}{\sqrt{3}}\sin\frac{2}{3}\pi(r-s)(2n)$$

$$\left[\tilde{N}_{2n\,2n} + \frac{1}{4}\left(a_{2n}S^b_{2n} + b_{2n}S^a_{2n}\right)\right]\,. \tag{3.44}$$

By explicit substitution we see that

$$\Omega^s_{2n\,2n} = \sum_{r=1}^{3} e^{2i\pi r/3}\left[-\frac{2}{3}e^{2i\pi(r-s)}(2n)\tilde{N}_{2n\,2n}\right.$$

$$-2\left(\delta_{rs}-\frac{1}{2}\cos\frac{2}{3}\pi(r-s)+\frac{i}{6}\sin\frac{2}{3}\pi(r-s)\right)\Big]=0, \tag{3.45}$$

since

$$\sum_{r=1}^{3}e^{4i\pi r/3}=\sum_{r=1}^{3}\left(\delta_{rs}-\frac{1}{2}\cos\frac{2}{3}\pi(r-s)+\frac{i}{6}\sin\frac{2}{3}\pi(r-s)\right)=0. \tag{3.46}$$

This completes the proof of (3.37) for $k = even \geq 0$. Still need to consider $k = odd \geq 1$. For this case, (3.36) gives

$$\Omega^{s}_{2n\,2k+1}=\sum_{r=1}^{3}e^{2i\pi r/3}\Bigg[2n\tilde{N}^{rs}_{2n\,2k+1}-\frac{2i}{\sqrt{3}}\sum_{m=0}^{\infty}B_{2n\,2m+1}$$
$$\left((2m+1)\tilde{N}^{rs}_{2m+1\,2k+1}-\delta^{rs}\delta_{2m+1\,2k+1}\right)\Bigg]\,. \tag{3.47}$$

Now we proceed to evaluate

$$\sum_{m=0}^{\infty}B_{2n\,2m+1}(2m+1)\tilde{N}^{rs}_{2m+1\,2k+1}\,. \tag{3.48}$$

By direct substitution, we have

$$\sum_{m=0}^{\infty}B_{2n\,2m+1}(2m+1)\tilde{N}^{rs}_{2m+1\,2k+1}=\left(\delta^{rs}-\cos\frac{2}{3}\pi(r-s)\right)B_{2n\,2k+1}$$
$$-\frac{2}{3}\cos\frac{2}{3}\pi(r-s)\sum_{m=0}^{\infty}B_{2n\,2m+1}(2m+1)\tilde{N}_{2m+1\,2k+1}\,. \tag{3.49}$$

The sum in the above equation can be evaluated using many results from ref. [2] and the fact

$$\frac{2m+1}{(2m+1)\pm(2k+1)}=1\mp\frac{2k+1}{(2m+1)\pm(2k+1)}\,. \tag{3.50}$$

After a rather lengthy algebraic exercise in the use of various identities from ref. [2] we obtain

$$\sum_{m=0}^{\infty}B_{2n\,2m+1}(2m+1)\tilde{N}_{2m+1\,2k+1}=\frac{\sqrt{3}}{2}(-)^{k+n}a_{2k+1}b_{2n}-\frac{3}{2}B_{2n\,2k+1}$$
$$+\frac{\pi\sqrt{3}}{2}\left(\frac{2k+1}{2}\right)\left[B_{2n\,2k+1}a_{2k+1}b_{2n}-B_{2k+1\,2n}b_{2k+2}a_{2n}\right]\,. \tag{3.51}$$

Using (4.9) and recalling the fact

$$\tilde{N}_{2n\,2k+1} = \frac{(-)^{n+k}}{2}\left[\frac{b_{2n}a_{2k+1}+b_{2k+1}a_{2n}}{2n-(2k+1)} + \frac{b_{2n}a_{2k+1}-b_{2k+1}a_{2n}}{2n+(2k+1)}\right]$$
$$= -\frac{\pi}{2}\left(B_{2k+1\,2n}b_{2n}a_{2k+1} - B_{2n\,2k+1}b_{2k+1}a_{2n}\right) . \tag{3.52}$$

