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Abstract

A kinetic approach is used for a description of a plasma in the presence of a large amplitude wave (ie relativistic electrons). The case of a "small" momentum spread around an average momentum is considered and corrections to the cold plasma case (zero spread) are given.

A plasma is relativistic either in the presence of a wave of amplitude so large that the electron quiver velocity is close to the velocity of light c or when its temperature is large ($T \gtrsim mc^2$). The latter case has been reviewed by Mikhailovskii[1]. The case of a large amplitude wave has been solved by Tsytovich[2], see also Akhiezer et al[3], for the cold plasma case and Mori and Katsouleas[4] have tried to incorporate thermal effects using a water-bag model. This is not fully consistent: the problem is that in the presence of a large amplitude wave the plasma cannot be assumed to be close to thermal equilibrium and a plasma temperature for such a case cannot be defined. In the present work we propose a kinetic approach, based on the Vlasov equation, to investigate the effects of a "small" spread of electron momenta around an average momentum (cold plasma case). The assumed model is the same as for the cold plasma case: a collisionless, unmagnetized plasma with immobile ions; the wave field is $\underline{B}^L = \underline{B}^L(\xi), \underline{E}^L = \underline{E}^L(\xi)$ where $\xi = x' - Vt'$ in the laboratory frame with x' the direction of wave propagation; the electron distribution function is $f^L = f^L(\xi, \underline{p}')$, where $\underline{p}'$ is the electron momentum, and is assumed to satisfy the Vlasov relativistic equation. The wave is sub-luminous ($V < c$) or super-luminous ($V > c$): the case $V = c$ is not considered here. The Vlasov-Maxwell system of equations is most easily solved in a reference frame ($xyzt$) moving along the $x \equiv x'$ axis with velocity V_o relative to the lab frame, where:

$$V_o = \begin{cases} V \text{ for } V < c \\ c^2 V^{-1} \text{ for } V > c \end{cases} \quad \xi = \begin{cases} x\gamma_o^{-1} \\ -ct(\gamma_o^2 - 1)^{-1/2} \end{cases} \quad ; \gamma_o = (1-\beta_o^2)^{-1/2}, \beta_o = V_o/c \quad (1)$$

In the two cases the Vlasov Maxwell system becomes, in the new frame:

Super-luminous

$$\underline{B} = 0 \tag{2}$$

$$\rho + \gamma_o \rho_o = 0 \tag{3}$$

$$\frac{d}{dt}\underline{E}(t) = -4\pi(\underline{j} - \beta_o \gamma_o c \rho_o \hat{u}_x) \tag{4}$$

$$\frac{\partial f(\underline{p}, t)}{\partial t} - e\underline{E}(t).\frac{\partial f(t, \underline{p})}{\partial \underline{p}} = 0 \tag{5}$$

Sub-luminous

$$E_y = E_z = B_x = 0 \tag{6}$$

$$j_x - \beta_o \gamma_o c \rho_o = 0 \tag{7}$$

$$\frac{d}{dx}E_x(x) = 4\pi(\rho + \gamma_o \rho_o) \tag{8}$$

$$\frac{d}{dx}B_y(x) = \frac{4\pi}{c} j_y \tag{9}$$

$$\frac{d}{dx}B_z(x) = -\frac{4\pi}{c} j_z \tag{10}$$

$$v_x \frac{\partial f(x, \underline{p})}{\partial x} - e\left(E_x + \frac{v_y}{c}B_z - \frac{v_z}{c}B_y\right)\frac{\partial f(x, \underline{p})}{\partial p_x} + e\frac{v_z}{c}\left(B_z \frac{\partial f}{\partial p_y} - B_y \frac{\partial f}{\partial p_z}\right) = 0 \tag{11}$$

where the four-current is:

$$(c\rho, \underline{j}) = -e \int d^3p(c, \underline{v}(\underline{p}))f(\underline{p}) \tag{12}$$

and ρ_o is the immobile ion background (in the lab frame).

