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On Taylor series approximations for trust-region and regularized subproblems in optimization

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On Taylor series approximations for trust-region and regularized subproblems in optimization[1]

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ABSTRACT

Dollar, Gould and Robinson have recently built on the pioneering work of Gay, Moré and Sorensen for solving trust-region and regularisation subproblems which arise in unconstrained optimization: enhancements include the use of high-order polynomial approximation. In this paper, we consider the Taylor-series expansions that Dollar *et al.* have used and give conditions for when these will overestimate or underestimate the function that is being considered. Full proofs are provided.

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1 Introduction

Given a symmetric matrix $H \in \mathbb{R}^{n\times n}$, a vector $c \in \mathbb{R}^n$ and positive scalars Δ, σ and $p > 2$, we are interested in computing solutions of the optimization problems

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \;\; q(x) \stackrel{\text{def}}{=} c^T x + \tfrac{1}{2} x^T H x \;\text{ subject to }\; \|x\|_M \leq \Delta \tag{1.1}$$

and

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \;\; r(x) \stackrel{\text{def}}{=} c^T x + \tfrac{1}{2} x^T H x + \tfrac{\sigma}{p}\|x\|_M^p, \tag{1.2}$$

where the M-norm of x is $\|x\|_M \stackrel{\text{def}}{=} \sqrt{x^T M x}$. In [1], we built on the pioneering work of Gay, Moré and Sorensen [2], methods which obtain the solution of a sequence of parametrized linear systems by factorization are used. Enhancements using high-order polynomial approximation and inverse iteration ensure that the resulting method is both globally and asymptotically at least superlinearly convergent in all cases, including in the notorious hard case. In this report, we give the full proof for Lemma B.6 in [1].

Let $x(\lambda)$ satisfy

$$(H + \lambda I)x(\lambda) = -c,$$

then

$$\pi(\lambda;\beta) := \|x(\lambda)\|^\beta = \left(\sum_{i=1}^{n} \frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^{\frac{\beta}{2}},$$

where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the eigenvalues of the generalized eigenvalue problem $Hx = \lambda M x$, [1]. Let

$$\lambda_{\text{S}} \stackrel{\text{def}}{=} \max 0, -\lambda_1.$$

For problem (1.1) we often wish to find $\lambda \in (\lambda_{\text{S}}, \infty)$ which solves the "secular" equation

$$\pi(\lambda;\beta) = \Delta^\beta$$

and for problem (1.2) we often wish to find $\lambda \in (\lambda_{\text{S}}, \infty)$ which solves the nonlinear secular equation

$$\pi(\lambda;\beta) = (\lambda/\sigma)^{\frac{\beta}{p-2}},$$

where $\beta \neq 0$ for both these problems. One possibility is to approximate $\pi(\lambda;\beta)$ by Taylor series expansions of various degrees. To this end, we would like to know whether the Taylor series expansion overestimates or underestimates $\pi(\lambda;\beta)$. If we define the k-th order Taylor approximation

$$\pi_k(\delta;\beta) \stackrel{\text{def}}{=} \sum_{j=0}^{k} \frac{\pi^{(j)}(\lambda_C;\beta)}{j!}\delta^j$$

to $\pi(\lambda_C + \delta;\beta)$, we see from Taylor's theorem that the error is

$$\pi(\lambda_C+\delta;\beta) - \pi_k(\delta;\beta) = \frac{1}{(k+1)!}\pi^{(k+1)}(\lambda_C+\psi;\beta)\delta^{k+1}$$

for some ψ strictly between 0 and δ so long as $\delta > \lambda_{\rm S} - \lambda_C$. Thus, we are interested in knowing when $\pi^{(k+1)}(\lambda;\beta)$ is guaranteed to be positive or negative.

Differentiating $\pi(\lambda;\beta) = \|x(\lambda)\|_M^\beta \equiv [\pi(\lambda)]^{\frac{\beta}{2}}$ with respect to λ, using the chain rule, we obtain

$$\begin{aligned}
\pi^{(1)}(\lambda;\beta) &= \tfrac{\beta}{2}\,[\pi(\lambda)]^{\frac{\beta}{2}-1}\,\pi^{(1)}(\lambda),\\
\pi^{(2)}(\lambda;\beta) &= \tfrac{\beta}{2}\,[\pi(\lambda)]^{\frac{\beta}{2}-1}\,\pi^{(2)}(\lambda) + \tfrac{\beta}{2}\left(\tfrac{\beta}{2}-1\right)[\pi(\lambda)]^{\frac{\beta}{2}-2}\left[\pi^{(1)}(\lambda)\right]^2,\\
\pi^{(3)}(\lambda;\beta) &= \tfrac{\beta}{2}\,[\pi(\lambda)]^{\frac{\beta}{2}-3}\Big([\pi(\lambda)]^2\,\pi^{(3)}(\lambda) + 3\left(\tfrac{\beta}{2}-1\right)\pi(\lambda)\pi^{(1)}(\lambda)\pi^{(2)}(\lambda) \qquad (1.3)\\
&\quad + \left(\tfrac{\beta}{2}-1\right)\left(\tfrac{\beta}{2}-2\right)\left[\pi^{(1)}(\lambda)\right]^3\Big)\\
\text{and }\ \pi^{(4)}(\lambda;\beta) &= \tfrac{\beta}{2}\,[\pi(\lambda)]^{\frac{\beta}{2}-4}\Big([\pi(\lambda)]^3\,\pi^{(4)}(\lambda) + 4\left(\tfrac{\beta}{2}-1\right)[\pi(\lambda)]^2\,\pi^{(1)}(\lambda)\pi^{(3)}(\lambda) \qquad (1.4)\\
&\quad +3\left(\tfrac{\beta}{2}-1\right)[\pi(\lambda)]^2\left[\pi^{(2)}(\lambda)\right]^2 + 6\left(\tfrac{\beta}{2}-1\right)\left(\tfrac{\beta}{2}-2\right)\pi(\lambda)\left[\pi^{(1)}(\lambda)\right]^2\pi^{(2)}(\lambda)\\
&\quad + \left(\tfrac{\beta}{2}-1\right)\left(\tfrac{\beta}{2}-2\right)\left(\tfrac{\beta}{2}-3\right)\left[\pi^{(1)}(\lambda)\right]^4\Big).
\end{aligned}$$

From this we may deduce the following result.

Lemma 1.1 Let $\pi(\lambda;\beta) = \|x(\lambda)\|_M^\beta$, where $x(\lambda)$ satisfies $(H+\lambda I)x(\lambda) = -c$. Suppose that $\lambda > \lambda_{\rm S}$. Then

(i) $\pi^{(1)}(\lambda;\beta) < 0$ for all $\beta > 0$ while $\pi^{(1)}(\lambda;\beta) > 0$ for all $\beta < 0$;

(ii) $\pi^{(2)}(\lambda;\beta) > 0$ for all $\beta > 0$ while $\pi^{(2)}(\lambda;\beta) < 0$ for all $\beta \in [-1, 0)$;

(iii) $\pi^{(3)}(\lambda;\beta) < 0$ for all $\beta > 0$ while $\pi^{(3)}(\lambda;\beta) > 0$ for all $\beta \in [-\frac{2}{3}, 0)$; and

(iv) $\pi^{(4)}(\lambda;\beta) > 0$ for all $\beta > 0$ while $\pi^{(4)}(\lambda;\beta) < 0$ for all $\beta \in [-\frac{2}{5}, 0)$.

The first two statements are proved within [1] but only a sketch of the methodology for proving the last two statements is given. In Section 2 we give the proof for the third statement and in Section 3 we provide the proof for the fourth statement.

Notation: perm(i,j,k,l) is defined to be the set of distinct permuations of the arugu- ments. For example,

$$\operatorname{perm}(2,2,4,4) = \{(2,2,4,4),(2,4,2,4),(2,4,4,2),(4,2,2,4),(4,2,4,2),(4,4,2,2)\}.$$

2 Third derivative proof

Since $\pi(\lambda) = \sum_{i=1}^{n} \gamma_i^2 / (\lambda + \lambda_i)^2$ it follows that

$$\pi^{(1)}(\lambda) = -2 \sum_{i=1}^{n} \frac{\gamma_i^2}{(\lambda+\lambda_i)^3} \tag{2.5}$$

$$\pi^{(2)}(\lambda) = 6 \sum_{i=1}^{n} \frac{\gamma_i^2}{(\lambda+\lambda_i)^4} \tag{2.6}$$

$$\pi^{(3)}(\lambda) = -24 \sum_{i=1}^{n} \frac{\gamma_i^2}{(\lambda+\lambda_i)^5} \tag{2.7}$$

$$\pi^{(4)}(\lambda) = 120 \sum_{i=1}^{n} \frac{\gamma_i^2}{(\lambda+\lambda_i)^6} \tag{2.8}$$

Lemma 2.1 Let $\beta > 0$, then $\pi^{(3)}(\lambda;\beta) < 0$ for all $\lambda \in (\lambda_{\mathrm{S}}, \infty)$; let $\beta \in [-\frac{2}{3}, 0)$, then $\pi^{(3)}(\lambda;\beta) > 0$ for all $\lambda \in (\lambda_{\mathrm{S}}, \infty)$;

Proof. Substituting (2.5)–(2.7) into (1.3) and rearranging we obtain

$$\pi^{(3)}(\lambda;\beta) = \beta \left[\pi(\lambda)\right]^{\beta-6} \eta(\lambda;\beta),$$

where

$$\begin{aligned}\eta(\lambda;\beta) &= (-\beta^2+6\beta-8)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)^3 - 12\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^2\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^5}\right)\\ &\quad +(18-9\beta)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^4}\right).\end{aligned}$$

From Lemma 2.2 we deduce that $\pi^{(3)}(\lambda;\beta) < 0$ for all $\lambda \in (\lambda_{\mathrm{S}}, \infty)$ and $\beta > 0$; $\pi^{(3)}(\lambda;\beta) > 0$ for all $\lambda \in (\lambda_{\mathrm{S}}, \infty)$ and $\beta \in [-\frac{2}{3}, 0)$. This completes the proof. $\square$

Lemma 2.2 Let

$$\eta(\lambda;\beta) = (-\beta^2+6\beta-8)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)^3 - 12\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^2\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^5}\right)$$
$$+(18-9\beta)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^4}\right).$$

The expression $\eta(\lambda;\beta)$ is negative for all $\lambda\in(\lambda_{\mathrm{S}},\infty)$ and $\beta\geq-\frac{2}{3}$.

Proof. The proof is by induction. Let p be such that $\gamma_p\neq 0$ and $\gamma_i=0$ for $i=1,\ldots,p-1$. Define

$$\eta(\lambda;\beta;q) \stackrel{\text{def}}{=} (-\beta^2+6\beta-8)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)^3 - 12\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^2\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^5}\right)$$
$$+(18-9\beta)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^4}\right).$$

Note that $\eta(\lambda;\beta,n)=\eta(\lambda;\beta)$.

Now $\eta(\lambda;\beta;p)=(-\beta^2-3\beta-2)\,\frac{\gamma_p^6}{(\lambda+\lambda_p)^9}<0$ for all $\lambda\in(\lambda_{\mathrm{S}},\infty)$ and $\beta\geq-\frac{2}{3}$.

Assume that $\eta(\lambda;\beta;k-1)<0$ for all $\lambda\in(\lambda_{\mathrm{S}},\infty)$. Now

$$\begin{aligned}\eta(\lambda\ \beta;k) &= \eta(\lambda;\beta;k-1)+\left(-\beta^2-3\beta-2\right)\frac{\gamma_k^6}{(\lambda+\lambda_k)^9}\\ &\quad+\sum_{i=p}^{k-1}\frac{\gamma_k^4\gamma_i^2\eta_1(\lambda;\lambda_k;\lambda_i;\beta)}{(\lambda+\lambda_k)^7(\lambda+\lambda_i)^5}\\ &\quad+\sum_{i,j=p}^{k-1}\frac{\gamma_k^2\gamma_i^2\gamma_j^2\eta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)}{(\lambda+\lambda_k)^5(\lambda+\lambda_i)^5(\lambda+\lambda_j)^5},\end{aligned}$$

where

$$
\begin{aligned}
\eta_1(\lambda;\lambda_k;\lambda_i;\beta) &= (-9\beta-6)(\lambda+\lambda_i)^3 + \left(-3\beta^2+9\beta-6\right)(\lambda+\lambda_i)^2(\lambda+\lambda_k) \\
&\quad + (18-9\beta)(\lambda+\lambda_i)(\lambda+\lambda_k)^2 - 12(\lambda+\lambda_k)^3, \\
\eta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= \frac{18-9\beta}{2}(\lambda+\lambda_k)^3\left((\lambda+\lambda_i)(\lambda+\lambda_j)^2 + (\lambda+\lambda_i)^2(\lambda+\lambda_j)\right) \\
&\quad + \frac{18-9\beta}{2}(\lambda+\lambda_k)^2\left((\lambda+\lambda_i)(\lambda+\lambda_j)^3 + (\lambda+\lambda_i)^3(\lambda+\lambda_j)\right) \\
&\quad + \frac{18-9\beta}{2}(\lambda+\lambda_k)\left((\lambda+\lambda_i)^2(\lambda+\lambda_j)^3 + (\lambda+\lambda_i)^3(\lambda+\lambda_j)^2\right) \\
&\quad + \left(-3\beta^2+18\beta-24\right)(\lambda+\lambda_i)^2(\lambda+\lambda_j)^2(\lambda+\lambda_k)^2 \\
&\quad - 12(\lambda+\lambda_i)^3(\lambda+\lambda_j)^3 - 12(\lambda+\lambda_i)^3(\lambda+\lambda_k)^3 \\
&\quad - 12(\lambda+\lambda_j)^3(\lambda+\lambda_k)^3.
\end{aligned}
$$

From Lemmas A.3 and A.4 we deduce that $\eta(\lambda;\beta;k) < 0$ for all $\lambda \in (\lambda_{\mathrm{S}}, \infty)$ and $\beta \geq -\frac{2}{3}$.

$\square$

3 Fourth derivative proof

Lemma 3.1 Let $\beta > 0$, then $\pi^{(4)}(\lambda;\beta) > 0$ for all $\lambda \in (\lambda_{\mathrm{S}}, \infty)$; let $\beta \in [-\frac{2}{5}, 0)$, then $\pi^{(4)}(\lambda;\beta) < 0$ for all $\lambda \in (\lambda_{\mathrm{S}}, \infty)$;

Proof. Substituting (2.5)–(2.8) into (1.4) we obtain

$$\pi^{(4)}(\lambda;\beta) = \beta\,[\pi(\lambda)]^{\beta-8}\,\zeta(\lambda;\beta),$$

where

$$\begin{aligned}\zeta(\lambda;\beta) &= (\beta-2)(\beta-4)(\beta-6)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)^4\\ &\quad +60\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^3\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^6}\right)\\ &\quad +27(\beta-2)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^2\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^4}\right)^2\\ &\quad +48(\beta-2)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^2\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^5}\right)\\ &\quad +18(\beta-2)(\beta-4)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)^2\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^4}\right)\end{aligned}$$

From Lemma 3.2 we deduce that $\pi^{(4)}(\lambda;\beta)$ is positive for all $\lambda \in (\lambda_{\mathrm{S}},\infty)$ and $\beta > 0$; $\pi^{(4)}(\lambda;\beta)$ is negative for all $\lambda \in (\lambda_{\mathrm{S}},\infty)$ and $\beta \in [-\frac{2}{5},0)$.

□

Lemma 3.2 Let

$$\begin{aligned}\zeta(\lambda;\beta) &= (\beta-2)(\beta-4)(\beta-6)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)^4\\ &\quad +60\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^3\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^6}\right)\\ &\quad +27(\beta-2)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^2\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^4}\right)^2\\ &\quad +48(\beta-2)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^2\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^5}\right)\\ &\quad +18(\beta-2)(\beta-4)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)^2\left(\sum_{i=1}^{n}\frac{\gamma_i^2}{(\lambda+\lambda_i)^4}\right).\end{aligned}$$

The expression $\zeta(\lambda;\beta)$ is positive for all $\lambda \in (\lambda_{\mathrm{S}},\infty)$ and $\beta \geq -\frac{2}{5}$.