Eq. (4.6) becomes

$$\Omega^s_{2n\,2k+1} = -\frac{i\pi}{3}(2n)\left(B_{2k+1\,2n}b_{2n}a_{2k+1} - B_{2n\,2k+1}b_{2k+1}a_{2n}\right)e^{-2i\pi s/3}\sum_{r=1}^{3}e^{4i\pi r/3}, \tag{3.53}$$

which is identically zero. In arriving at the above result we have also used the fact

$$\frac{\sqrt{3}}{2}\left[(-)^{k+n}a_{2k+1}b_{2n} + \pi\left(\frac{2k+1}{2}\right)\left(B_{2n\,2k+1}a_{2k+1}b_{2n} - B_{2k+1\,2n}b_{2k+1}a_{2n}\right)\right]$$
$$= -\frac{\pi\sqrt{3}}{4}\left[(2n)a_{2k+1}b_{2n} + (2k+1)b_{2k+1}a_{2n}\right]B_{2k+1\,2n} , \tag{3.54}$$

which can be verified by direct substitution. This completes the proof for the overlap equation of $\Pi^r_c(\sigma)$.

Exactly the same procedure is used to to prove the comma overlaps for $b^r(\sigma)$ and $\Pi^r_b(\sigma)$, given in appendix A, in the Oscillator Hilbert space of the full string. They are seen to hold too. This completes the demonstration of the basic comma overlaps on the fermionic ghost three-point vertex of WSFT. In the next section we consider the comma $BRST$ operators, Q^L and Q^R in the full string representation.

4 BRST Operator

In ref. [1] it was shown that the identity

$$P^L_0 + P^R_0 = p_0 , \tag{4.1}$$

holds in the full string Oscillator Hilbert space. However, there we were not in a position to address the identity which relates the $BRST$ operator

(Q_L) to the corresponding $BRST$ operator (Q_R). Recall that Witten's action has a local gauge invariance [3] which requires

$$\left[Q^1 + Q^2 + Q^3\right] |V_3^{Witten} >= 0\,, \tag{4.2}$$

where Q is the BRST charge. This fact was checked explicitly[10] using oscillators in ref. [5]. The transformation to the comma theory requires the stronger condition

$$\left(Q_R^r + Q_L^{r+1}\right) |V_3^{comma} >= 0\,, \tag{4.3}$$

where Q_L and Q_R are the comma $BRST$ charges and $r = 1, 2, 3,\ with\, 4 \equiv 1$. Now it is not obvious that the Witten's vertex satisfies the comma overlaps (4.3). In the full string theory, the $BRST$ current may be expanded in modes

$$j(\sigma) = Q_0 + \sqrt{2}\sum_{n=1}^{\infty} J_n \cos n\sigma\,; \quad \sigma \in [o, \pi], \tag{4.4}$$

where

$$J_n = \frac{1}{\sqrt{2}}\left(Q_n + Q_{-n}\right)\,, \tag{4.5}$$

and, up to normal ordering ambiguities,

$$Q_n \simeq \sum_k c_{-k}\mathcal{L}_{n+k}\,. \tag{4.6}$$

$\mathcal{L}_n$ is defined in the standard way (see eq. (1.11)). The zero mode Q_0 is just the usual $BRST$ operator Q (see eq. (1.10)). In the comma theory the $BRST$ currents are defined in the usual way

$$j^r(\sigma) = \begin{cases} j(\sigma)\ ,\ if\ r = 1 \equiv L\,, \\ j(\pi - \sigma)\ ,\ if\ r = 2 \equiv R\ , \\ \sigma \in [0, \frac{\pi}{2})\,, \end{cases} \tag{4.7}$$

with

$$lim^{L,R}_{\sigma \to \pi/2}(\sigma) = j(\pi/2)\,. \tag{4.8}$$

The mode expansion of the comma $BRST$ currents reads

$$j^r(\sigma) = Q^r + \sqrt{2}\sum_{n=1}^{\infty} j^r_{2n} \cos 2n\sigma\,; \quad \sigma \in [0, \pi/2)\,. \tag{4.9}$$

[10] Eq. (4.2) was also proven, using the comma formulation of WSFT in ref. [4].

with $r = 1, 2$ refers to the left, right parts of the string respectively. The comma modes are given by

$$Q^r = Q + (-)^r \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n}{2n-1} \left(Q_{2n-1} + Q_{-2n+1}\right) , \tag{4.10}$$

$$j^r_{2n} = \frac{1}{\sqrt{2}} \left(Q_{2n} + Q_{-2n}\right) + \sqrt{2}(-)^r \sum_{m=1}^{\infty} B_{2n\, 2m-1} \left(Q_{2m-1} + Q_{-2m+1}\right) , \tag{4.11}$$

and their inverse is given by

$$Q = \frac{1}{2} \left(Q^L + Q^R\right) , \tag{4.12}$$

$$j_{2n} = \frac{1}{2} \sum_{r=1}^{2} \left(j^L_{2n} + j^R_{2n}\right) , \quad n \geq 1, \tag{4.13}$$

$$j_{2n-1} = -\frac{\sqrt{2}}{\pi} \frac{(-)^n}{2n-1} \left(Q_L + Q_R\right) - \sum_{m=1}^{\infty} B_{2m\ 2n-1} \left(j^L_{2m} + j^R_{2m}\right) . \ n \geq 1 , \tag{4.14}$$