Introducing the potentials:

$$\underline{A}(t) = -e\int_{t_o}^{t} \underline{E}(t')dt'; \phi(x) = -\int_{x_o}^{x} E_x(x')dx' \tag{13}$$

the solution to the Vlasov equations can be found, via Fourier transform, in general (Transverse or Longitudinal wave) for the super-luminous case and for the case of a Longitudinal wave for the sub-luminous case. It is:

$$f(t, \underline{p}) = f_o\left(\underline{p} - \frac{e}{c}\underline{A}(t)\right) \qquad \text{(super - luminous)} \tag{14}$$

$$f(x,\underline{p}) = f_o\left(\frac{1}{c}\left[(\varepsilon - e\phi(x))^2 - p_\perp^2 c^2 - m^2c^4\right]^{1/2}, p_y, p_z\right) \text{(sub - luminous)} \quad (15)$$

where

$$\varepsilon = \left(m^2c^4 + p_x c^2 + p_\perp^2 c^2\right)^{1/2}, p_\perp^2 = p_y^2 + p_z^2 \quad (16)$$

Here f_o is the distribution function at an arbitrary time t_o ($\underline{A}(t_o) = 0$) in the first case and at an arbitrary point $x_o(\phi(x_o) = 0)$ in the second case. We stress again that f_o is not the "equilibrium" distribution function as no thermal equilibrium situation exists in the presence of the wave. To find the four-current it is now convenient to introduce the new variables:

$$\underline{q} = \underline{p} - \frac{e}{c}\underline{A} \text{(super - luminous)} \quad (17)$$

$$q_x = \frac{1}{c}\left(\left(\varepsilon(\underline{p}) - e\phi\right)^2 - \underline{p}_\perp^2 c^2 - m^2c^4\right)^{1/2} \text{sign} p_x \text{(sub - luminous)} \quad (18)$$

$$q_y = p_y, q_z = p_z$$

and from equation (12) we have:

$$(c\rho, \underline{j}) = -e\int d^3q f_o(q)\left(c, \frac{\left(\underline{q} + \frac{e}{c}\underline{A}\right)c}{\left[m^2c^2 + \left(q + \frac{e}{c}\underline{A}\right)^2\right]^{1/2}}\right) \text{(super - luminous)} \quad (19)$$

$$(c\rho, j_x) = -e\int_{D(\phi)} d^3q(c, v_x(q))\frac{V_x(q)}{v_x(q)} f_o(q) \text{(sub - luminous)} \quad (20)$$

where $D(\phi)$ is the domain of definition of the new variables:

$$|q_x| \geq \frac{1}{c}\left(\left[\left(m^2c^4 + p_\perp^2c^2\right)^{1/2} - e\phi\right]^2 - p_\perp^2c^2 - m^2c^4\right)^{1/2} \quad (21)$$

and $V_x(q)/v_x(q)$ is the Jacobian of the transformation (18) with:

$$v_x(q) = \frac{c^2 q_x}{\varepsilon(\underline{q})}; V_x(q) = \frac{c^2 \boldsymbol{P}_x(q_x)}{\mathcal{E}(\underline{p}(\underline{q}))}, \mathcal{E}(\underline{p}) = \varepsilon(\underline{p}) - e\phi \quad (22)$$

The "solution" (19) or (20) for the four-current must then be used with equations (3,4) in the super-luminous case and (7,8) in the sub-luminous case.

For the cold-plasma case $f_o(\underline{p}) = n_o\delta(\underline{p} - \underline{p}_o)$ the well known solutions, given eg. by Akhiezer et al[3], are recovered.

We look for corrections to the cold plasma results assuming that the electron momenta have a "small" spread around an average momentum $\underline{p}_o$, defined at $t = t_o$ (or $x = x_o$) as:

$$\underline{p}_o = \int d^3p\underline{p}f_o(\underline{p}) / \int d^3pf_o(\underline{p}) \tag{23}$$

We assume that the distribution function at time $t = t_o$ (or $x = x_o$) is peaked around $\underline{p}_o$, so that:

$$\int d^3pf_o(\underline{p})(p_i - p_{oi})(p_j - p_{oj}) = \left[mc^2 \int d^3pf_o(\underline{p})\right] c_{ij}, \quad |c_{ij}| \ll 1$$

$$\int d^3pf_o(\underline{p})(p_{i1} - p_{oi1}) \cdots (p_{in} - p_{oin}) = 0 \text{ for } \eta > 2 \tag{24}$$

The second moments c_{ij} represent (small) "thermal" corrections to the cold plasma case, taking into account a momentum spread around the average. We can now expand the integrands around $\underline{p}_o$ in equations (19) and (20), integrate term by term using equations (23,24) and then use equations (3,4) or (7,8) to find:

(a) super-luminous wave:

$$\frac{d^2u_x}{dt^2} = -\omega_o^2\gamma_o\left\{\beta_o + u_x(1+\underline{u}^2)^{-1/2} + \frac{1}{2}c_{ij}\frac{\partial^2}{\partial u_i\partial u_j}\left[u_x(1+\underline{u}^2)^{-1/2}\right]\right\} \tag{25}$$

$$\frac{d^2u_\alpha}{dt^2} = -\omega_o^2\gamma_o\left\{u_\alpha(1+\underline{u}^2)^{-1/2} + \frac{1}{2}c_{ij}\frac{\partial^2}{\partial u_i\partial u_j}\left[u_\alpha(1+\underline{u}^2)^{-1/2}\right]\right\} \alpha = y, z \tag{26}$$

where

$$\underline{u} = \frac{1}{mc}\left(\underline{p}_o + \frac{e}{c}\underline{A}\right) \;;\; \underline{E}(t) = -\frac{mc}{e}\frac{d\underline{u}}{dt} \tag{27}$$