Proof. The proof is by induction. Let p be such that $\gamma_p \neq 0$ and $\gamma_i = 0$ for $i = 1, \ldots, p-1$. Define

$$\begin{aligned}
\zeta(\lambda;\beta;q) &= (\beta-2)(\beta-4)(\beta-6)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)^4 \\
&\quad +60\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^3\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^6}\right) \\
&\quad +27(\beta-2)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^2\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^4}\right)^2 \\
&\quad +48(\beta-2)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)^2\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^5}\right) \\
&\quad +18(\beta-2)(\beta-4)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^2}\right)\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^3}\right)^2\left(\sum_{i=p}^{q}\frac{\gamma_i^2}{(\lambda+\lambda_i)^4}\right).
\end{aligned}$$

Now $\zeta(\lambda;\beta;p) = (\beta^3+6\beta^2+11\beta+6)\frac{\gamma_p^8}{(\lambda+\lambda_p)^{12}} > 0$ for all $\lambda \in (\lambda_{\mathrm{S}}, \infty)$ and $\beta \geq -\frac{2}{5}$.

Assume that $\zeta(\lambda;\beta;k-1) > 0$ for all $\lambda \in (\lambda_{\mathrm{S}}, \infty)$. Now

$$\begin{aligned}
\zeta(\lambda;\beta;k) &= \zeta(\lambda;\beta;k-1) + \left(\beta^3+6\beta^2+11\beta+6\right)\frac{\gamma_1^8}{(\lambda+\lambda_k)^{12}} \\
&\quad +\sum_{i=1}^{k-1}\frac{\gamma_k^6\gamma_i^2\zeta_1(\lambda;\lambda_k;\lambda_i;\beta)}{(\lambda+\lambda_k)^{10}(\lambda+\lambda_i)^6} \\
&\quad +\sum_{i,j=1}^{k-1}\frac{\gamma_k^4\gamma_i^2\gamma_j^2\zeta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)}{(\lambda+\lambda_k)^8(\lambda+\lambda_i)^6(\lambda+\lambda_j)^6} \\
&\quad +\sum_{i,j,l=1}^{k-1}\frac{\gamma_k^2\gamma_i^2\gamma_j^2\gamma_l^2\zeta_3(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)}{(\lambda+\lambda_k)^6(\lambda+\lambda_i)^6(\lambda+\lambda_j)^6}(\lambda+\lambda_l)^6,
\end{aligned}$$

where

$$\begin{aligned}
\zeta_1(\lambda;\lambda_k;\lambda_i;\beta) &= \left(18\beta^2+42\beta+24\right)(\lambda+\lambda_i)^4+60\,(\lambda+\lambda_k)^4\\
&\quad+\left(4\beta^3-12\beta^2+8\beta\right)(\lambda+\lambda_i)^3(\lambda+\lambda_k)-48\,(2-\beta)\,(\lambda+\lambda_i)\,(\lambda+\lambda_k)^3\\
&\quad+\left(18\beta^2-54\beta+36\right)(\lambda+\lambda_i)^2(\lambda+\lambda_k)^2,\\
\zeta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (75\beta+30)\,(\lambda+\lambda_i)^4(\lambda+\lambda_j)^4\\
&\quad+90\,(\lambda+\lambda_i)^4(\lambda+\lambda_k)^4+90\,(\lambda+\lambda_j)^4(\lambda+\lambda_k)^4\\
&\quad+\left(18\beta^2-60\beta+48\right)(\lambda+\lambda_i)^3(\lambda+\lambda_j)^4(\lambda+\lambda_k)\\
&\quad+\left(18\beta^2-60\beta+48\right)(\lambda+\lambda_i)^4(\lambda+\lambda_j)^3(\lambda+\lambda_k)\\
&\quad-48\,(2-\beta)\,(\lambda+\lambda_i)\,(\lambda+\lambda_j)^4(\lambda+\lambda_k)^3\\
&\quad-48\,(2-\beta)\,(\lambda+\lambda_i)^4(\lambda+\lambda_j)\,(\lambda+\lambda_k)^3\\
&\quad-24\,(2-\beta)\,(\lambda+\lambda_i)\,(\lambda+\lambda_j)^3(\lambda+\lambda_k)^4\\
&\quad-24\,(2-\beta)\,(\lambda+\lambda_i)^3(\lambda+\lambda_j)\,(\lambda+\lambda_k)^4\\
&\quad-27\,(2-\beta)\,(\lambda+\lambda_i)^2(\lambda+\lambda_j)^2(\lambda+\lambda_k)^4\\
&\quad+\left(9\beta^2-36\right)(\lambda+\lambda_i)^2(\lambda+\lambda_j)^4(\lambda+\lambda_k)^2\\
&\quad+\left(9\beta^2-36\right)(\lambda+\lambda_i)^4(\lambda+\lambda_j)^2(\lambda+\lambda_k)^2\\
&\quad+18\left(\beta^2-6\beta+8\right)(\lambda+\lambda_i)^2(\lambda+\lambda_j)^3(\lambda+\lambda_k)^3\\
&\quad+18\left(\beta^2-6\beta+8\right)(\lambda+\lambda_i)^3(\lambda+\lambda_j)^2(\lambda+\lambda_k)^3\\
&\quad+\left(6\beta^3-54\beta^2+156\beta-144\right)(\lambda+\lambda_i)^3(\lambda+\lambda_j)^3(\lambda+\lambda_k)^2,
\end{aligned}$$

and

$$\begin{aligned}
\zeta_3(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 60\sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\mathrm{perm}(0,4,4,4)}}(\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3}(\lambda+\lambda_l)^{n_4}\\
&\quad+(16\beta-32)\sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\mathrm{perm}(1,3,4,4)}}(\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3}(\lambda+\lambda_l)^{n_4}\\
&\quad+(18\beta-36)\sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\mathrm{perm}(2,2,4,4)}}(\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3}(\lambda+\lambda_l)^{n_4}\\
&\quad+\left(6\beta^2-36\beta+48\right)\sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\mathrm{perm}(2,3,3,4)}}(\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3}(\lambda+\lambda_l)^{n_4}\\
&\quad+\left(4\beta^3-48\beta^2+176\beta-192\right)(\lambda+\lambda_k)^3(\lambda+\lambda_i)^3(\lambda+\lambda_j)^3(\lambda+\lambda_l)^3.
\end{aligned}$$

From Lemmas B.5–B.7 we deduce that $\zeta(\lambda;\beta;k)>0$ for all $\lambda\in(\lambda_{\mathrm{S}},\infty)$ and $\beta\geq-\frac{2}{5}$. This completes the proof.

$\square$

Appendix A

Lemma A.3 Let

$$\begin{aligned}\eta_1(\lambda;\lambda_k;\lambda_i;\beta) &= (-9\beta-6)(\lambda+\lambda_i)^3 + \left(-3\beta^2+9\beta-6\right)(\lambda+\lambda_i)^2(\lambda+\lambda_k)\\ &\quad + (18-9\beta)(\lambda+\lambda_i)(\lambda+\lambda_k)^2 - 12(\lambda+\lambda_k)^3\end{aligned}$$

and assume that $\lambda_i \le \lambda_k$. If $\lambda_i \le 0$, the expression $\eta_1(\lambda;\lambda_k;\lambda_i;\beta)$ is non-positive for all $\lambda \in [-\lambda_i, \infty)$ and $\beta \ge -1$; if $\lambda_i > 0$, the expression $\eta_1(\lambda;\lambda_k;\lambda_i;\beta)$ is negative for all $\lambda \in [0,\infty)$ and $\beta > -1$;

Proof. Differentiating $\eta_1(\lambda;\lambda_k;\lambda_i;\beta)$ with respect to λ we obtain

$$\begin{aligned}\eta_1(\lambda;\lambda_k;\lambda_i;\beta) &= -12(\lambda+\lambda_k)^3 + (18-9\beta)(\lambda+\lambda_k)^2(\lambda+\lambda_i)\\ &\quad + \left(-3\beta^2+9\beta-6\right)(\lambda+\lambda_k)(\lambda+\lambda_i)^2\\ &\quad + (-9\beta-6)(\lambda+\lambda_i)^3, \qquad (A.1)\\ \eta_1^{(1)}(\lambda;\lambda_k;\lambda_i;\beta) &= (-18-9\beta)(\lambda+\lambda_k)^2 + (24-6\beta^2)(\lambda+\lambda_k)(\lambda+\lambda_i) \qquad (A.2)\\ &\quad +(-3\beta^2-18\beta-24)(\lambda+\lambda_i)^2,\\ \eta_1^{(2)}(\lambda;\lambda_k;\lambda_i;\beta) &= (-6\beta^2-18\beta-12)(\lambda+\lambda_k) \qquad (A.3)\\ &\quad +(-12\beta^2-36\beta-24)(\lambda+\lambda_i),\\ \eta_1^{(3)}(\lambda;\lambda_k;\lambda_i;\beta) &= -18\beta^2-54\beta-36. \qquad (A.4)\end{aligned}$$

Clearly $\eta_1^{(3)}(\lambda;\lambda_k;\lambda_i;\beta)$ is negative for all $\beta > -1$.

Consider the case $\lambda_i \le 0$. Subsituting in $\lambda = -\lambda_i$ into (A.1)–(A.3) we obtain

$$\begin{aligned}\eta_1(-\lambda_i;\lambda_k;\lambda_i;\beta) &= -12(\lambda_k-\lambda_i)^3 \le 0,\\ \eta_1^{(1)}(-\lambda_i;\lambda_k;\lambda_i;\beta) &= (-18-9\beta)(\lambda_k-\lambda_i)^2 \le 0 \text{ for all } \beta \ge -2\\ \eta_1^{(2)}(-\lambda_i;\lambda_k;\lambda_i;\beta) &= (-6\beta^2-18\beta-12)(\lambda_k-\lambda_i) \le 0 \text{ for all } \beta > -1.\end{aligned}$$

Hence, $\eta_1(\lambda;\lambda_k;\lambda_i;\beta)$ is non-positive for all $\lambda \ge -\lambda_i$ and $\beta \ge -1$. Since $\lambda_{\mathrm{S}} \ge -\lambda_i$, this completes the proof for the case $\lambda_i \le 0$.

Consider the case $\lambda_i > 0$. Subsituting in $\lambda = 0$ into (A.1)–(A.3) we obtain

$$\begin{aligned}\eta_1(0;\lambda_k;\lambda_i;\beta) &= -12\lambda_k^3 + (18-9\beta)\lambda_k^2\lambda_i + \left(-3\beta^2+9\beta-6\right)\lambda_k\lambda_i^2 + (-9\beta-6)\lambda_i^3,\\ \eta_1^{(1)}(0;\lambda_k;\lambda_i;\beta) &= (-18-9\beta)\lambda_k^2 + (24-6\beta^2)\lambda_k\lambda_i + (-3\beta^2-18\beta-24)\lambda_i^2,\\ \eta_1^{(2)}(0;\lambda_k;\lambda_i;\beta) &= (-6\beta^2-18\beta-12)\lambda_k + (-12\beta^2-36\beta-24)\lambda_i.\end{aligned}$$

We wish to show that the above equations are all non-positive for all $\lambda_k \ge \lambda_i$ and $\beta \ge -1$.

Define $\eta_{10}(\lambda_k;\lambda_i;\beta) = \eta_1(0;\lambda_k;\lambda_i;\beta)$. Differentiating $\eta_{10}(\lambda_k;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{10}(\lambda_k;\lambda_i;\beta) &= -12\lambda_k^3 + (18-9\beta)\,\lambda_k^2\lambda_i + \left(-3\beta^2+9\beta-6\right)\lambda_k\lambda_i^2 \qquad \text{(A.5)}\\
&\quad + (-9\beta-6)\,\lambda_i^3,\\
\eta_{10}^{(1)}(\lambda_k;\lambda_i;\beta) &= -36\lambda_k^2 + (-18\beta+36)\lambda_k\lambda_i + \left(-3\beta^2+9\beta-6\right)\lambda_i^2, \qquad \text{(A.6)}\\
\eta_{10}^{(2)}(\lambda_k;\lambda_i;\beta) &= -72\lambda_k + (36-18\beta)\,\lambda_i, \qquad \text{(A.7)}\\
\eta_{10}^{(3)}(\lambda_k;\lambda_i;\beta) &= -72.
\end{aligned}$$

Substituting $\lambda_k = \lambda_i$ into (A.5)–(A.7) we obtain

$$\begin{aligned}
\eta_{10}(\lambda_i;\lambda_i;\beta) &= (-3\beta^2-9\beta-6)\lambda_i^3 < 0 \text{ for all } \beta > -1,\\
\eta_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= (-3\beta^2-9\beta-6)\lambda_i^3 < 0 \text{ for all } \beta > -1,\\
\eta_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= (-18\beta-36)\lambda_i^3 < 0 \text{ for all } \beta > -2.
\end{aligned}$$

Hence, $\eta_1(0;\lambda_k;\lambda_i;\beta)$ is negative for all $\lambda_k \geq \lambda_i$ and $\beta > -1$.

Define $\eta_{11}(\lambda_k;\lambda_i;\beta) = \eta_1^{(1)}(0;\lambda_k;\lambda_i;\beta)$. Differentiating $\eta_{11}(\lambda_k;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{11}(\lambda_k;\lambda_i;\beta) &= (-18-9\beta)\lambda_k^2 + (24-6\beta^2)\lambda_k\lambda_i + (-3\beta^2-18\beta-24)\lambda_i^2, \qquad \text{(A.8)}\\
\eta_{11}^{(1)}(\lambda_k;\lambda_i;\beta) &= (-36-18\beta)\lambda_k + (24-6\beta)\lambda_i, \qquad \text{(A.9)}\\
\eta_{11}^{(2)}(\lambda_k;\lambda_i;\beta) &= -36-18\beta \leq 0 \text{ for all } \beta \geq -2.
\end{aligned}$$

Substituting $\lambda_k = \lambda_i$ into (A.8) and (A.9) we obtain

$$\begin{aligned}
\eta_{11}(\lambda_i;\lambda_i;\beta) &= (-9\beta^2-27\beta-18)\lambda_i^2 < 0 \text{ for all } \beta > -1,\\
\eta_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= (-6\beta^2-18\beta-12)\lambda_i < 0 \text{ for all } \beta > -1.
\end{aligned}$$

Hence, $\eta_1^{(1)}(0;\lambda_k;\lambda_i;\beta)$ is negative for all $\lambda_k \geq \lambda_i$ and $\beta > -1$.

Define $\eta_{12}(\lambda_k;\lambda_i;\beta) = \eta_1^{(2)}(0;\lambda_k;\lambda_i;\beta)$. Differentiating $\eta_{12}(\lambda_k;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{12}(\lambda_k;\lambda_i;\beta) &= (-6\beta^2-18\beta-12)\lambda_k + (-12\beta^2-36\beta-24)\lambda_i, \qquad \text{(A.10)}\\
\eta_{12}^{(1)}(\lambda_k;\lambda_i;\beta) &= -6\beta^2-18\beta-12 < 0 \text{ for all } \beta > -1.
\end{aligned}$$

Substituting $\lambda_k = \lambda_i$ into (A.10) we obtain

$$\eta_{12}(\lambda_i;\lambda_i;\beta) = (-18\beta^2-54\beta-36)\lambda_i < 0 \text{ for all } \beta > -1.$$

Hence, $\eta_1^{(2)}(0;\lambda_k;\lambda_i;\beta)$ is negative for all $\lambda_k \geq \lambda_i$ and $\beta \geq -1$. This implies that $\eta_1(\lambda;\lambda_k;\lambda_i;\beta)$ is negative for all $\lambda \geq 0$, $\lambda_k \geq \lambda_i$ and $\beta > -1$.

This concludes the proof.