Now we need to prove that

$$\left(Q^1_R + Q^2_L\right) |V^W_3 >= \left(Q^2_R + Q^3_L\right) |V^W_3 >= \left(Q^3_R + Q^1_L\right) |V^W_3 >= 0, \tag{4.15}$$

i.e., to show that (4.15) continues to hold in the Witten's *cubic* vertex representation. The left (right) operators Q_L (Q_R) entering (4.15) and given by (4.10) are precisely (up to an overall factor) those obtained in ref. [11] where eq. (4.15) has been verified, in a great detail. Therefore there is no need to repeat the calculation here. A final word about the anomalous symmetries of $|V_3>$ generated by K^{matter}_n and K^{ghost}_n in the comma representation. These problems have already been addressed in ref. [4].

5 Conclusions

In this paper and in [1, 2] we have completed the proof of the comma overlap equations in the Hilbert space of the full string developed in ref. [5, 6, 7, 8, 9]. We have shown that Witten's cubic vertex (matter plus ghost) indeed solves the comma overlaps thus establishing the equivalence of the two theories at least on the level of the vertices. We have seen that multiplication $(*)$ in WSFT is associative [1] and that Q is a derivation. The invariance under (K_n) has been discussed in refs. [1, 4]. A challenging problem remains; is the construction of a conformal operator connecting the *comma* $3-vertex$ and Witten's $cubic-vertex$ (if it exists). Work in this direction is in progress.

Appendix A

The comma antighost coordinate, $b^r(\sigma)$, is defined exactly as in (1.12) with $c(\sigma)$ being replaced by $b(\sigma)$. Thus we obtain

$$b^r(\sigma) = \sqrt{2}\sum_{n=1} h^r_{2n-1}\sin(2n-1)\sigma\,, \quad \sigma\in[0,\pi/2), \tag{A.1}$$

with

$$h^r_{2n-1} = \frac{1}{\sqrt{2}}\left(b_{2n-1}-b_{-2n+1}\right) - \sqrt{2}(-)^r\sum_{m=1}^{\infty} B_{2n-1\,2m}\,\left(b_{2m}-b_{-2m}\right). \tag{A.2}$$

The inverse relations read

$$\frac{1}{\sqrt{2}}\left(b_{2n}-b_{-2n}\right) = \sum_{r=1}^{2}\sum_{m=1}^{\infty}(-)^r\left(\frac{2n}{2m-1}\right)B_{2n\,2m-1}\,h^r_{2m-1}\,, \tag{A.3}$$

$$\frac{1}{\sqrt{2}}\left(b_{2n-1}-b_{-2n+1}\right) = \frac{1}{2}\sum_{r=1}^{2}h^r_{2n-1}\,. \tag{A.4}$$

Observe that eqs. (A.3) and (A.4) are readily consistent with the requirement that the neighboring points on the string are not allowed to wonder to far from each other while the midpoint vary freely. Likewise, for the conjugate momentum $\pi^r_b(\sigma)$, we obtain

$$\pi^r_b(\sigma) = \sqrt{2}\sum_{n=1} z^r_{2n-1}\sin(2n-1)\sigma\,, \quad \sigma\in[0,\pi/2), \tag{A.5}$$

with

$$z^r_{2n-1} = \frac{1}{\sqrt{2}}\left(c_{2n-1}-c_{-2n+1}\right) - \sqrt{2}(-)^r\sum_{m=1}^{\infty} B_{2n-1\,2m}\,\left(c_{2m}-c_{-2m}\right). \tag{A.6}$$

The inverse relations read

$$\frac{1}{\sqrt{2}}\left(c_{2n}-c_{-2n}\right) = \sum_{r=1}^{2}\sum_{m=1}^{\infty}(-)^r\left(\frac{2n}{2m-1}\right)B_{2n\,2m-1}\,z^r_{2m-1}\,, \tag{A.7}$$

$$\frac{1}{\sqrt{2}}\left(c_{2n-1}-c_{-2n+1}\right)=\frac{1}{2}\sum_{r=1}^{2} z^{r}_{2n-1}\,. \tag{A.8}$$

Again eqs. (A.7) and (A.8) are consistent with the constraint on the comma modes. Quantization is carried out as before. Introducing the hated operators

$$\hat{h}^r_n=\frac{i}{\sqrt{2}}h^r_{2n-1}\,,\qquad \hat{z}^r_n=\frac{i}{\sqrt{2}}z^r_{2n-1} \tag{A.9}$$

we see that the desired anticommutation relations are indeed satisfied.