(b) Sub-luminous wave:

$$c^2\frac{d^2u}{dx^2} = -\omega_o^2\gamma_o\left\{1 - \beta_o u(u^2-1)^{-1/2} + \beta_o\left[k_2(u^2-1)^{-3/2}(\frac{1}{u_o} - u) - k_1u(u^2-1)^{-5/2}\left(1 - \frac{u^2}{u_o^2}\right)\right]\right\} \tag{28}$$

where

$$u = \frac{(p_o^2c^2 + m^2c^4)^{1/2} + e\phi}{mc^2} \;;\; E = -\frac{mc^2}{e}\frac{du}{dx} \tag{29}$$

and

$$k_1 = \frac{3}{2}c_{xx}, k_2 = \frac{k_1}{u_o^2} + \frac{1}{2}(c_{yy} + c_{zz}) \tag{30}$$

The constant u_o can be determined from equation (7) and the expansion for j_x:

$$u_o = \left(1 - V_x^2(p_o)/c^2\right)^{-1/2} \quad ; \; j_x = -en_o\left(1 - \frac{k_2}{u_o^2}\right)V_x(p_o) = \beta_o\gamma_o\rho_o c \tag{31}$$

For the super-luminous wave consider now the two cases of transverse or longitudinal wave:

1) Super-luminous, transverse wave: $E_x = 0, u_x = \text{constant} \equiv \eta, \underline{u}^2 = \eta^2 + u_\perp^2$

Then equation (25) becomes:

$$(1+\underline{u}^2)^{5/2}\beta_o + \eta(1+\underline{u})^2 - \frac{3}{2}c_{xx}\eta(1+u_\perp^2) - c_{xy}u_y(1+u_\perp^2-2\eta^2) - c_{xz}u_z(1+u_\perp^2-2\eta^2)+$$

$$+ 3c_{yz}\eta u_y u_z - \frac{1}{2}c_{yy}\eta(1+\eta^2+u_z^2-2u_y^2) - \frac{1}{2}c_{zz}\eta(1+\eta^2+u_y^2-2u_z^2) = 0 \tag{32}$$

As for the cold plasma case, linearly polarized waves are not allowed: if for instance $E_y = 0$ (ie. u_y = constant) then from (32) it follows u_z = constant ie. $E_z = 0$.

Consider circularly polarized waves. Assume a solution:

$$\begin{aligned} E_y^L &= -E_o\cos\omega_L\tau \\ E_z^L &= -E_o\sin\omega_L\tau \end{aligned} \quad ; \tau \simeq t' - \frac{x'}{V} \tag{33}$$

corresponding in the moving frame to:

$$\begin{aligned} E_y &= -\tfrac{E_o}{\gamma_o}\cos\omega t \;,\; u_y = a\sin\omega t \;,\; u_\perp^2 = a^2 \\ E_z &= +\tfrac{E_o}{\gamma_o}\sin\omega t \;,\; u_z = a\cos\omega t \;,\; a = \tfrac{eE_o}{mc\omega\gamma_o} = \tfrac{eE_o}{mc\omega_L} \end{aligned} \tag{34}$$

Then from (32) it is:

$$c_{xy} = c_{xz} = c_{yz} = 0 \; ; \; c_{yy} = c_{zz} \equiv c_\perp \; ; \; c_{xx} \equiv c_\parallel$$

$$(1+\eta^2+a^2)^{5/2}\beta_o + \eta(1+\eta^2+a^2)^2 - \frac{3}{2}c_\parallel\eta(1+a^2) - c_\perp\eta(1+\eta^2 - \frac{1}{2}a^2) = 0 \tag{35}$$

which is an equation for $\eta = \eta(c_\perp, c_{||}, \beta_o, a^2)$ and a dispersion relation follows from (26) for the assumed solution:

$$\omega^2 = \omega_o^2\gamma_o\left\{(1+\eta^2+a^2)^{-1/2} + \frac{1}{2}(1+\eta^2+a^2)^{-5/2}\left[c_{||}(2\eta^2-1-a^2) - c_\perp(4+4\eta^2+a^2)\right]\right\}. \tag{36}$$

Eliminating η between (35) and (36) we find, to lowest order in the thermal coefficients $c_{||}, c_\perp$, for the dispersion relation in the Lab system:

$$\omega_L^2 = \omega_o^2\frac{\beta^2}{\beta^2-1}(1+a^2)^{-1/2} - \frac{1}{2}\omega_o^2(1+a^2)^{-3/2}\left[c_{||} + \frac{c_\perp}{\beta^2-1}\right] - \frac{2}{3}\omega_o^2(1+a^2)^{-5/2}c_\perp \tag{37}$$

showing that the frequency is lower than in the cold plasma case ($c_{||} = c_\perp = 0$).