□

Lemma A.4 Let

$$\begin{aligned}
\eta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) \;=\;& \frac{18-9\beta}{2}\left(\lambda+\lambda_k\right)^3\left(\left(\lambda+\lambda_i\right)\left(\lambda+\lambda_j\right)^2+\left(\lambda+\lambda_i\right)^2\left(\lambda+\lambda_j\right)\right)\\
&+\frac{18-9\beta}{2}\left(\lambda+\lambda_k\right)^2\left(\left(\lambda+\lambda_i\right)\left(\lambda+\lambda_j\right)^3+\left(\lambda+\lambda_i\right)^3\left(\lambda+\lambda_j\right)\right)\\
&+\frac{18-9\beta}{2}\left(\lambda+\lambda_k\right)\left(\left(\lambda+\lambda_i\right)^2\left(\lambda+\lambda_j\right)^3+\left(\lambda+\lambda_i\right)^3\left(\lambda+\lambda_j\right)^2\right)\\
&+\left(-3\beta^2+18\beta-24\right)\left(\lambda+\lambda_i\right)^2\left(\lambda+\lambda_j\right)^2\left(\lambda+\lambda_k\right)^2\\
&-12\left(\lambda+\lambda_i\right)^3\left(\lambda+\lambda_j\right)^3-12\left(\lambda+\lambda_i\right)^3\left(\lambda+\lambda_k\right)^3\\
&-12\left(\lambda+\lambda_j\right)^3\left(\lambda+\lambda_k\right)^3
\end{aligned}$$

and $\lambda_k \geq \max(\lambda_i, \lambda_j)$.
If $\lambda_i \leq 0$, the expression $\eta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda \in [-\lambda_i, \infty)$ and $\beta \geq -\frac{2}{3}$; if $\lambda_i > 0$, the expression $\eta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda \in [0,\infty)$ and $\beta \geq -\frac{2}{3}$.

Proof. Without loss of generality assume that $\lambda_i \leq \lambda_j$.

Differentiating $\eta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ we obtain

$$\begin{aligned}
\eta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) \;=\;& \left(\frac{18-9\beta}{2}\right)\sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(1,2,3)}}\left(\lambda+\lambda_i\right)^{n_1}\left(\lambda+\lambda_j\right)^{n_2}\left(\lambda+\lambda_k\right)^{n_3} \qquad \text{(A.11)}\\
&+\left(-3\beta^2+18\beta-24\right)\left(\lambda+\lambda_i\right)^2\left(\lambda+\lambda_j\right)^2\left(\lambda+\lambda_k\right)^2\\
&-12\sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(0,3,3)}}\left(\lambda+\lambda_i\right)^{n_1}\left(\lambda+\lambda_j\right)^{n_2}\left(\lambda+\lambda_k\right)^{n_3},\\
\eta_2^{(1)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) \;=\;& \left(\frac{-54-9\beta}{2}\right)\sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(0,2,3)}}\left(\lambda+\lambda_i\right)^{n_1}\left(\lambda+\lambda_j\right)^{n_2}\left(\lambda+\lambda_k\right)^{n_3} \qquad \text{(A.12)}\\
&+(36-18\beta)\sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(1,1,3)}}\left(\lambda+\lambda_i\right)^{n_1}\left(\lambda+\lambda_j\right)^{n_2}\left(\lambda+\lambda_k\right)^{n_3}\\
&+(-6\beta^2+9\beta+6)\sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(1,2,2)}}\left(\lambda+\lambda_i\right)^{n_1}\left(\lambda+\lambda_j\right)^{n_2}\left(\lambda+\lambda_k\right)^{n_3},
\end{aligned}$$

$$\begin{aligned}\eta_2^{(2)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (-18-27\beta)\sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(0,1,3)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3} \\ &\quad +(-6\beta^2-18\beta-156)\sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(0,2,2)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3} \\ &\quad +(-24\beta^2-18\beta+132)\sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(1,1,2)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3},\end{aligned} \tag{A.13}$$

$$\begin{aligned}\eta_2^{(3)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (-36-54\beta)\left((\lambda+\lambda_k)^3+(\lambda+\lambda_j)^3+(\lambda+\lambda_i)^3\right) \\ &\quad +(-144\beta^2-108\beta+792)(\lambda+\lambda_i)(\lambda+\lambda_j)(\lambda+\lambda_k) \\ &\quad +(-36\beta^2-135\beta-234)\sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(0,1,2)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3},\end{aligned} \tag{A.14}$$

$$\begin{aligned}\eta_2^{(4)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(-72\beta^2-432\beta-576\right)\left((\lambda+\lambda_k)^2+(\lambda+\lambda_j)^2+(\lambda+\lambda_i)^2\right) \\ &\quad +(-288\beta^2-648\beta-144)\sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(0,1,1)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3},\end{aligned} \tag{A.15}$$

$$\eta_2^{(5)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) = \left(-720\beta^2-2160\beta-1440\right)\left((\lambda+\lambda_k)+(\lambda+\lambda_j)+(\lambda+\lambda_i)\right), \tag{A.16}$$

$$\eta_2^{(6)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) = -2160\beta^2-6480\beta-4320. \tag{A.17}$$

Clearly $\eta_2^{(6)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ is negative for all $\beta > -1$.

Consider the case $\lambda_i \leq 0$. Substituting in $\lambda = -\lambda_i$ into (A.11)–(A.16) we obtain

$$\begin{aligned}\eta_2(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= -12(\lambda_k-\lambda_i)^3(\lambda_j-\lambda_i)^3 \\ &\leq 0, \\ \eta_2^{(1)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(\frac{-54-9\beta}{2}\right)\left((\lambda_k-\lambda_i)^3(\lambda_j-\lambda_i)^2+(\lambda_k-\lambda_i)^2(\lambda_j-\lambda_i)^3\right)2 \\ &\leq 0 \text{ for all } \beta \geq -6, \\ \eta_2^{(2)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= (-18-27\beta)\left((\lambda_k-\lambda_i)(\lambda_j-\lambda_i)^3+(\lambda_k-\lambda_i)^3(\lambda_j-\lambda_i)\right)2 \\ &\quad +(-6\beta^2-18\beta-156)(\lambda_k-\lambda_i)^2(\lambda_j-\lambda_i)^2 \\ &\leq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\ \eta_2^{(3)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= (-36-54\beta)\left((\lambda_k-\lambda_i)^3+(\lambda_j-\lambda_i)^3\right) \\ &\quad +(-36\beta^2-135\beta-234)\left((\lambda_k-\lambda_i)(\lambda_j-\lambda_i)^2+(\lambda_k-\lambda_i)^2(\lambda_j-\lambda_i)\right) \\ &\leq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\ \eta_2^{(4)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(-72\beta^2-432\beta-576\right)\left((\lambda_k-\lambda_i)^2+(\lambda_j-\lambda_i)^2\right) \\ &\quad +(-288\beta^2-648\beta-144)(\lambda_k-\lambda_i)(\lambda_j-\lambda_i), \\ \eta_2^{(5)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(-720\beta^2-2160\beta-1440\right)\left((\lambda_k-\lambda_i)+(\lambda_j-\lambda_i)\right) \\ &\leq 0 \text{ for all } \beta \geq -1.\end{aligned}$$

It remains for us to show that $\eta_2^{(4)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\beta \geq -\frac{2}{3}$ when $\lambda_i \leq 0$.

Define $\hat{\eta_2}(\lambda_k;\lambda_j;\lambda_i;\beta) = \eta_2^{(4)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta)$. Differentiating $\hat{\eta_2}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\hat{\eta_2}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(-72\beta^2 - 432\beta - 576\right)\left((\lambda_k-\lambda_i)^2 + (\lambda_j-\lambda_i)^2\right) \qquad \text{(A.18)}\\
&\quad +(-288\beta^2 - 648\beta - 144)(\lambda_k-\lambda_i)(\lambda_j-\lambda_i),\\
\hat{\eta_2}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(-144\beta^2 - 864\beta - 1152\right)(\lambda_k-\lambda_i) \qquad \text{(A.19)}\\
&\quad +(-288\beta^2 - 648\beta - 144)(\lambda_j-\lambda_i),\\
\hat{\eta_2}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= -144\beta^2 - 864\beta - 1152\\
&\leq 0 \text{ for all } \beta \geq -2.
\end{aligned}$$

Substituting $\lambda_k = \lambda_j$ into (A.18) and (A.19) we obtain

$$\begin{aligned}
\hat{\eta_2}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 2\left(-432\beta^2 - 1512\beta - 1296\right)(\lambda_j-\lambda_i)^2\\
&\leq 0 \text{ for all } \beta \geq -\frac{3}{2},\\
\hat{\eta_2}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= \left(-432\beta^2 - 1512\beta - 1296\right)(\lambda_j-\lambda_i)\\
&\leq 0 \text{ for all } \beta \geq -\frac{3}{2}.
\end{aligned}$$

Hence, $\eta_2^{(4)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\beta \geq -\frac{2}{3}$ when $\lambda_i \leq 0$. From this we deduce that $\eta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda \geq -\lambda_i$ and $\beta \geq -\frac{2}{3}$.

Consider the case $\lambda_i > 0$. Let us define the functions $\eta_{20}(\lambda_k;\lambda_j;\lambda_i;\beta)$, $\eta_{21}(\lambda_k;\lambda_j;\lambda_i;\beta)$, $\eta_{22}(\lambda_k;\lambda_j;\lambda_i;\beta)$, $\eta_{23}(\lambda_k;\lambda_j;\lambda_i;\beta)$, $\eta_{24}(\lambda_k;\lambda_j;\lambda_i;\beta)$ and $\eta_{25}(\lambda_k;\lambda_j;\lambda_i;\beta)$ as follows:

$$\begin{aligned}
\eta_{20}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \eta_2(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\eta_{21}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \eta_2^{(1)}(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\eta_{22}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \eta_2^{(2)}(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\eta_{23}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \eta_2^{(3)}(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\eta_{24}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \eta_2^{(4)}(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\eta_{25}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \eta_2^{(5)}(0;\lambda_k;\lambda_j;\lambda_i;\beta).
\end{aligned}$$

We will show that the above equations are all negative for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{3}$. This combined with $\eta_2^{(6)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) < 0$ for all $\beta \geq -\frac{2}{3}$ will prove that $\eta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda \geq 0$ and $\beta \geq -\frac{2}{3}$.

Differentiating $\eta_{20}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{20}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(\frac{18-9\beta}{2}\right)\sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(1,2,3)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_k^{n_3} \\
&\quad + \left(-3\beta^2+18\beta-24\right)\lambda_i^2\lambda_j^2\lambda_k^2 \\
&\quad -12\sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(0,3,3)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_k^{n_3}, \\
\eta_{20}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= -36\lambda_k^2\left(\lambda_j^3+\lambda_i^3\right) + \left(\frac{54-27\beta}{2}\right)\lambda_k^2\lambda_j\lambda_i\left(\lambda_j+\lambda_i\right) + (18-9\beta)\,\lambda_k\left(\lambda_j^3+\lambda_i^3\right) \\
&\quad + \left(\frac{-18-9\beta}{2}\right)\lambda_j^2\lambda_i^2\left(\lambda_j+\lambda_i\right) + \left(-6\beta^2+36\beta-48\right)\lambda_k\lambda_j^2\lambda_i^2, \\
\eta_{20}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= -72\lambda_k\left(\lambda_j^3+\lambda_i^3\right) + (54-27\beta)\,\lambda_k\lambda_j\lambda_i\left(\lambda_j+\lambda_i\right) \\
&\quad + (18-9\beta)\,\lambda_j\lambda_i\left(\lambda_j^2+\lambda_i^2\right) + \left(-6\beta^2+36\beta-48\right)\lambda_j^2\lambda_i^2, \\
\eta_{20}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= -72\left(\lambda_j^3+\lambda_i^3\right) + (54-27\beta)\,\lambda_j\lambda_i\left(\lambda_j+\lambda_i\right).
\end{aligned}$$

Define $\eta_{203}(\lambda_j;\lambda_i;\beta) = \eta_{20}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{203} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{203} with respect to λ_j gives

$$\begin{aligned}
\eta_{203}(\lambda_j;\lambda_i;\beta) &= -72\left(\lambda_j^3+\lambda_i^3\right) + (54-27\beta)\,\lambda_j\lambda_i\left(\lambda_j+\lambda_i\right), \\
\eta_{203}^{(1)}(\lambda_j;\lambda_i;\beta) &= -216\lambda_j^2 + (-54\beta+108)\lambda_j\lambda_i + (54-27\beta)\,\lambda_i^2, \\
\eta_{203}^{(2)}(\lambda_j;\lambda_i;\beta) &= -432\lambda_j + (108-54\beta)\,\lambda_i, \\
\eta_{203}^{(3)}(\lambda_j;\lambda_i;\beta) &= -432 < 0.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{203}(\lambda_i;\lambda_i;\beta) &= (-36-54\beta)\,\lambda_i^3 \leq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\
\eta_{203}^{(1)}(\lambda_i;\lambda_i;\beta) &= (-81-54\beta)\lambda_i^2 \leq 0 \text{ for all } \beta \geq -\frac{3}{2}, \\
\eta_{203}^{(2)}(\lambda_i;\lambda_i;\beta) &= (-324-54\beta)\,\lambda_i \leq 0 \text{ for all } \beta \geq -6.
\end{aligned}$$

Hence, $\eta_{20}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{202}(\lambda_j;\lambda_i;\beta) = \eta_{20}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that η_{202} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{202} with respect to λ_j gives

$$\begin{aligned}
\eta_{202}(\lambda_j;\lambda_i;\beta) &= -72\lambda_j^3 + (72-36\beta)\,\lambda_j^2\lambda_i + \left(-6\beta^2+9\beta+6\right)\lambda_j\lambda_i^2 + (-54-9\beta)\,\lambda_i^3, \\
\eta_{202}^{(1)}(\lambda_j;\lambda_i;\beta) &= -216\lambda_j^2 + (144-72\beta)\,\lambda_j\lambda_i + \left(-6\beta^2+9\beta+6\right)\lambda_i^2, \\
\eta_{202}^{(2)}(\lambda_j;\lambda_i;\beta) &= -432\lambda_j + (144-72\beta)\,\lambda_i, \\
\eta_{202}^{(3)}(\lambda_j;\lambda_i;\beta) &= -432 < 0.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{202}(\lambda_i;\lambda_i;\beta) &= \left(-6\beta^2-36\beta-48\right)\lambda_i^3 \leq 0 \text{ for all } \beta\geq -2,\\
\eta_{202}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-6\beta^2-63\beta-66\right)\lambda_i^2 \leq 0 \text{ for all } \beta\geq -1.1803,\\
\eta_{202}^{(2)}(\lambda_i;\lambda_i;\beta) &= (-288-72\beta)\,\lambda_i \leq 0 \text{ for all } \beta\geq -4.
\end{aligned}$$

Hence, $\eta_{20}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_j\geq\lambda_i$ and $\beta\geq-\frac{2}{3}$.

Define $\eta_{201}(\lambda_j;\lambda_i;\beta)=\eta_{20}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that η_{201} is non-positive for all $\lambda_j\geq\lambda_i$. Differentiating η_{201} with respect to λ_j gives

$$\begin{aligned}
\eta_{201}(\lambda_j;\lambda_i;\beta) &= -36\lambda_j^3+\left(\frac{90-45\beta}{2}\right)\lambda_j^2\lambda_i+(-6\beta^2+18\beta-12)\lambda_j\lambda_i^2+\left(\frac{18-27\beta}{2}\right)\lambda_i^3,\\
\eta_{201}^{(1)}(\lambda_j;\lambda_i;\beta) &= -108\lambda_j^2+(90-45\beta)\lambda_j\lambda_i+\left(-6\beta^2+18\beta-12\right)\lambda_i^2,\\
\eta_{201}^{(2)}(\lambda_j;\lambda_i;\beta) &= -216\lambda_j+(90-45\beta)\,\lambda_i,\\
\eta_{201}^{(3)}(\lambda_j;\lambda_i;\beta) &= -216<0.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{201}(\lambda_i;\lambda_i;\beta) &= \left(-6\beta^2-18\beta-12\right)\lambda_i^3 \leq 0 \text{ for all } \beta\geq -1,\\
\eta_{201}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-6\beta^2-27\beta-30\right)\lambda_i^2 \leq 0 \text{ for all } \beta\geq -2,\\
\eta_{201}^{(2)}(\lambda_i;\lambda_i;\beta) &= (-126-45\beta)\,\lambda_i \leq 0 \text{ for all } \beta\geq -2.8.
\end{aligned}$$

Hence, $\eta_{20}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_j\geq\lambda_i$ and $\beta\geq-\frac{2}{3}$.