The comma overlaps, for $b^r(\sigma)$, are are

$$\left[B^L(\sigma)+e^{2\pi i/3}B^R(\sigma)\right]|V_3^{comma}>=0\,,\ \sigma\in[0,\pi/2), \tag{A.10}$$

$$\left[B_3^L(\sigma)+B_3^R(\sigma)\right]|V_3^{comma}>=0\,,\ \sigma\in[0,\pi/2), \tag{A.11}$$

where $B^r(\sigma)\equiv B_1^r(\sigma)=\overline{B_2^r(\sigma)}$ and

$$B_k^r(\sigma)=\frac{1}{\sqrt{3}}\sum_{l=1}^{3} b_l^r(\sigma)e^{2\pi ilk/3},\ r=L,\,R,\quad k=1,2,3\,. \tag{A.12}$$

Similarly , in the complex space of the commas, the overlaps for $\pi_b^r(\sigma)$ are given by the same equations with $B^r(\sigma)\to\Pi_b^r(\sigma)$ and the relative sign flipped.

Appendix B

The integration-identity overlap

Let us proceed to construct the ghost Fock space vector, $|I^{ghost}>$. We first introduce the comma ghost modes, γ^r_n and β^r_n,

$$\gamma^r_n\equiv\frac{1}{\sqrt{2}}(\hat{y}^r_n+i\hat{z}^r_n)\,,\qquad n\geq 1, \tag{B.1}$$

$$\beta^r_n\equiv\frac{1}{\sqrt{2}}(\hat{g}^r_n+i\hat{h}^r_n)\,,\qquad n\geq 1\,, \tag{B.2}$$

with $\gamma^r_{-n}=\gamma^{r\,\dagger}_n,\ \ \beta^r_{-n}=\beta^{r\,\dagger}_n\,,\quad n\geq 1$. The zero modes are defined by

$$\gamma^r_0\equiv\hat{c}^r_0\,,\qquad \beta^r_0\equiv\hat{b}^r_0. \tag{B.3}$$

It is a straight forward to check that the desired anticommutation relations

$$\{\gamma^r_n,\beta^s_m\}=\delta^{rs}\delta_{n+m,0}, \tag{B.4}$$

are satisfied. Now we are ready to state the overlaps needed to construct the ghost Fock space vector, $|I^{ghost}>$ in the comma representation. Recall that the correct prescription is that $c^r(\sigma)$, $\pi_b^r(\sigma)$ are treated like momenta and $b^r(\sigma)$, $\pi_c^r(\sigma)$ are treated like coordinates, i.e., we identify

$$c^L(\sigma) = -c^R(\sigma), \qquad \pi_b^L(\sigma) = -\pi_b^R(\sigma)\,, \tag{B.5}$$

$$b^L(\sigma) = b^R(\sigma), \qquad \pi_c^L(\sigma) = \pi_c^R(\sigma)\,. \tag{B.6}$$

The overlap equations that $|I^{ghost}>$ must satisfy are then

$$\left[(\beta_n^L + \beta_n^R) + (\beta_{-n}^L + \beta_{-n}^R)\right] |I^{ghost}>= 0, \tag{B.7}$$

$$\left[(\gamma_n^L + \gamma_n^R) - (\gamma_{-n}^L + \gamma_{-n}^R)\right] |I^{ghost}>= 0, \tag{B.8}$$

$$\left[(\beta_n^L - \beta_n^R) - (\beta_{-n}^L - \beta_{-n}^R)\right] |I^{ghost}>= 0, \tag{B.9}$$

$$\left[(\gamma_n^L - \gamma_n^R) + (\gamma_{-n}^L - \gamma_{-n}^R)\right] |I^{ghost}>= 0\,. \tag{B.10}$$