2) Super-luminous, longitudinal wave: $E_y = E_z = 0, u_y, u_z$ constants

The system can now be reduced to $u_y = u_z = 0$ and

$$\frac{d^2u_x}{dt^2} = -\frac{\partial W}{\partial u_x}\,;\; W(u_x) = \omega_o^2\gamma_o\left[\beta_o u_x + (1+u_x^2)^{1/2} + c_{||}(1+u_x^2)^{-3/2} + c_\perp(1+u_x^2)^{-1/2}\right] + W_o \tag{38}$$

where

$$c_{||} = c_{xx}\ ,\ c_\perp = \frac{1}{2}(c_{yy}+c_{zz})$$

As for the cold plasma case ($c_{||} = c_\perp = 0$) the solution can be found in terms of elliptic integrals.

To conclude the super-luminous case it should be mentioned that the "thermal" coefficients $c_\perp c_{||}$, which represent the corrections to the cold plasma results, can be expressed in terms of more "physical" quantities, namely the parallel and perpendicular (with respect to wave propagation) average kinetic energies (or "temperatures"). Defining:

$$T_\perp = \left(<(q_y-p_y)^2> + <(q_z-p_z)^2>\right)/2m = \left(<q_y^2> - <q_y>^2 + <q_z^2> - <q_z>^2\right)/2m$$

$$T_L = \left(<(q_x-p_x)^2>\right)/2m = (<q_x^2> - <p_x>^2)/2m.$$

where $\underline{p} =< \underline{q} >$ and $< q_i^m >= \int d^3q \; q_i^n f / \int d^3q f$ and calculating the averages using the assumption (24) one finds:

$$\begin{matrix} T_\perp = mc^2 c_\perp \\ T_L = \frac{1}{2} mc^2 \gamma_o \left(c_\parallel + c_\perp \frac{a^2}{1+a^2} \beta_o^2 \gamma_o^2 \right) \end{matrix} \text{(T - Wave)} \quad \begin{matrix} T_\perp = mc^2 c_\perp \\ T_L = \frac{1}{2} mc^2 c_\parallel / \gamma_o^2 \end{matrix} \text{(L - Wave)} \quad (39)$$

The "thermal" state of the plasma, far from equilibrium in the presence of a large amplitude wave, is given through the $T_\perp$ and T_L coefficients, with $T_{L,\perp} \ll mc^2$ for the validity of the expansions in the present model.

Finally let us consider the case of the sub-luminous wave. For the cold plasma case $k_1 = k_2 = 0$ equation (28) can be written as:

$$\frac{d^2u}{d\tau^2} = -\frac{\partial U(u)}{\partial u} \; , U(u) = \omega_o^2 \gamma_o \left[u - \beta_o (u^2 - 1)^{1/2} \right] \; , \tau = x/c$$

which is formally equivalent to the $1 - D$ motion of a particle ("coordinate" u) in a potential $U(u)$, interpreting τ as "time".

It is well known that extending this analogy to "conservation of mechanical energy", that is:

$$\frac{1}{2} \left(\frac{du}{d\tau} \right)^2 + U(u) = U(u_o)$$

it is possible to find the maximum "velocity" of particle "motion", corresponding to a wave-breaking field

$$E \max = \sqrt{2} \frac{mc}{e} \omega_o (\gamma_o - 1)^{1/2}$$

For the present case equation (28) can again be written in the form (40) introducing a generalized potential $U(u, k_1, k_2)$ but the same analysis is no longer possible because as the field increases the domain $D(\phi)$ changes (see equations 20,21) with the result that part of the region where $f_o \neq 0$ could be left out of the integration for some value of ϕ and the result of the integral (20) would no longer be valid.

The conclusion is that a discussion of wave-breaking for a "warm" plasma requires a detailed knowledge of the distribution function and is not possible in any "fluid" approximation.

As for the wave-forms it can be shown that equation (28). in analogy with the cold plasma case of Tsytovich[2], can be solved with elliptic integrals.

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