Define $\eta_{200}(\lambda_j;\lambda_i;\beta)=\eta_{20}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that η_{200} is negative for all $\lambda_j\geq\lambda_i$. Differentiating η_{200} with respect to λ_j gives

$$\begin{aligned}
\eta_{200}(\lambda_j;\lambda_i;\beta) &= -12\lambda_j^3+(18-9\beta)\,\lambda_j^2\lambda_i+(-3\beta^2+9\beta-6)\lambda_j\lambda_i^2+(-6-9\beta)\,\lambda_i^3,\\
\eta_{200}^{(1)}(\lambda_j;\lambda_i;\beta) &= -36\lambda_j^2+(36-18\beta)\lambda_j\lambda_i+\left(-3\beta^2+9\beta-6\right)\lambda_i^2,\\
\eta_{200}^{(2)}(\lambda_j;\lambda_i;\beta) &= -72\lambda_j+(36-18\beta)\,\lambda_i,\\
\eta_{200}^{(3)}(\lambda_j;\lambda_i;\beta) &= -72<0.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{200}(\lambda_i;\lambda_i;\beta) &= \left(-3\beta^2-9\beta-6\right)\lambda_i^3 < 0 \text{ for all } \beta> -1,\\
\eta_{200}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-3\beta^2-9\beta-6\right)\lambda_i^2 < 0 \text{ for all } \beta> -1,\\
\eta_{200}^{(2)}(\lambda_i;\lambda_i;\beta) &= (-36-18\beta)\,\lambda_i \leq 0 \text{ for all } \beta\geq -2.
\end{aligned}$$

Hence, $\eta_{20}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_j\geq\lambda_i$ and $\beta\geq-\frac{2}{3}$. This implies that $\eta_2(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_k\geq\lambda_j$ and $\beta\geq-\frac{2}{3}$.

Differentiating $\eta_{21}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{21}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(\frac{-54-9\beta}{2}\right) \sum_{\substack{(n_1,n_2,n_3)\\ \in\text{perm}(0,2,3)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_k^{n_3} \\
&\quad +(36-18\beta) \sum_{\substack{(n_1,n_2,n_3)\\ \in\text{perm}(1,1,3)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_k^{n_3} \\
&\quad +(-6\beta^2+9\beta+6) \sum_{\substack{(n_1,n_2,n_3)\\ \in\text{perm}(1,2,2)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3}, \\
\eta_{21}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(\frac{-162-27\beta}{2}\right)\left(\lambda_k^2\lambda_j^2+\lambda_k^2\lambda_i^2\right)+(108-54\beta)\,\lambda_k^2\lambda_j\lambda_i \\
&\quad +(-54-9\beta)\left(\lambda_k\lambda_j^3+\lambda_k\lambda_i^3\right)+\left(-12\beta^2+18\beta+12\right)\left(\lambda_k\lambda_j^2\lambda_i+\lambda_k\lambda_j\lambda_i^2\right) \\
&\quad +(36-18\beta)\left(\lambda_j^3\lambda_i+\lambda_j\lambda_i^3\right)+\left(-6\beta^2+9\beta+6\right)\lambda_j^2\lambda_i^2, \\
\eta_{21}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (-162-27\beta)\left(\lambda_k\lambda_j^2+\lambda_k\lambda_i^2\right)+(216-108\beta)\,\lambda_k\lambda_j\lambda_i \\
&\quad (-54-9\beta)\left(\lambda_j^3+\lambda_i^3\right)+\left(-12\beta^2+18\beta+12\right)\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right), \\
\eta_{21}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (-162-27\beta)\left(\lambda_j^2+\lambda_i^2\right)+(216-108\beta)\,\lambda_j\lambda_i.
\end{aligned}$$

Define $\eta_{213}(\lambda_j;\lambda_i;\beta)=\eta_{21}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{213} is non-positive for all $\lambda_j\geq\lambda_i$. Differentiating η_{213} with respect to λ_j gives

$$\begin{aligned}
\eta_{213}(\lambda_j;\lambda_i;\beta) &= (-162-27\beta)\left(\lambda_j^2+\lambda_i^2\right)+(216-108\beta)\,\lambda_j\lambda_i, \\
\eta_{213}^{(1)}(\lambda_j;\lambda_i;\beta) &= (-324-54\beta)\,\lambda_j+(216-108\beta)\,\lambda_i, \\
\eta_{213}^{(2)}(\lambda_j;\lambda_i;\beta) &= (-324-54\beta)\leq 0 \text{ for all } \beta\geq -6.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{213}(\lambda_i;\lambda_i;\beta) &= (-108-162\beta)\,\lambda_i^2\leq 0 \text{ for all } \beta\geq -\frac{2}{3}, \\
\eta_{213}^{(1)}(\lambda_i;\lambda_i;\beta) &= (-108-162\beta)\,\lambda_i\leq 0 \text{ for all } \beta\geq -\frac{2}{3}.
\end{aligned}$$

Hence, $\eta_{21}^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_k\geq\lambda_j\geq\lambda_i$ and $\beta\geq -\frac{2}{3}$.

Define $\eta_{212}(\lambda_j;\lambda_i;\beta)=\eta_{21}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{212} is non-positive for all $\lambda_j\geq\lambda_i$. Differentiating η_{212} with respect to λ_j gives

$$\begin{aligned}
\eta_{212}(\lambda_j;\lambda_i;\beta) &= (-216-36\beta)\,\lambda_j^3+\left(-12\beta^2-90\beta+228\right)\lambda_j^2\lambda_i \\
&\quad +\left(-12\beta^2-9\beta-150\right)\lambda_j\lambda_i^2+(-54-9\beta)\,\lambda_i^3, \\
\eta_{212}^{(1)}(\lambda_j;\lambda_i;\beta) &= (-648-108\beta)\,\lambda_j^2+\left(-24\beta^2-180\beta+456\right)\lambda_j\lambda_i+\left(-12\beta^2-9\beta-150\right)\lambda_i^2, \\
\eta_{212}^{(2)}(\lambda_j;\lambda_i;\beta) &= (-1296-216\beta)\,\lambda_j+\left(-24\beta^2-180\beta+456\right)\lambda_i, \\
\eta_{212}^{(3)}(\lambda_j;\lambda_i;\beta) &= -1296-216\beta\leq 0 \text{ for all } \beta\geq -6.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{212}(\lambda_i;\lambda_i;\beta) &= \left(-24\beta^2-144\beta-192\right)\lambda_i^3 \le 0 \text{ for all } \beta \ge -2,\\
\eta_{212}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-36\beta^2-297\beta-342\right)\lambda_i^2 \le 0 \text{ for all } \beta \ge -1.3835,\\
\eta_{212}^{(2)}(\lambda_i;\lambda_i;\beta) &= \left(-24\beta^2-396\beta-840\right)\lambda_i \le 0 \text{ for all } \beta \ge -2.5.
\end{aligned}$$

Hence, $\eta_{21}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{3}$.

Define $\eta_{211}(\lambda_j;\lambda_i;\beta) = \eta_{21}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that η_{211} is non-positive for all $\lambda_j \ge \lambda_i$. Differentiating η_{211} with respect to λ_j gives

$$\begin{aligned}
\eta_{211}(\lambda_j;\lambda_i;\beta) &= \left(\frac{-270-45\beta}{2}\right)\lambda_j^3 + \left(-12\beta^2-54\beta+156\right)\lambda_j^2\lambda_i\\
&\quad +(-18\beta^2+\frac{27}{2}\beta-63)\lambda_j\lambda_i^2 + \left(-27\beta-18\right)\lambda_i^3,\\
\eta_{211}^{(1)}(\lambda_j;\lambda_i;\beta) &= \left(\frac{-810-135\beta}{2}\right)\lambda_j^2 + \left(-24\beta^2-108\beta+312\right)\lambda_j\lambda_i\\
&\quad +\left(-18\beta^2+\frac{27}{2}\beta-63\right)\lambda_i^2,\\
\eta_{211}^{(2)}(\lambda_j;\lambda_i;\beta) &= \left(-810-135\beta\right)\lambda_j + \left(-24\beta^2-108\beta+312\right)\lambda_i,\\
\eta_{211}^{(3)}(\lambda_j;\lambda_i;\beta) &= -810-135\beta \le 0 \text{ for all } \beta \ge -6.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{211}(\lambda_i;\lambda_i;\beta) &= \left(-30\beta^2-90\beta-60\right)\lambda_i^3 \le 0 \text{ for all } \beta \ge -1,\\
\eta_{211}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-42\beta^2-162\beta-156\right)\lambda_i^2 \le 0 \text{ for all } \beta \ge -\frac{13}{7},\\
\eta_{211}^{(2)}(\lambda_i;\lambda_i;\beta) &= (-24\beta^2-243\beta-498)\lambda_i \le 0 \text{ for all } \beta \ge -2.8537.
\end{aligned}$$

Hence, $\eta_{21}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{3}$.

Define $\eta_{210}(\lambda_j;\lambda_i;\beta) = \eta_{21}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that η_{210} is negative for all $\lambda_j \ge \lambda_i$. Differentiating η_{210} with respect to λ_j gives

$$\begin{aligned}
\eta_{210}(\lambda_j;\lambda_i;\beta) &= \left(-54-9\beta\right)\lambda_j^3 + \left(-6\beta^2-27\beta+78\right)\lambda_j^2\lambda_i\\
&\quad + \left(-12\beta^2+9\beta-42\right)\lambda_j\lambda_i^2 + \left(-18-27\beta\right)\lambda_i^3,\\
\eta_{210}^{(1)}(\lambda_j;\lambda_i;\beta) &= \left(-162-27\beta\right)\lambda_j^2 + \left(-12\beta^2-54\beta+156\right)\lambda_j\lambda_i\\
&\quad + \left(-12\beta^2+9\beta-42\right)\lambda_i^2,\\
\eta_{210}^{(2)}(\lambda_j;\lambda_i;\beta) &= \left(-324-54\beta\right)\lambda_j + \left(-12\beta^2-54\beta+156\right)\lambda_i,\\
\eta_{210}^{(3)}(\lambda_j;\lambda_i;\beta) &= -324-54\beta \le 0 \text{ for all } \beta \ge -6.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{210}(\lambda_i;\lambda_i;\beta) &= \left(-18\beta^2-54\beta-36\right)\lambda_i^3 < 0 \text{ for all } \beta > -1,\\
\eta_{210}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-24\beta^2-72\beta-48\right)\lambda_i^2 < 0 \text{ for all } \beta > -1,\\
\eta_{210}^{(2)}(\lambda_i;\lambda_i;\beta) &= \left(-12\beta^2-108\beta-168\right)\lambda_i \le 0 \text{ for all } \beta \ge -2.
\end{aligned}$$

Hence, $\eta_{21}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$. This implies that $\eta_2^{(1)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_k \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Differentiating $\eta_{22}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{22}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (-18-27\beta) \sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(0,1,3)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_k^{n_3} \\
&\quad +(-6\beta^2-18\beta-156)\left(\lambda_k^2\lambda_j^2+\lambda_k^2\lambda_i^2+\lambda_j^2\lambda_i^2\right) \\
&\quad +(-24\beta^2-18\beta+132)\left(\lambda_k^2\lambda_j\lambda_i+\lambda_k\lambda_j^2\lambda_i+\lambda_k\lambda_j\lambda_i^2\right), \\
\eta_{22}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (-54-81\beta)\left(\lambda_k^2\lambda_j+\lambda_k^2\lambda_i\right)+\left(-12\beta^2-312-36\beta\right)\left(\lambda_k\lambda_j^2+\lambda_k\lambda_i^2\right) \\
&\quad +\left(-36\beta-48\beta^2+264\right)\lambda_k\lambda_j\lambda_i+(-27\beta-18)\left(\lambda_j^3+\lambda_i^3\right) \\
&\quad +\left(-24\beta^2+132-18\beta\right)\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right), \\
\eta_{22}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (-108-162\beta)\left(\lambda_k\lambda_j+\lambda_k\lambda_i\right)+\left(-12\beta^2-312-36\beta\right)\left(\lambda_j^2+\lambda_i^2\right) \\
&\quad +\left(-36\beta-48\beta^2+264\right)\lambda_j\lambda_i, \\
\eta_{22}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (-108-162\beta)\left(\lambda_j+\lambda_i\right) \leq 0 \text{ for all } \beta \geq -\frac{2}{3}.
\end{aligned}$$

Define $\eta_{222}(\lambda_j;\lambda_i;\beta)=\eta_{22}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{222} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{222} with respect to λ_j gives

$$\begin{aligned}
\eta_{222}(\lambda_j;\lambda_i;\beta) &= \left(-420-198\beta-12\beta^2\right)\lambda_j^2+(-48\beta^2-198\beta+156)\lambda_j\lambda_i \\
&\quad 7+\left(-12\beta^2-36\beta-312\right)\lambda_i^2, \\
\eta_{222}^{(1)}(\lambda_j;\lambda_i;\beta) &= \left(-396\beta-840-24\beta^2\right)\lambda_j+\left(-48\beta^2-198\beta+156\right)\lambda_i, \\
\eta_{222}^{(2)}(\lambda_j;\lambda_i;\beta) &= -396\beta-840-24\beta^2 \leq 0 \text{ for all } \beta \geq -2.5.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{222}(\lambda_i;\lambda_i;\beta) &= \left(-576-432\beta-72\beta^2\right)\lambda_i^2 \leq 0 \text{ for all } \beta \geq -2, \\
\eta_{222}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-594\beta-684-72\beta^2\right)\lambda_i \leq 0 \text{ for all } \beta \geq -1.3835.
\end{aligned}$$

Hence, $\eta_{22}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{221}(\lambda_j;\lambda_i;\beta)=\eta_{22}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{221} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{221} with respect to λ_j gives

$$\begin{aligned}
\eta_{221}(\lambda_j;\lambda_i;\beta) &= \left(-144\beta-12\beta^2-384\right)\lambda_j^3+\left(-72\beta^2+342-135\beta\right)\lambda_j^2\lambda_i \\
&\quad +\left(-180-36\beta^2-54\beta\right)\lambda_j\lambda_i^2+(-27\beta-18)\lambda_i^3, \\
\eta_{221}^{(1)}(\lambda_j;\lambda_i;\beta) &= \left(-36\beta^2-1152-432\beta\right)\lambda_j^2+\left(-270\beta+684-144\beta^2\right)\lambda_j\lambda_i \\
&\quad +\left(-180-36\beta^2-54\beta\right)\lambda_i^2, \\
\eta_{221}^{(2)}(\lambda_j;\lambda_i;\beta) &= \left(-864\beta-72\beta^2-2304\right)\lambda_j+\left(-270\beta+684-144\beta^2\right)\lambda_i, \\
\eta_{221}^{(3)}(\lambda_j;\lambda_i;\beta) &= -864\beta-72\beta^2-2304 \leq 0 \text{ for all } \beta \geq -4.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{221}(\lambda_i;\lambda_i;\beta) &= \left(-360\beta-120\beta^2-240\right)\lambda_i^3 \le 0 \text{ for all } \beta \ge -1,\\
\eta_{221}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-216\beta^2-648-756\beta\right)\lambda_i^2 \le 0 \text{ for all } \beta \ge -1.5,\\
\eta_{221}^{(2)}(\lambda_i;\lambda_i;\beta) &= \left(-1134\beta-216\beta^2-1620\right)\lambda_i \le 0 \text{ for all } \beta.
\end{aligned}$$

Hence, $\eta_{22}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{3}$.