Expressing $|I^{ghost}>$ as the exponential of quadratic form in creation operators acting on the Fock space vacuum ($|\Omega>^r$) annihilated by all $\gamma_{n\geq 1}^r$, $\beta_{n\geq 1}^r$,

$$|I^{ghost}>= e^{\beta_{-n}^r \theta_{nm}^{rs} \gamma_{-m}^s} |\Omega>^L |\Omega>^R, \tag{B.11}$$

and using eqs. (B.7) through (B.10) to fix the quadratic form, we obtain

$$\theta_{nm}^{LR} = \theta_{nm}^{RL} = \delta_{nm}\,, \qquad \theta_{nm}^{LL} = \theta_{nm}^{RR} = 0\,. \tag{B.12}$$

No zero modes appear in the $|I^{ghost}>$ since,

$$\left[\gamma_0^L + \gamma_0^R\right] |I^{ghost}>= \left[\beta_0^L - \beta_0^R\right] |I^{ghost}>= 0\,. \tag{B.13}$$

Since the zero modes γ_0^r, β_0^r anticommute, the vacuum state, $|\Omega>^r$, is a doublet of states, $|\pm>^r$. Now we have to determine which vacuum appears in (B.11). Eq. (B.13) implies that $\gamma_0^r = 0$ but that $\beta_0^r \neq 0$. Therefore we choose the vacuum to satisfy

$$\gamma_0^r |\Omega>^r = 0\,, \qquad r = L, R\,. \tag{B.14}$$

Construction of the cubic-string vertex

Now we are ready to construct the ghost interaction vertex. To simplify the calculation we introduce the variables

$$\Gamma_k^r = \frac{1}{\sqrt{3}} \sum_{l=1}^{3} \gamma_l^r e^{2i\pi kl/3}, \quad r = L, R, \quad k = 1, 2, 3, \tag{B.15}$$

and likewise for $\mathcal{B}_k^r = \mathcal{B}_k^r(\beta_1^r, \beta_2^r, \beta_3^r)$. Now the overlap equation in (2.6), (A.10), and their conjugate momenta yield

$$\left[(\mathcal{B}_n^L + e\mathcal{B}_n^R) + (\mathcal{B}_{-n}^L + e\mathcal{B}_{-n}^R)\right] |V_3^{\mathcal{B}\Gamma} >= 0, \tag{B.16}$$

$$\left[(\mathcal{B}_n^L - e\mathcal{B}_n^R) - (\mathcal{B}_{-n}^L - e\mathcal{B}_{-n}^R)\right] |V_3^{\mathcal{B}\Gamma} >= 0, \tag{B.17}$$

$$\left[\Gamma_n^L - e\Gamma_n^R) + (\Gamma_{-n}^L - e\Gamma_{-n}^R)\right] |V_3^{\mathcal{B}\Gamma} >= 0, \tag{B.18}$$

$$\left[\Gamma_n^L + e\Gamma_n^R) - (\Gamma_{-n}^L + e\Gamma_{-n}^R)\right] |V_3^{\mathcal{B}\Gamma} >= 0, \tag{B.19}$$

with $e \equiv e^{2i\pi/3}$, whereas the overlap eqs. (2.7), (A.11) and their conjugate momenta satisfy the usual (identity-type) equations. The most general solution has the form

$$|V_3^{\mathcal{B}\Gamma} >= exp\left[-\Gamma_{-n}^{3,r}\theta_{nm}^{rs}\mathcal{B}_{-m}^{3,s} + \overline{\mathcal{B}_{-n}^r} K_{nm}^{rs}\Gamma_{-m}^s + \overline{\Gamma_{-n}^r}\tilde{K}_{nm}^{rs}\mathcal{B}_{-m}^s\right] \prod_{i=1}^{3} |\Omega >_i^L |\Omega >_i^R . \tag{B.20}$$

Now using the overlaps, after a little algebra we obtain

$$\left(\tilde{K}_{nm}^{Ls} + e\tilde{K}_{nm}^{Rs}\right) + \left(\delta^{Ls}\delta_{nm} + e\delta^{Rs}\delta_{nm}\right) = 0, \tag{B.21}$$

$$\left(\tilde{K}_{nm}^{Ls} - e\tilde{K}_{nm}^{Rs}\right) - \left(\delta^{Ls}\delta_{nm} - e\delta^{Rs}\delta_{nm}\right) = 0, \tag{B.22}$$

and the same equations for K_{nm}^{rs} with $e \to -e$. These equations are trivial to solve. Hence, one obtains

$$K_{nm}^{Ls} = -\tilde{K}_{nm}^{Ls} = e\delta^{Rs}\,\delta_{nm}\,, \quad s = L, R, \tag{B.23}$$

$$K_{nm}^{Rs} = -\tilde{K}_{nm}^{Rs} = e\delta^{Ls}\,\delta_{nm}\,, \quad s = L, R. \tag{B.24}$$