Define $\eta_{220}(\lambda_j;\lambda_i;\beta) = \eta_{22}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that η_{220} is negative for all $\lambda_j \ge \lambda_i$. Differentiating η_{220} with respect to λ_j gives

$$\begin{aligned}
\eta_{220}(\lambda_j;\lambda_i;\beta) &= \left(-6\beta^2-72\beta-192\right)\lambda_j^3 + \left(228-90\beta-48\beta^2\right)\lambda_j^2\lambda_i\\
&\quad + \left(-180-36\beta^2-54\beta\right)\lambda_j\lambda_i^2 + \left(-36-54\beta\right)\lambda_i^3,\\
\eta_{220}^{(1)}(\lambda_j;\lambda_i;\beta) &= \left(-216\beta-576-18\beta^2\right)\lambda_j^2 + \left(-96\beta^2-180\beta+456\right)\lambda_j\lambda_i\\
&\quad + \left(-180-36\beta^2-54\beta\right)\lambda_i^2,\\
\eta_{220}^{(2)}(\lambda_j;\lambda_i;\beta) &= \left(-36\beta^2-1152-432\beta\right)\lambda_j + \left(-96\beta^2-180\beta+456\right)\lambda_i,\\
\eta_{220}^{(3)}(\lambda_j;\lambda_i;\beta) &= -36\beta^2-1152-432\beta \le 0 \text{ for all } \beta \ge -4.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{220}(\lambda_i;\lambda_i;\beta) &= \left(-90\beta^2-270\beta-180\right)\lambda_i^3 < 0 \text{ for all } \beta > -1,\\
\eta_{220}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-450\beta-300-150\beta^2\right)\lambda_i^2 < 0 \text{ for all } \beta > -1,\\
\eta_{220}^{(2)}(\lambda_i;\lambda_i;\beta) &= \left(-132\beta^2-696-612\beta\right)\lambda_i \le 0 \text{ for all } \beta \ge -2.
\end{aligned}$$

Hence, $\eta_{22}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{3}$. This implies that $\eta_2^{(2)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_k \ge \lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{3}$.

Differentiating $\eta_{23}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{23}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (-36-54\beta)\left(\lambda_k^3+\lambda_j^3+\lambda_i^3\right) + (-144\beta^2-108\beta+792)\lambda_k\lambda_j\lambda_i\\
&\quad +(-36\beta^2-135\beta-234)\sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(0,1,2)}} \lambda_k^{n_1}\lambda_j^{n_2}\lambda_i^{n_3},\\
\eta_{23}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (-108-162\beta)\lambda_k^2 + \left(-468-72\beta^2-270\beta\right)(\lambda_k\lambda_j+\lambda_k\lambda_i)\\
&\quad + \left(-36\beta^2-135\beta-234\right)\left(\lambda_j^2+\lambda_i^2\right) + \left(-144\beta^2-108\beta+792\right)\lambda_j\lambda_i,\\
\eta_{23}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (-216-324\beta)\lambda_k + \left(-468-72\beta^2-270\beta\right)(\lambda_i+\lambda_j)\\
&\le 0 \text{ for all } \beta \ge -\frac{2}{3}.
\end{aligned}$$

Define $\eta_{231}(\lambda_j;\lambda_i;\beta) = \eta_{23}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{231} is non-positive for all $\lambda_j \ge \lambda_i$. Differentiating η_{231} with respect to λ_j gives

$$\begin{aligned}
\eta_{231}(\lambda_j;\lambda_i;\beta) &= \left(-810-567\beta-108\beta^2\right)\lambda_j^2 + \left(-216\beta^2+324-378\beta\right)\lambda_j\lambda_i\\
&\quad + \left(-36\beta^2-135\beta-234\right)\lambda_i^2,\\
\eta_{231}^{(1)}(\lambda_j;\lambda_i;\beta) &= \left(-1134\beta-216\beta^2-1620\right)(\lambda_j+\lambda_i) \le 0 \text{ for all } \beta.
\end{aligned}$$

Now

$$\eta_{231}(\lambda_i;\lambda_i;\beta) \;=\; \left(-720-1080\beta-360\beta^2\right)\lambda_i^2 \le 0 \text{ for all } \beta \ge -1.$$

Hence, $\eta_{23}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{3}$.

Define $\eta_{230}(\lambda_j;\lambda_i;\beta) = \eta_{23}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{230} is negative for all $\lambda_j \ge \lambda_i$. Differentiating η_{231} with respect to λ_j gives

$$\begin{aligned}
\eta_{230}(\lambda_j;\lambda_i;\beta) &= \left(-378\beta-72\beta^2-540\right)\lambda_j^3+\left(-216\beta^2+324-378\beta\right)\lambda_j^2\lambda_i \\
&\quad +\left(-72\beta^2-270\beta-468\right)\lambda_j\lambda_i^2+\left(-36-54\beta\right)\lambda_i^3, \\
\eta_{230}^{(1)}(\lambda_j;\lambda_i;\beta) &= \left(-1134\beta-216\beta^2-1620\right)\lambda_j^2+\left(-756\beta+648-432\beta^2\right)\lambda_j\lambda_i \\
&\quad +\left(-72\beta^2-270\beta-468\right)\lambda_i^2, \\
\eta_{230}^{(2)}(\lambda_j;\lambda_i;\beta) &= \left(-3240-2268\beta-432\beta^2\right)\lambda_j+\left(-756\beta+648-432\beta^2\right)\lambda_i, \\
\eta_{230}^{(3)}(\lambda_j;\lambda_i;\beta) &= \left(-3240-2268\beta-432\beta^2\right) \le 0 \text{ for all } \beta.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{230}(\lambda_i;\lambda_i;\beta) &= \left(-720-1080\beta-360\beta^2\right)\lambda_i^3 < 0 \text{ for all } \beta > -1, \\
\eta_{230}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-2160\beta-720\beta^2-1440\right)\lambda_i^2 < 0 \text{ for all } \beta > -1, \\
\eta_{230}^{(2)}(\lambda_i;\lambda_i;\beta) &= \left(-2592-3024\beta-864\beta^2\right)\lambda_i \le 0 \text{ for all } \beta \ge -1.5.
\end{aligned}$$

Hence, $\eta_{23}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{3}$. This implies that $\eta_2^{(3)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_k \ge \lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{3}$.

Differentiating $\eta_{24}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{24}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(-72\beta^2-432\beta-576\right)\left(\lambda_k^2+\lambda_j^2+\lambda_i^2\right) \\
&\quad +\left(-288\beta^2-648\beta-144\right)\left(\lambda_k\lambda_j+\lambda_k\lambda_i+\lambda_j\lambda_i\right), \\
\eta_{24}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(-864\beta-144\beta^2-1152\right)\lambda_k+\left(-288\beta^2-648\beta-144\right)\lambda_j \\
&\quad +\left(-144-288\beta^2-648\beta\right)\lambda_i, \\
\eta_{24}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= -864\beta-144\beta^2-1152 \le 0 \text{ for all } \beta \ge -2.
\end{aligned}$$

Define $\eta_{241}(\lambda_j;\lambda_i;\beta) = \eta_{24}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{241} is non-positive for all $\lambda_j \ge \lambda_i$. Differentiating η_{241} with respect to λ_j gives

$$\begin{aligned}
\eta_{241}(\lambda_j;\lambda_i;\beta) &= \left(-1296-1512\beta-432\beta^2\right)\lambda_j+\left(-288\beta^2-648\beta-144\right)\lambda_i, \\
\eta_{241}^{(1)}(\lambda_j;\lambda_i;\beta) &= -1296-1512\beta-432\beta^2 \le 0 \text{ for all } \beta \ge -1.5.
\end{aligned}$$

Now

$$\eta_{241}(\lambda_i;\lambda_i;\beta) \;=\; \left(-2160\beta-720\beta^2-1440\right)\lambda_i \le 0 \text{ for all } \beta \ge -1.$$

Hence, $\eta_{24}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-positive for all $\lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{3}$.

Define $\eta_{240}(\lambda_j;\lambda_i;\beta) = \eta_{24}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{240} is negative for all $\lambda_j \geq \lambda_i$. Differentiating η_{240} with respect to λ_j gives

$$\begin{aligned}
\eta_{240}(\lambda_j;\lambda_i;\beta) &= \left(-1296 - 1512\beta - 432\beta^2\right)\lambda_j^2 + \left(-1296\beta - 576\beta^2 - 288\right)\lambda_j\lambda_i \\
&\quad + \left(-576 - 432\beta - 72\beta^2\right)\lambda_i^2, \\
\eta_{240}^{(1)}(\lambda_j;\lambda_i;\beta) &= \left(-2592 - 3024\beta - 864\beta^2\right)\lambda_j + \left(-1296\beta - 576\beta^2 - 288\right)\lambda_i, \\
\eta_{240}^{(2)}(\lambda_j;\lambda_i;\beta) &= -2592 - 3024\beta - 864\beta^2 \leq 0 \text{ for all } \beta \geq -1.5.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{240}(\lambda_i;\lambda_i;\beta) &= \left(-2160 - 3240\beta - 1080\beta^2\right)\lambda_i^2 < 0 \text{ for all } \beta > -1, \\
\eta_{240}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(-2880 - 4320\beta - 1440\beta^2\right)\lambda_i < 0 \text{ for all } \beta > -1.
\end{aligned}$$

Hence, $\eta_{24}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$. This implies that $\eta_2^{(4)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_k \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Differentiating $\eta_{25}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{25}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \left(-1440 - 2160\beta - 720\beta^2\right)\lambda_k + \left(-720\beta^2 - 1440 - 2160\beta\right)\lambda_j \\
&\quad + \left(-1440 - 720\beta^2 - 2160\beta\right)\lambda_i, \\
\eta_{25}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= -1440 - 2160\beta - 720\beta^2 \leq 0 \text{ for all } \beta \geq -1.
\end{aligned}$$

Define $\eta_{250}(\lambda_j;\lambda_i;\beta) = \eta_{25}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that η_{250} is negative for all $\lambda_j \geq \lambda_i$. Differentiating η_{250} with respect to λ_j gives

$$\begin{aligned}
\eta_{250}(\lambda_j;\lambda_i;\beta) &= \left(-2880 - 4320\beta - 1440\beta^2\right)\lambda_j + \left(-720\beta^2 - 2160\beta - 1440\right)\lambda_i, \\
\eta_{250}^{(1)}(\lambda_j;\lambda_i;\beta) &= -2880 - 4320\beta - 1440\beta^2 \leq 0 \text{ for all } \beta \geq -1.
\end{aligned}$$

Now

$$\eta_{240}(\lambda_i;\lambda_i;\beta) = \left(-4320 - 6480\beta - 2160\beta^2\right)\lambda_i < 0 \text{ for all } \beta > -1.$$

Hence, $\eta_{25}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$. This implies that $\eta_2^{(5)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is negative for all $\lambda_k \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

This completes the proof.

□

Appendix B

Lemma B.5 Let

$$\begin{aligned}\zeta_1(\lambda;\lambda_k;\lambda_i;\beta) &= \left(18\beta^2+42\beta+24\right)(\lambda+\lambda_i)^4+60\,(\lambda+\lambda_k)^4\\ &\quad+\left(4\beta^3-12\beta^2+8\beta\right)(\lambda+\lambda_i)^3(\lambda+\lambda_k)\\ &\quad-48\,(2-\beta)(\lambda+\lambda_i)(\lambda+\lambda_k)^3\\ &\quad+\left(18\beta^2-54\beta+36\right)(\lambda+\lambda_i)^2(\lambda+\lambda_k)^2\end{aligned}$$

and $\lambda_k \geq \lambda_i$. If $\lambda_i \leq 0$, the expression $\zeta_1(\lambda;\lambda_k;\lambda_i;\beta)$ is non-negative for all $\lambda \in [-\lambda_i,\infty)$ and $\beta \geq -1$; if $\lambda_i > 0$, the expression $\zeta_1(\lambda;\lambda_k;\lambda_i;\beta)$ is positive for all $\lambda \in [0,\infty)$ and $\beta > -1$.

Proof. Differentiating $\zeta_1(\lambda;\lambda_k;\lambda_i;\beta)$ with respect to λ we obtain

$$\begin{aligned}\zeta_1(\lambda;\lambda_k;\lambda_i;\beta) &= \left(18\beta^2+42\beta+24\right)(\lambda+\lambda_i)^4+60\,(\lambda+\lambda_k)^4 &\text{(B.1)}\\ &\quad+\left(4\beta^3-12\beta^2+8\beta\right)(\lambda+\lambda_i)^3(\lambda+\lambda_k)\\ &\quad-48\,(2-\beta)(\lambda+\lambda_i)(\lambda+\lambda_k)^3\\ &\quad+\left(18\beta^2-54\beta+36\right)(\lambda+\lambda_i)^2(\lambda+\lambda_k)^2,\\ \zeta_1^{(1)}(\lambda;\lambda_k;\lambda_i;\beta) &= (144+48\beta)(\lambda+\lambda_k)^3\\ &\quad+\left(-216+36\beta+36\beta^2\right)(\lambda+\lambda_k)^2(\lambda+\lambda_i) &\text{(B.2)}\\ &\quad+\left(72+12\beta^3-84\beta\right)(\lambda+\lambda_k)(\lambda+\lambda_i)^2\\ &\quad+\left(60\beta^2+176\beta+96+4\beta^3\right)(\lambda+\lambda_i)^3,\\ \zeta_1^{(2)}(\lambda;\lambda_k;\lambda_i;\beta) &= \left(216+180\beta+36\beta^2\right)(\lambda+\lambda_k)^2\\ &\quad+\left(24\beta^3+72\beta^2-96\beta-288\right)(\lambda+\lambda_k)(\lambda+\lambda_i) &\text{(B.3)}\\ &\quad+\left(180\beta^2+444\beta+360+24\beta^3\right)(\lambda+\lambda_i)^2,\\ \zeta_1^{(3)}(\lambda;\lambda_k;\lambda_i;\beta) &= \left(24\beta^3+144\beta^2+264\beta+144\right)(\lambda+\lambda_k)\\ &\quad+\left(432\beta^2+792\beta+432+72\beta^3\right)(\lambda+\lambda_i)\\ &\geq 0 \text{ for all } \beta \geq -1.\end{aligned}$$

Consider the case $\lambda_i \leq 0$. Substituting $\lambda = -\lambda_i$ into (B.1)–(B.3) we obtain

$$\begin{aligned}\zeta_1(-\lambda_i;\lambda_k;\lambda_i;\beta) &= 60\,(\lambda_k-\lambda_i)^4 \geq 0 \text{ for all } \beta,\\ \zeta_1^{(1)}(-\lambda_i;\lambda_k;\lambda_i;\beta) &= (144+48\beta)(\lambda_k-\lambda_i)^3 \geq 0 \text{ for all } \beta \geq -3,\\ \zeta_1^{(2)}(-\lambda_i;\lambda_k;\lambda_i;\beta) &= \left(216+180\beta+36\beta^2\right)(\lambda_k-\lambda_i)^2 \geq 0 \text{ for all } \beta \geq -2.\end{aligned}$$

Hence, $\zeta_1(\lambda;\lambda_k;\lambda_i;\beta)$ is non-negative for all $\lambda \geq -\lambda_i$ and $\beta > -1$.

Consider the case $\lambda_i > 0$. Substituting $\lambda = 0$ into (B.1)–(B.3) we obtain

$$\begin{aligned}
\zeta_1(0;\lambda_k;\lambda_i;\beta) &= \left(18\beta^2+42\beta+24\right)\lambda_i^4+60\lambda_k^4 \\
&\quad +\left(4\beta^3-12\beta^2+8\beta\right)\lambda_i^3\lambda_k-48\left(2-\beta\right)\lambda_i\lambda_k^3 \\
&\quad +\left(18\beta^2-54\beta+36\right)\lambda_i^2\lambda_k^2, \\
\zeta_1^{(1)}(0;\lambda_k;\lambda_i;\beta) &= \left(144+48\beta\right)\lambda_k^3+\left(-216+36\beta+36\beta^2\right)\lambda_k^2\lambda_i \\
&\quad +\left(72+12\beta^3-84\beta\right)\lambda_k\lambda_i^2+\left(60\beta^2+176\beta+96+4\beta^3\right)\lambda_i^3, \\
\zeta_1^{(2)}(0;\lambda_k;\lambda_i;\beta) &= \left(216+180\beta+36\beta^2\right)\lambda_k^2+\left(24\beta^3+72\beta^2-96\beta-288\right)\lambda_k\lambda_i \\
&\quad +\left(180\beta^2+444\beta+360+24\beta^3\right)\lambda_i^2.
\end{aligned}$$

We will show that the above equations are all positive for all $\lambda_k \geq \lambda_i$ and $\beta > -1$. This will then imply that $\zeta_1(\lambda;\lambda_k;\lambda_i;\beta)$ is positive for all $\lambda \geq 0$ and $\beta > -1$.

Define $\zeta_{10}(\lambda_k;\lambda_i;\beta) = \zeta_1(0;\lambda_k;\lambda_i;\beta)$. Differentiating $\zeta_{10}(\lambda_k;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{10}(\lambda_k;\lambda_i;\beta) &= \left(18\beta^2+42\beta+24\right)\lambda_i^4+60\lambda_k^4 && \text{(B.4)}\\
&\quad +\left(4\beta^3-12\beta^2+8\beta\right)\lambda_i^3\lambda_k-48\left(2-\beta\right)\lambda_i\lambda_k^3 \\
&\quad +\left(18\beta^2-54\beta+36\right)\lambda_i^2\lambda_k^2, \\
\zeta_{10}^{(1)}(\lambda_k;\lambda_i;\beta) &= 240\lambda_k^3+\left(-288+144\beta\right)\lambda_k^2\lambda_i && \text{(B.5)}\\
&\quad +\left(72-108\beta+36\beta^2\right)\lambda_k\lambda_i^2+\left(4\beta^3-12\beta^2+8\beta\right)\lambda_i^3, \\
\zeta_{10}^{(2)}(\lambda_k;\lambda_i;\beta) &= 720\lambda_k^2+\left(288\beta-576\right)\lambda_k\lambda_i && \text{(B.6)}\\
&\quad +\left(72-108\beta+36\beta^2\right)\lambda_i^2, \\
\zeta_{10}^{(3)}(\lambda_k;\lambda_i;\beta) &= 1440\lambda_k+\left(288\beta-576\right)\lambda_i, && \text{(B.7)}\\
\zeta_{10}^{(4)}(\lambda_k;\lambda_i;\beta) &= 1440.
\end{aligned}$$

Substituting $\lambda_k = \lambda_i$ into (B.4)–(B.7) we obtain

$$\begin{aligned}
\zeta_{10}(\lambda_i;\lambda_i;\beta) &= \left(24+44\beta+24\beta^2+4\beta^3\right)\lambda_i^4 > 0 \text{ for all } \beta > -1, \\
\zeta_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(24+44\beta+24\beta^2+4\beta^3\right)\lambda_i^3 > 0 \text{ for all } \beta > -1, \\
\zeta_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= \left(216+180\beta+36\beta^2\right)\lambda_i^2 > 0 \text{ for all } \beta > -2, \\
\zeta_{10}^{(3)}(\lambda_i;\lambda_i;\beta) &= \left(864+288\beta\right)\lambda_i > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, $\zeta_1(0;\lambda_k;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_i$ and $\beta > -1$.

Define $\zeta_{11}(\lambda_k;\lambda_i;\beta) = \zeta_1^{(1)}(0;\lambda_k;\lambda_i;\beta)$. Differentiating $\zeta_{11}(\lambda_k;\lambda_i;\beta)$ with respect to λ_k

we obtain

$$\begin{aligned}
\zeta_{11}(\lambda_k;\lambda_i;\beta) &= (144+48\beta)\,\lambda_k^3+\left(-216+36\beta+36\beta^2\right)\lambda_k^2\lambda_i \\
&\quad+\left(72+12\beta^3-84\beta\right)\lambda_k\lambda_i^2+\left(60\beta^2+176\beta+96+4\beta^3\right)\lambda_i^3, && \text{(B.8)}\\
\zeta_{11}^{(1)}(\lambda_k;\lambda_i;\beta) &= (432+144\beta)\,\lambda_k^2+\left(72\beta^2-432+72\beta\right)\lambda_k\lambda_i \\
&\quad+\left(72+12\beta^3-84\beta\right)\lambda_i^2, && \text{(B.9)}\\
\zeta_{11}^{(2)}(\lambda_k;\lambda_i;\beta) &= (864+288\beta)\,\lambda_k+\left(72\beta^2-432+72\beta\right)\lambda_i, && \text{(B.10)}\\
\zeta_{11}^{(3)}(\lambda_k;\lambda_i;\beta) &= 864+288\beta\geq 0 \text{ for all } \beta\geq -3.
\end{aligned}$$

Substituting $\lambda_k=\lambda_i$ into (B.8)–(B.10) we obtain

$$\begin{aligned}
\zeta_{11}(\lambda_i;\lambda_i;\beta) &= \left(96+176\beta+96\beta^2+16\beta^3\right)\lambda_i^3>0 \text{ for all } \beta>-1,\\
\zeta_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(72+132\beta+72\beta^2+12\beta^3\right)\lambda_i^2>0 \text{ for all } \beta>-1,\\
\zeta_{11}^{(2)}(\lambda_i;\lambda_i;\beta) &= \left(432+360\beta+72\beta^2\right)\lambda_i>0 \text{ for all } \beta>-2.
\end{aligned}$$

Hence, $\zeta_1^{(1)}(0;\lambda_k;\lambda_i;\beta)$ is positive for all $\lambda_k\geq\lambda_i$ and $\beta>-1$.

Define $\zeta_{12}(\lambda_k;\lambda_i;\beta)=\zeta_1^{(2)}(0;\lambda_k;\lambda_i;\beta)$. Differentiating $\zeta_{12}(\lambda_k;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{12}(\lambda_k;\lambda_i;\beta) &= \left(216+180\beta+36\beta^2\right)\lambda_k^2 \\
&\quad+\left(24\beta^3+72\beta^2-96\beta-288\right)\lambda_k\lambda_i \\
&\quad+\left(180\beta^2+444\beta+360+24\beta^3\right)\lambda_i^2, && \text{(B.11)}\\
\zeta_{12}^{(1)}(\lambda_k;\lambda_i;\beta) &= \left(432+360\beta+72\beta^2\right)\lambda_k+\left(24\beta^3+72\beta^2-96\beta-288\right)\lambda_i, && \text{(B.12)}\\
\zeta_{12}^{(2)}(\lambda_k;\lambda_i;\beta) &= 432+360\beta+72\beta^2\geq 0 \text{ for all } \beta\geq -2.
\end{aligned}$$

Substituting $\lambda_k=\lambda_i$ into (B.11) and (B.12) we obtain

$$\begin{aligned}
\zeta_{12}(\lambda_i;\lambda_i;\beta) &= \left(288+528\beta+288\beta^2+48\beta^3\right)\lambda_i^3>0 \text{ for all } \beta>-1,\\
\zeta_{12}^{(1)}(\lambda_i;\lambda_i;\beta) &= \left(24\beta^3+144\beta^2+264\beta+144\right)\lambda_i^2>0 \text{ for all } \beta>-1.
\end{aligned}$$

Hence, $\zeta_1^{(2)}(0;\lambda_k;\lambda_i;\beta)$ is positive for all $\lambda_k>\lambda_i$ and $\beta\geq -1$. This implies that $\zeta_1(\lambda;\lambda_k;\lambda_i;\beta)$ is positive for all $\lambda\geq 0$, $\lambda_k\geq\lambda_i$ and $\beta>-1$.

This completes the proof.

□

Lemma B.6 Let

$$\tilde{s}(a,b) = \sum_{\substack{(n_1,n_2)\\ \in \mathrm{perm}(a,b)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}$$

and

$$\begin{aligned}
\zeta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) \quad = \quad & (75\beta+30)\,\tilde{s}(4,4) \\
& +90\,(\lambda+\lambda_k)^4\,\tilde{s}(0,4) \\
& +\left(18\beta^2-60\beta+48\right)(\lambda+\lambda_k)\,\tilde{s}(3,4) \\
& -48\,(2-\beta)\,(\lambda+\lambda_k)^3\,\tilde{s}(1,4) \\
& -24\,(2-\beta)\,(\lambda+\lambda_k)^4\,\tilde{s}(1,3) \\
& -27\,(2-\beta)\,(\lambda+\lambda_k)^4\,\tilde{s}(2,2) \\
& +\left(9\beta^2-36\right)(\lambda+\lambda_k)^2\,\tilde{s}(2,4) \\
& +18\left(\beta^2-6\beta+8\right)(\lambda+\lambda_k)^3\,\tilde{s}(2,3) \\
& +\left(6\beta^3-54\beta^2+156\beta-144\right)(\lambda+\lambda_k)^2\,\tilde{s}(3,3)
\end{aligned}$$

and $\lambda_k \geq \max(\lambda_i,\lambda_j)$. If $\min(\lambda_i,\lambda_j) \leq 0$, the expression $\zeta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda \in [\max(-\lambda_i,-\lambda_j),\infty)$ and $\beta \geq -\frac{2}{5}$; if $\min(\lambda_i,\lambda_j) > 0$, the expression $\zeta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda \in [0,\infty)$ and $\beta \geq -\frac{2}{5}$.

Proof.

Without loss of generality assume that $\lambda_i \leq \lambda_j$.

Differentiating $\zeta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ we obtain

$$\begin{aligned}
\zeta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) \quad = \quad & (75\beta+30)\,\tilde{s}(4,4) \\
& +90\,(\lambda+\lambda_k)^4\,\tilde{s}(0,4) \\
& +\left(18\beta^2-60\beta+48\right)(\lambda+\lambda_k)\,\tilde{s}(3,4) \\
& -48\,(2-\beta)\,(\lambda+\lambda_k)^3\,\tilde{s}(1,4) \\
& -24\,(2-\beta)\,(\lambda+\lambda_k)^4\,\tilde{s}(1,3) \\
& -27\,(2-\beta)\,(\lambda+\lambda_k)^4\,\tilde{s}(2,2) \\
& +\left(9\beta^2-36\right)(\lambda+\lambda_k)^2\,\tilde{s}(2,4) \\
& +18\left(\beta^2-6\beta+8\right)(\lambda+\lambda_k)^3\,\tilde{s}(2,3) \\
& +\left(6\beta^3-54\beta^2+156\beta-144\right)(\lambda+\lambda_k)^2\,\tilde{s}(3,3),
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
\zeta_2^{(1)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (312+24\beta)\,(\lambda+\lambda_k)^4\,\tilde{s}(1,3) \\
&\quad +126\,(\beta-2)\,(\lambda+\lambda_k)^4\,\tilde{s}(1,2) \\
&\quad +24\,(2\beta+11)\,(\lambda+\lambda_k)^3\,\tilde{s}(0,4) \\
&\quad +36\left(\beta^2+\beta-8\right)(\lambda+\lambda_k)^3\,\tilde{s}(1,3) \\
&\quad +108\left(\beta^2-5\beta+6\right)(\lambda+\lambda_k)^3\,\tilde{s}(2,2) \\
&\quad +18\left(\beta^2+8\beta-20\right)(\lambda+\lambda_k)^2\,\tilde{s}(1,4) \\
&\quad +18\left(\beta^3-4\beta^2+8\beta-8\right)(\lambda+\lambda_k)^2\,\tilde{s}(2,3) \\
&\quad +36\left(2\beta^2-5\beta+2\right)(\lambda+\lambda_k)\,\tilde{s}(2,4) \\
&\quad +12\left(\beta^3+3\beta^2-14\beta+8\right)(\lambda+\lambda_k)\,\tilde{s}(3,3) \\
&\quad +6\left(3\beta^2+40\beta+28\right)(\lambda+\lambda_k)\,\tilde{s}(3,3),
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
\zeta_2^{(2)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (198\beta+684)\,(\lambda+\lambda_k)^4\,\tilde{s}(1,2) \\
&\quad +504\,(\beta-2)\,(\lambda+\lambda_k)^4\,\tilde{s}(1,1) \\
&\quad +(2016+36\beta^2+360\beta)\,(\lambda+\lambda_k)^3\,\tilde{s}(1,3) \\
&\quad +(-576+324\beta^2-360\beta)\,(\lambda+\lambda_k)^3\,\tilde{s}(1,2) \\
&\quad +(432+18\beta^2+288\beta)\,(\lambda+\lambda_k)^2\,\tilde{s}(0,4) \\
&\quad +(36\beta^2+1080\beta+36\beta^3-2592)\,(\lambda+\lambda_k)^2\,\tilde{s}(1,3) \\
&\quad +(-756\beta-108\beta^2+1080+108\beta^3)\,(\lambda+\lambda_k)^2\,\tilde{s}(2,2) \\
&\quad +(-72\beta+180\beta^2-576)\,(\lambda+\lambda_k)\,\tilde{s}(1,4) \\
&\quad +(72\beta^3+252\beta^2+288-936\beta)\,(\lambda+\lambda_k)\,\tilde{s}(2,3) \\
&\quad +(126\beta^2+576+540\beta)\tilde{s}(2,4) \\
&\quad +(180\beta^2+1752\beta+12\beta^3+1440)\tilde{s}(3,3),
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
\zeta_2^{(3)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (900\beta+360)\,(\lambda+\lambda_k)^4\,((\lambda+\lambda_j)+(\lambda+\lambda_i)) \\
&\quad +(432\beta^2+8208+1512\beta)\,(\lambda+\lambda_k)^3\,\tilde{s}(0,2) \\
&\quad +(-6336+576\beta+1296\beta^2)\,(\lambda+\lambda_k)^3\,\tilde{s}(1,1) \\
&\quad +(3312\beta+5184+36\beta^3+216\beta^2)\,(\lambda+\lambda_k)^2\,\tilde{s}(0,3) \\
&\quad +(-7344+864\beta^2+324\beta^3+648\beta)\,(\lambda+\lambda_k)^2\,\tilde{s}(1,2) \\
&\quad +(216\beta^2+288+504\beta)\,(\lambda+\lambda_k)\,\tilde{s}(0,4) \\
&\quad +(1296\beta^2+216\beta^3-6912)\,(\lambda+\lambda_k)\,\tilde{s}(1,3) \\
&\quad +(648\beta^3-7128\beta+1296\beta^2+3888)\,(\lambda+\lambda_k)\,\tilde{s}(2,2) \\
&\quad +(576+1008\beta+432\beta^2)\tilde{s}(1,4) \\
&\quad +(1296\beta^2+6480\beta+6912+108\beta^3)\tilde{s}(2,3),
\end{aligned} \tag{B.16}$$

$$\begin{aligned}
\zeta_2^{(4)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (1800\beta+720)\,(\lambda+\lambda_k)^4 && \text{(B.17)}\\
&+(7200\beta+2160\beta^2+11520)\,(\lambda+\lambda_k)^3\,\tilde{s}(0,1)\\
&+(432\beta^3+2808\beta^2+15120\beta+32832)\,(\lambda+\lambda_k)^2\,\tilde{s}(0,2)\\
&+(-48384+4320\beta+7344\beta^2+1296\beta^3)\,(\lambda+\lambda_k)^2\,\tilde{s}(1,1)\\
&+(2592\beta^2+4608+8640\beta+288\beta^3)\,(\lambda+\lambda_k)\,\tilde{s}(0,3)\\
&+(8208\beta^2+2592\beta^3-12960\beta-27648)\,(\lambda+\lambda_k)\,\tilde{s}(1,2),\\
\zeta_2^{(5)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (25920+21600\beta+4320\beta^2)\,(\lambda+\lambda_k)^3 && \text{(B.18)}\\
&+(51840+56160\beta+19440\beta^2+2160\beta^3)\,(\lambda+\lambda_k)^2\,\tilde{s}(0,1)\\
&+(4320\beta^3+51840+43200\beta+21600\beta^2)\,(\lambda+\lambda_k)\,\tilde{s}(0,2)\\
&+(-43200\beta+12960\beta^3-207360+47520\beta^2)\,(\lambda+\lambda_k)\,\tilde{s}(1,1)\\
&+(720\beta^3+10800\beta^2+17280+31680\beta)\tilde{s}(0,3)\\
&+(90720+6480\beta^3+101520\beta+43200\beta^2)\tilde{s}(1,2),\\
\zeta_2^{(6)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (4320\beta^3+177120\beta+181440+51840\beta^2)\,(\lambda+\lambda_k)^2 && \text{(B.19)}\\
&+(155520\beta+25920\beta^3+129600\beta^2)\,(\lambda+\lambda_k)\,\tilde{s}(0,1)\\
&+(239760\beta+194400+12960\beta^3+97200\beta^2)\tilde{s}(0,2)\\
&+(220320\beta^2+38880\beta^3+362880\beta+155520)\tilde{s}(1,1),\\
\zeta_2^{(7)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (665280\beta+362880\beta^2+60480\beta^3+362880)\,(\lambda+\lambda_k)\\
&+(544320\beta^2+997920\beta+544320+90720\beta^3)\tilde{s}(0,1)\\
&\geq 0 \text{ for all } \beta\geq -1.
\end{aligned}$$