It is left to specify which vacuum state occurs in (B.20). The overlap equations for the zero modes dictate the choice

$$|\Omega >_i^r \equiv |\Gamma_0^{ir} = 0 > \ . \tag{B.25}$$

Rewriting $|V_3^{B\Gamma}>$ in the original creation operators, i.e., $\beta^\dagger$, $\gamma^\dagger$ we obtain

$$|V_3^{\beta\gamma}>= exp\left(\beta_{-n}^{1,L}\gamma_{-n}^{3,R}+\beta_{-n}^{2,L}\gamma_{-n}^{1,R}+\beta_{-n}^{3,L}\gamma_{-n}^{2,R}+\beta_{-n}^{1,R}\gamma_{-n}^{2,L}\right.$$

$$\left.+\beta_{-n}^{2,R}\gamma_{-n}^{3,L}+\beta_{-n}^{3,R}\gamma_{-n}^{1,L}\right)\prod_{i=1}^{3}|\gamma_0^{i,L}=0>|\gamma_0^{i,R}=0> \, . \tag{B.26}$$

This is a very elegant form as compared to that in the full string formulation of WSFT.

We conclude this appendix by giving explicitly the equations relating the full-string ghost modes $(c_n\,,b_n)$ and the comma ghost modes $(\gamma_n^r\,,\beta_n^r\,,\ r=L,R)$. For the full-string modes we have ,

$$c_0=\frac{1}{\sqrt{2}}\sum_{r=1}^{2}\gamma_0^r\,, \tag{B.27}$$

$$c_{2n}=\frac{1}{2}\sum_{r=1}^{2}\frac{1}{\sqrt{2}}\left(\beta_n^r+\beta_{-n}^r\right)-\sum_{r=1}^{2}(-)^r\sum_{m=1}^{\infty}\left(\frac{2n}{2m-1}\right)B_{2n\,2m-1}$$
$$\frac{1}{\sqrt{2}}\left(\gamma_m^r-\gamma_{-m}^r\right)\,,\qquad n\geq 1\,, \tag{B.28}$$

$$c_{2n-1}=\frac{\sqrt{2}}{\pi}\frac{(-)^n}{2n-1}\sum_{r=1}^{2}(-)^r\gamma_0^r-\frac{1}{2}\sum_{r=1}^{2}\frac{1}{\sqrt{2}}\left(\gamma_n^r-\gamma_{-n}^r\right)-\sum_{r=1}^{2}(-)^r\sum_{m=1}^{\infty}\left(\frac{2n-1}{2m}\right)$$
$$B_{2n-1\,2m}\frac{1}{\sqrt{2}}\left(\beta_m^r+\beta_{-m}^r\right)\,,\qquad n\geq 1\,, \tag{B.29}$$

and $c_{-n}\equiv c_n^\dagger\,,\ n\geq 1$. The same equations are obtained for b_n with $\gamma\rightleftharpoons\beta$. The inverse relations are

$$\gamma_0^r=c_0+\frac{2}{\pi}(-)^r\sum_{n=1}^{\infty}\frac{(-)^n}{2n-1}\left(c_{2n-1}+c_{-2n+1}\right)\,, \tag{B.30}$$

$$\gamma_n^r=\frac{1}{2\sqrt{2}}\left(b_{2n}+b_{-2n}\right)+\frac{1}{\sqrt{2}}(-)^r\sum_{m=1}^{\infty}B_{2n\,2m-1}\left(b_{2n-1}+b_{-2n+1}\right)$$
$$-\frac{1}{2\sqrt{2}}\left(c_{2n-1}-c_{-2n+1}\right)+\frac{1}{\sqrt{2}}(-)^r\sum_{m=1}^{\infty}B_{2n-1\,2m}\left(c_{2n}-c_{-2n}\right)\,,\ \ n\geq 1\,, \tag{B.31}$$

and $\gamma_{-n}\equiv\gamma_n^\dagger\,,n\geq 1$. Likewise we obtain the same equations for β_n^r with $c\rightleftharpoons b$. One can check that all the desired anticommutation relations are indeed satisfied. Observe that the change of representation does not conserve species i.e., it mixes ghosts and antighosts degrees of freedom.

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