Consider the case $\lambda_i \leq 0$. Substituting in $\lambda = -\lambda_i$ into (B.13)–(B.19) we obtain

$$\begin{aligned}
\zeta_2(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= 90\,(\lambda_j-\lambda_i)^4\,(\lambda_k-\lambda_i)^4\\
&\geq 0,\\
\zeta_2^{(1)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= (312+24\beta)\,(\lambda_k-\lambda_i)^4\,(\lambda_j-\lambda_i)^3\\
&+24\,(2\beta+11)\,(\lambda_k-\lambda_i)^3\,(\lambda_j-\lambda_i)^4\\
&\geq 0 \text{ for all } \beta\geq -5.5,\\
\zeta_2^{(2)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= (198\beta+684)\,(\lambda_k-\lambda_i)^4\,(\lambda_j-\lambda_i)^2\\
&+(2016+36\beta^2+360\beta)\,(\lambda_k-\lambda_i)^3\,(\lambda_j-\lambda_i)^3\\
&+(432+18\beta^2+288\beta)\,(\lambda_k-\lambda_i)^2\,(\lambda_j-\lambda_i)^4\\
&\geq 0 \text{ for all } \beta\geq -1.6754,\\
\zeta_2^{(3)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= (900\beta+360)\,(\lambda_k-\lambda_i)^4\,(\lambda_j-\lambda_i)\\
&+(432\beta^2+8208+1512\beta)\,(\lambda_k-\lambda_i)^3\,(\lambda_j-\lambda_i)^2\\
&+(3312\beta+5184+36\beta^3+216\beta^2)\,(\lambda_k-\lambda_i)^2\,(\lambda_j-\lambda_i)^3\\
&+(216\beta^2+288+504\beta)\,(\lambda_k-\lambda_i)\,(\lambda_j-\lambda_i)^4\\
&\geq 0 \text{ for all } \beta\geq -0.4,
\end{aligned}$$

$$\begin{aligned}
\zeta_2^{(4)}(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta) &= (1800\beta+720)\left(\lambda_k-\lambda_i\right)^4 \\
&\quad +(7200\beta+2160\beta^2+11520)\left(\lambda_k-\lambda_i\right)^3\left(\lambda_j-\lambda_i\right) \\
&\quad +(432\beta^3+2808\beta^2+15120\beta+32832)\left(\lambda_k-\lambda_i\right)^2\left(\lambda_j-\lambda_i\right)^2 \\
&\quad +(2592\beta^2+4608+8640\beta+288\beta^3)\left(\lambda_k-\lambda_i\right)\left(\lambda_j-\lambda_i\right)^3 \\
&\geq 0 \text{ for all } \beta \geq -0.4, \\
\zeta_2^{(5)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= (25920+21600\beta+4320\beta^2)\left(\lambda_k-\lambda_i\right)^3 \\
&\quad +(51840+56160\beta+19440\beta^2+2160\beta^3)\left(\lambda_k-\lambda_i\right)^2\left(\lambda_j-\lambda_i\right) \\
&\quad +(4320\beta^3+51840+43200\beta+21600\beta^2)\left(\lambda_k-\lambda_i\right)\left(\lambda_j-\lambda_i\right)^2 \\
&\quad +(720\beta^3+10800\beta^2+17280+31680\beta)\left(\lambda_j-\lambda_i\right)^3 \\
&\geq 0 \text{ for all } \beta \geq -0.7085, \\
\zeta_2^{(6)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta) &= (4320\beta^3+177120\beta+181440+51840\beta^2)\left(\lambda_k-\lambda_i\right)^2 \\
&\quad +(155520\beta+25920\beta^3+129600\beta^2)\left(\lambda_k-\lambda_i\right)\left(\lambda_j-\lambda_i\right) \\
&\quad +(239760\beta+194400+12960\beta^3+97200\beta^2)\left(\lambda_j-\lambda_i\right)^2 .
\end{aligned}$$

Define $\hat{\zeta}_2(\lambda_k;\lambda_j;\lambda_i;\beta) = \zeta_2^{(6)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta)$. We wish to show that $\hat{\zeta}_2(\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\beta \geq -\frac{2}{5}$. Differentiating $\hat{\zeta}_2(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\hat{\zeta}_2(\lambda_k;\lambda_j;\lambda_i;\beta) &= (4320\beta^3+177120\beta+181440+51840\beta^2)\left(\lambda_k-\lambda_i\right)^2 && \text{(B.20)} \\
&\quad +(155520\beta+25920\beta^3+129600\beta^2)\left(\lambda_k-\lambda_i\right)\left(\lambda_j-\lambda_i\right) \\
&\quad +(239760\beta+194400+12960\beta^3+97200\beta^2)\left(\lambda_j-\lambda_i\right)^2 , \\
\hat{\zeta}_2^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= 2(4320\beta^3+177120\beta+181440+51840\beta^2)\left(\lambda_k-\lambda_i\right) && \text{(B.21)} \\
&\quad +(155520\beta+25920\beta^3+129600\beta^2)\left(\lambda_j-\lambda_i\right) \\
\hat{\zeta}_2^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= 2(4320\beta^3+177120\beta+181440+51840\beta^2) \\
&\geq 0 \text{ for all } \beta \geq -2.
\end{aligned}$$

Substituting $\lambda_k = \lambda_j$ into (B.20) and (B.21) we obtain

$$\begin{aligned}
\hat{\zeta}_2(\lambda_k;\lambda_j;\lambda_i;\beta) &= (375840+43200\beta^3+278640\beta^2+572400\beta)\left(\lambda_j-\lambda_i\right)^2 \\
&\geq 0 \text{ for all } \beta \geq -1.45, \\
\hat{\zeta}_2^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (34560\beta^3+509760\beta+233280\beta^2+362880)\left(\lambda_k-\lambda_i\right) \\
&\geq 0 \text{ for all } \beta \geq -1.75.
\end{aligned}$$

Hence, $\zeta_2^{(6)}(-\lambda_i;\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\beta \geq -\frac{2}{5}$ when $\lambda_i \leq 0$. From this we deduce that $\zeta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda \geq -\lambda_i$.

Consider the case $\lambda_i > 0$. Let us define the functions $\zeta_{20}(\lambda_k;\lambda_j;\lambda_i;\beta)$, $\zeta_{21}(\lambda_k;\lambda_j;\lambda_i;\beta)$, $\zeta_{22}(\lambda_k;\lambda_j;\lambda_i;\beta)$, $\zeta_{23}(\lambda_k;\lambda_j;\lambda_i;\beta)$, $\zeta_{24}(\lambda_k;\lambda_j;\lambda_i;\beta)$, $\zeta_{25}(\lambda_k;\lambda_j;\lambda_i;\beta)$, and $\zeta_{26}(\lambda_k;\lambda_j;\lambda_i;\beta)$

as follows:

$$\begin{aligned}
\zeta_{20}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \zeta_2(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\zeta_{21}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \zeta_2^{(1)}(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\zeta_{22}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \zeta_2^{(2)}(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\zeta_{23}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \zeta_2^{(3)}(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\zeta_{24}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \zeta_2^{(4)}(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\zeta_{25}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \zeta_2^{(5)}(0;\lambda_k;\lambda_j;\lambda_i;\beta),\\
\zeta_{26}(\lambda_k;\lambda_j;\lambda_i;\beta) &= \zeta_2^{(6)}(0;\lambda_k;\lambda_j;\lambda_i;\beta).
\end{aligned}$$

We will show that the above equations are all positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$ and, hence, prove that $\zeta_2(\lambda;\lambda_k;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda \geq 0$.

Differentiating $\zeta_{20}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{20}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (75\beta+30)\,\lambda_i^4\lambda_j^4\\
&\quad +90\lambda_i^4\lambda_k^4+90\lambda_j^4\lambda_k^4\\
&\quad +\left(18\beta^2-60\beta+48\right)\left(\lambda_i^3\lambda_j^4\lambda_k+\lambda_i^4\lambda_j^3\lambda_k\right)\\
&\quad -48\,(2-\beta)\left(\lambda_i\lambda_j^4\lambda_k^3+\lambda_i^4\lambda_j\lambda_k^3\right)\\
&\quad -24\,(2-\beta)\left(\lambda_i\lambda_j^3\lambda_k^4+\lambda_i^3\lambda_j\lambda_k^4\right)\\
&\quad -27\,(2-\beta)\,\lambda_i^2\lambda_j^2\lambda_k^4\\
&\quad +\left(9\beta^2-36\right)\left(\lambda_i^2\lambda_j^4\lambda_k^2+\lambda_i^4\lambda_j^2\lambda_k^2\right)\\
&\quad +18\left(\beta^2-6\beta+8\right)\left(\lambda_i^2\lambda_j^3\lambda_k^3+\lambda_i^3\lambda_j^2\lambda_k^3\right)\\
&\quad +\left(6\beta^3-54\beta^2+156\beta-144\right)\lambda_i^3\lambda_j^3\lambda_k^2,\\
\zeta_{20}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= 360\lambda_k^3\left(\lambda_i^4+\lambda_j^4\right)\\
&\quad +(96\beta-192)\,\lambda_k^3\left(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j\right)\\
&\quad +(108\beta-216)\,\lambda_k^3\lambda_i^2\lambda_j^2\\
&\quad +\left(54\beta^2-324\beta+432\right)\left(\lambda_i^3\lambda_j^2+\lambda_i^2\lambda_j^3\right)\lambda_k^2\\
&\quad +(144\beta-288)\,\lambda_k^2\left(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j\right)\\
&\quad +\left(18\beta^2-72\right)\lambda_k\left(\lambda_i^4\lambda_j^2+\lambda_i^2\lambda_j^4\right)\\
&\quad +(12\beta^3-108\beta^2+312\beta-288)\lambda_k\lambda_i^3\lambda_j^3\\
&\quad +\left(18\beta^2-60\beta+48\right)\left(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3\right),\\
\zeta_{20}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (288\beta-576)\,\lambda_k^2\left(\lambda_i^3\lambda_j+\lambda_i\lambda_j^3\right)\\
&\quad +(324\beta-648)\,\lambda_k^2\lambda_i^2\lambda_j^2\\
&\quad +1080\lambda_k^2\left(\lambda_i^4+\lambda_j^4\right)\\
&\quad +\left(108\beta^2-648\beta+864\right)\lambda_k\left(\lambda_i^3\lambda_j^2+\lambda_i^2\lambda_j^3\right)\\
&\quad +(288\beta-576)\,\lambda_k\left(\lambda_i^4\lambda_j+\lambda_i\lambda_j^4\right)\\
&\quad +\left(18\beta^2-72\right)\left(\lambda_i^4\lambda_j^2+\lambda_i^2\lambda_j^4\right)\\
&\quad +\left(12\beta^3-108\beta^2+312\beta-288\right)\lambda_i^3\lambda_j^3,
\end{aligned}$$

$$\begin{aligned}
\zeta_{20}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (576\beta-1152)\,\lambda_k\left(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j\right)\\
&\quad+(648\beta-1296)\,\lambda_k\lambda_i^2\lambda_j^2\\
&\quad+2160\lambda_k\left(\lambda_i^4+\lambda_j^4\right)\\
&\quad+\left(108\beta^2-648\beta+864\right)\left(\lambda_i^3\lambda_j^2+\lambda_i^2\lambda_j^3\right)\\
&\quad+(288\beta-576)\left(\lambda_i^4\lambda_j+\lambda_i\lambda_j^4\right),\\
\zeta_{20}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (576\beta-1152)\,\lambda_k\left(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j\right)\\
&\quad+(648\beta-1296)\,\lambda_i^2\lambda_j^2\\
&\quad+2160\left(\lambda_i^4+\lambda_j^4\right).
\end{aligned}$$

Define $\zeta_{204}(\lambda_j;\lambda_i;\beta)=\zeta_{20}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{204} is non-negative for all $\lambda_j\geq\lambda_i$. Differentiating ζ_{204} with respect to λ_j gives

$$\begin{aligned}
\zeta_{204}(\lambda_j;\lambda_i;\beta) &= (576\beta-1152)\,\lambda_k\left(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j\right)\\
&\quad(648\beta-1296)\,\lambda_i^2\lambda_j^2\\
&\quad 2160\left(\lambda_i^4+\lambda_j^4\right),\\
\zeta_{204}^{(1)}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^3+(-3456\lambda_i+1728\lambda_i\beta)\lambda_j^2+(1296\lambda_i^2\beta-2592\lambda_i^2)\lambda_j+(576\beta-1152)\lambda_i^3,\\
\zeta_{204}^{(2)}(\lambda_j;\lambda_i;\beta) &= 25920\lambda_j^2+(-6912\lambda_i+3456\lambda_i\beta)\lambda_j+1296\lambda_i^2\beta-2592\lambda_i^2,\\
\zeta_{204}^{(3)}(\lambda_j;\lambda_i;\beta) &= 51840\lambda_j-6912\lambda_i+3456\lambda_i\beta,\\
\zeta_{204}^{(4)}(\lambda_j;\lambda_i;\beta) &= 51840>0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{204}(\lambda_i;\lambda_i;\beta) &= (1800\beta+720)\lambda_i^4\geq 0 \text{ for all } \beta\geq-\frac{2}{5},\\
\zeta_{204}^{(1)}(\lambda_i;\lambda_i;\beta) &= (1440+3600\beta)\lambda_i^3\geq 0 \text{ for all } \beta\geq-\frac{2}{5},\\
\zeta_{204}^{(2)}(\lambda_i;\lambda_i;\beta) &= (16416+4752\beta)\lambda_i^2\geq 0 \text{ for all } \beta\geq-\frac{38}{11},\\
\zeta_{204}^{(3)}(\lambda_i;\lambda_i;\beta) &= (44928+3456\beta)\lambda_i\geq 0 \text{ for all } \beta\geq-13.
\end{aligned}$$

Hence, $\zeta_{20}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j\geq\lambda_i$ and $\beta\geq-\frac{2}{5}$.

Define $\zeta_{203}(\lambda_j;\lambda_i;\beta)=\zeta_{20}^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ζ_{203} is non-negative for all $\lambda_j\geq\lambda_i$. Differentiating ζ_{203} with respect to λ_j gives

$$\begin{aligned}
\zeta_{203}(\lambda_j;\lambda_i;\beta) &= 2160\lambda_j^4+(864\beta-1728)\lambda_j^3\lambda_i+(108\beta^2-432)\lambda_j^2\lambda_i^2\\
&\quad+(-72\beta+108\beta^2-288)\lambda_j\lambda_i^3+(1584+288\beta)\lambda_i^4,\\
\zeta_{203}^{(1)}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^3+(2592\beta-5184)\lambda_j^2\lambda_i+(-864+216\beta^2)\lambda_j\lambda_i^2\\
&\quad+(-72\beta+108\beta^2-288)\lambda_i^3,\\
\zeta_{203}^{(2)}(\lambda_j;\lambda_i;\beta) &= 25920\lambda_j^2+(5184\beta-10368)\lambda_j\lambda_i+(-864+216\beta^2)\lambda_i^2,\\
\zeta_{203}^{(3)}(\lambda_j;\lambda_i;\beta) &= 51840\lambda_j+5184\lambda_i\beta-10368\lambda_i,\\
\zeta_{203}^{(4)}(\lambda_j;\lambda_i;\beta) &= 51840>0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{203}(\lambda_i;\lambda_i;\beta) &= (1296+1080\beta+216\beta^2)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2,\\
\zeta_{203}^{(1)}(\lambda_i;\lambda_i;\beta) &= (2304+2520\beta+324\beta^2)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -1.05,\\
\zeta_{203}^{(2)}(\lambda_i;\lambda_i;\beta) &= (14688+5184\beta+216\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3.28,\\
\zeta_{203}^{(3)}(\lambda_i;\lambda_i;\beta) &= (41472+5184\beta)\lambda_i \geq 0 \text{ for all } \beta \geq -8.
\end{aligned}$$

Hence, $\zeta_{20}^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{202}(\lambda_j;\lambda_i;\beta) = \zeta_{20}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ζ_{202} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{202} with respect to λ_j gives

$$\begin{aligned}
\zeta_{202}(\lambda_j;\lambda_i;\beta) &= 1080\lambda_j^4+(576\beta-1152)\lambda_j^3\lambda_i+(144-324\beta+126\beta^2)\lambda_j^2\lambda_i^2\\
&\quad +(12\beta^3-48\beta)\lambda_j\lambda_i^3+(288\beta+432+18\beta^2)\lambda_i^4,\\
\zeta_{202}^{(1)}(\lambda_j;\lambda_i;\beta) &= 4320\lambda_j^3+(1728\beta-3456)\lambda_j^2\lambda_i+(-648\beta+252\beta^2+288)\lambda_j\lambda_i^2\\
&\quad +(12\beta^3-48\beta)\lambda_i^3,\\
\zeta_{202}^{(2)}(\lambda_j;\lambda_i;\beta) &= 12960\lambda_j^2+(-6912+3456\beta)\lambda_j\lambda_i+(-648\beta+252\beta^2+288)\lambda_i^2,\\
\zeta_{202}^{(3)}(\lambda_j;\lambda_i;\beta) &= 25920\lambda_j-6912\lambda_i+3456\lambda_i\beta,\\
\zeta_{202}^{(4)}(\lambda_j;\lambda_i;\beta) &= 25920 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{202}(\lambda_i;\lambda_i;\beta) &= (504+492\beta+144\beta^2+12\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2,\\
\zeta_{202}^{(1)}(\lambda_i;\lambda_i;\beta) &= (1152+1032\beta+252\beta^2+12\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2,\\
\zeta_{202}^{(2)}(\lambda_i;\lambda_i;\beta) &= (6336+2808\beta+252\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3.14,\\
\zeta_{202}^{(3)}(\lambda_i;\lambda_i;\beta) &= (19008+3456\beta)\lambda_i \geq 0 \text{ for all } \beta \geq -5.5.
\end{aligned}$$

Hence, $\zeta_{20}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{201}(\lambda_j;\lambda_i;\beta) = \zeta_{20}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ζ_{201} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{201} with respect to λ_j gives

$$\begin{aligned}
\zeta_{201}(\lambda_j;\lambda_i;\beta) &= 360\lambda_j^4+(-480+240\beta)\lambda_j^3\lambda_i+(72\beta^2-216\beta+144)\lambda_j^2\lambda_i^2\\
&\quad +(-36\beta^2+24\beta+12\beta^3)\lambda_j\lambda_i^3+(84\beta+36\beta^2+48)\lambda_i^4,\\
\zeta_{201}^{(1)}(\lambda_j;\lambda_i;\beta) &= 1440\lambda_j^3+(-1440+720\beta)\lambda_j^2\lambda_i+(288-432\beta+144\beta^2)\lambda_j\lambda_i^2\\
&\quad +(-36\beta^2+24\beta+12\beta^3)\lambda_i^3,\\
\zeta_{201}^{(2)}(\lambda_j;\lambda_i;\beta) &= 4320\lambda_j^2+(-2880+1440\beta)\lambda_j\lambda_i+(288-432\beta+144\beta^2)\lambda_i^2,\\
\zeta_{201}^{(3)}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j-2880\lambda_i+1440\lambda_i\beta,\\
\zeta_{201}^{(4)}(\lambda_j;\lambda_i;\beta) &= 8640 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{201}(\lambda_i;\lambda_i;\beta) &= (72+132\beta+72\beta^2+12\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -1,\\
\zeta_{201}^{(1)}(\lambda_i;\lambda_i;\beta) &= (288+312\beta+108\beta^2+12\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2,\\
\zeta_{201}^{(2)}(\lambda_i;\lambda_i;\beta) &= (1728+1008\beta+144\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3,\\
\zeta_{201}^{(3)}(\lambda_i;\lambda_i;\beta) &= (5760+1440\beta)\lambda_i \geq 0 \text{ for all } \beta \geq -4.
\end{aligned}$$

Hence, $\zeta_{20}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{200}(\lambda_j;\lambda_i;\beta) = \zeta_{20}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ζ_{200} is positive for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{200} with respect to λ_j gives

$$\begin{aligned}
\zeta_{200}(\lambda_j;\lambda_i;\beta) &= 90\lambda_j^4 + (-144\lambda_i + 72\lambda_i\beta)\lambda_j^3 + (54\lambda_i^2 + 27\lambda_i^2\beta^2 - 81\lambda_i^2\beta)\lambda_j^2\\
&\quad + (-18\lambda_i^3\beta^2 + 6\lambda_i^3\beta^3 + 12\lambda_i^3\beta)\lambda_j + 36\lambda_i^4 + 27\lambda_i^4\beta^2 + 63\lambda_i^4\beta,\\
\zeta_{200}^{(1)}(\lambda_j;\lambda_i;\beta) &= 360\lambda_j^3 + (-432\lambda_i + 216\lambda_i\beta)\lambda_j^2 + (-162\lambda_i^2\beta + 54\lambda_i^2\beta^2 + 108\lambda_i^2)\lambda_j\\
&\quad -18\lambda_i^3\beta^2 + 6\lambda_i^3\beta^3 + 12\lambda_i^3\beta,\\
\zeta_{200}^{(2)}(\lambda_j;\lambda_i;\beta) &= 1080\lambda_j^2 + (432\lambda_i\beta - 864\lambda_i)\lambda_j - 162\lambda_i^2\beta + 54\lambda_i^2\beta^2 + 108\lambda_i^2,\\
\zeta_{200}^{(3)}(\lambda_j;\lambda_i;\beta) &= 2160\lambda_j + 432\lambda_i\beta - 864\lambda_i,\\
\zeta_{200}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2160 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{200}(\lambda_i;\lambda_i;\beta) &= (36+66\beta+36\beta^2+6\beta^3)\lambda_i^4 > 0 \text{ for all } \beta > -1,\\
\zeta_{200}^{(1)}(\lambda_i;\lambda_i;\beta) &= (36+66\beta+36\beta^2+6\beta^3)\lambda_i^3 > 0 \text{ for all } \beta > -1,\\
\zeta_{200}^{(2)}(\lambda_i;\lambda_i;\beta) &= (324+270\beta+54\beta^2)\lambda_i^2 > 0 \text{ for all } \beta > -2,\\
\zeta_{200}^{(3)}(\lambda_i;\lambda_i;\beta) &= (1296+432\beta)\lambda_i > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, $\zeta_{20}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{21}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{21}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (312+24\beta)\,\lambda_k^4\left(\lambda_j^3+\lambda_i^3\right)+126\,(\beta-2)\,\lambda_k^4\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right)\\
&\quad +24\,(2\beta+11)\,\lambda_k^3\left(\lambda_j^4+\lambda_i^4\right)+36\left(\beta^2+\beta-8\right)\lambda_k^3\left(\lambda_j^3\lambda_i+\lambda_j\lambda_i^3\right)\\
&\quad +108\left(\beta^2-5\beta+6\right)\lambda_k^3\lambda_j^2\lambda_i^2+18\left(\beta^2+8\beta-20\right)\lambda_k^2\left(\lambda_j^4\lambda_i+\lambda_j\lambda_i^4\right)\\
&\quad +18\left(\beta^3-4\beta^2+8\beta-8\right)\lambda_k^2\left(\lambda_j^3\lambda_i^2+\lambda_j^2\lambda_i^3\right)\\
&\quad +36\left(2\beta^2-5\beta+2\right)\lambda_k\left(\lambda_j^4\lambda_i^2+\lambda_j^2\lambda_i^4\right)\\
&\quad +12\left(\beta^3+3\beta^2-14\beta+8\right)\lambda_k\lambda_j^3\lambda_i^3+6\left(3\beta^2+40\beta+28\right)\lambda_k\lambda_j^3\lambda_i^3,\\
\zeta_{21}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (96\beta+1248)\lambda_k^3\left(\lambda_j^3+\lambda_i^3\right)+(504\beta-1008)\lambda_k^3\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right)\\
&\quad +(48\beta+264)\lambda_k^2\left(\lambda_j^4+\lambda_i^4\right)+(-288\lambda_i+36\lambda_i\beta^2+72\lambda_i\beta)\lambda_k^2\left(\lambda_j^3\lambda_i+\lambda_j\lambda_i^3\right)\\
&\quad +(648-540\beta+108\beta^2)\lambda_k^2\lambda_j^2\lambda_i^2\\
&\quad +(36\lambda_i\beta^2+288\lambda_i\beta-720\lambda_i)\lambda_k\left(\lambda_j^4\lambda_i+\lambda_j\lambda_i^4\right)\\
&\quad +(36\beta^3+288\beta-144\beta^2-288)\lambda_k\left(\lambda_j^3\lambda_i^2+\lambda_j^2\lambda_i^3\right)\\
&\quad +(72\beta^2+72-180\beta)\left(\lambda_j^4\lambda_i^2+\lambda_j^2\lambda_i^4\right)+(36\beta^2-168\beta+96+12\beta^3)\lambda_j^3\lambda_i^3,\\
\zeta_{21}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (3744+288\beta)\lambda_k^2\left(\lambda_j^3+\lambda_i^3\right)+(1512\beta-3024)\lambda_k^2\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right)\\
&\quad +(288\beta+1584)\lambda_k\left(\lambda_j^4+\lambda_i^4\right)+(216\beta^2-1728+432\beta)\lambda_k\left(\lambda_j^3\lambda_i+\lambda_j\lambda_i^3\right)\\
&\quad +(-3240\beta+648\beta^2+3888)\lambda_k\lambda_j^2\lambda_i^2+(36\beta^2+288\beta-720)\left(\lambda_j^4\lambda_i+\lambda_j\lambda_i^4\right)\\
&\quad +(36\beta^3+288\beta-144\beta^2-288)\left(\lambda_j^3\lambda_i^2+\lambda_j^2\lambda_i^3\right),\\
\zeta_{21}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (7488+576\beta)\lambda_k\left(\lambda_j^3+\lambda_i^3\right)+(3024\beta-6048)\lambda_k\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right)\\
&\quad +(288\beta+1584)\left(\lambda_j^4+\lambda_i^4\right)+(216\beta^2-1728+432\beta)\left(\lambda_j^3\lambda_i+\lambda_j\lambda_i^3\right)\\
&\quad +(-3240\beta+648\beta^2+3888)\lambda_j^2\lambda_i^2,\\
\zeta_{21}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (7488+576\beta)\left(\lambda_j^3+\lambda_i^3\right)+(3024\beta-6048)\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right).
\end{aligned}$$

Define $\zeta_{214}(\lambda_j;\lambda_i;\beta)=\zeta_{21}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{214} is non-negative for all $\lambda_j\ge\lambda_i$. Differentiating ζ_{214} with respect to λ_j gives

$$\begin{aligned}
\zeta_{214}(\lambda_j;\lambda_i;\beta) &= (7488+576\beta)\left(\lambda_j^3+\lambda_i^3\right)+(3024\beta-6048)\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right),\\
\zeta_{214}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1728\beta+22464)\lambda_j^2+(6048\lambda_i\beta-12096\lambda_i)\lambda_j-6048\lambda_i^2+3024\lambda_i^2\beta,\\
\zeta_{214}^{(2)}(\lambda_j;\lambda_i;\beta) &= (3456\beta+44928)\lambda_j+6048\lambda_i\beta-12096\lambda_i,\\
\zeta_{214}^{(3)}(\lambda_j;\lambda_i;\beta) &= 3456\beta+44928\ge 0 \text{ for all } \beta\ge -13.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{214}(\lambda_i;\lambda_i;\beta) &= (7200\beta+2880)\lambda_i^3\ge 0 \text{ for all } \beta\ge-\frac{2}{5},\\
\zeta_{214}^{(1)}(\lambda_i;\lambda_i;\beta) &= (10800\beta+4320)\lambda_i^2\ge 0 \text{ for all } \beta\ge-\frac{2}{5},\\
\zeta_{214}^{(2)}(\lambda_i;\lambda_i;\beta) &= (9504\beta+32832)\lambda_i\ge 0 \text{ for all } \beta\ge-\frac{38}{11}.
\end{aligned}$$

Hence, $\zeta_{21}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{213}(\lambda_j;\lambda_i;\beta) = \zeta_{21}^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{213} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{213} with respect to λ_j gives

$$\begin{aligned}
\zeta_{213}(\lambda_j;\lambda_i;\beta) &= (864\beta+9072)\lambda_j^4 + (216\lambda_i\beta^2+3456\lambda_i\beta-7776\lambda_i)\lambda_j^3 \\
&\quad +(-216\lambda_i^2\beta-2160\lambda_i^2+648\lambda_i^2\beta^2)\lambda_j^2 \\
&\quad +(5760\lambda_i^3+1008\lambda_i^3\beta+216\lambda_i^3\beta^2)\lambda_j+1584\lambda_i^4+288\lambda_i^4\beta, \\
\zeta_{213}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3456\beta+36288)\lambda_j^3+(-23328\lambda_i+10368\lambda_i\beta+648\lambda_i\beta^2)\lambda_j^2 \\
&\quad +(1296\lambda_i^2\beta^2-432\lambda_i^2\beta-4320\lambda_i^2)\lambda_j+5760\lambda_i^3+1008\lambda_i^3\beta+216\lambda_i^3\beta^2, \\
\zeta_{213}^{(2)}(\lambda_j;\lambda_i;\beta) &= (10368\beta+108864)\lambda_j^2+(1296\lambda_i\beta^2-46656\lambda_i+20736\lambda_i\beta)\lambda_j \\
&\quad +1296\lambda_i^2\beta^2-432\lambda_i^2\beta-4320\lambda_i^2, \\
\zeta_{213}^{(3)}(\lambda_j;\lambda_i;\beta) &= (217728+20736\beta)\lambda_j+1296\lambda_i\beta^2-46656\lambda_i+20736\lambda_i\beta, \\
\zeta_{213}^{(4)}(\lambda_j;\lambda_i;\beta) &= 217728+20736\beta \geq 0 \text{ for all } \beta \geq -10.5.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{213}(\lambda_i;\lambda_i;\beta) &= (5400\beta+6480+1080\beta^2)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{213}^{(1)}(\lambda_i;\lambda_i;\beta) &= (14400\beta+14400+2160\beta^2)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -1.22, \\
\zeta_{213}^{(2)}(\lambda_i;\lambda_i;\beta) &= (30672\beta+57888+2592\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -2.35, \\
\zeta_{213}^{(3)}(\lambda_i;\lambda_i;\beta) &= (171072+41472\beta+1296\beta^2)\lambda_i \geq 0 \text{ for all } \beta \geq -4.86.
\end{aligned}$$

Hence, $\zeta_{21}^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{212}(\lambda_j;\lambda_i;\beta) = \zeta_{21}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ζ_{212} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{212} with respect to λ_j gives

$$\begin{aligned}
\zeta_{212}(\lambda_j;\lambda_i;\beta) &= (5328+576\beta)\lambda_j^4+(2232\lambda_i\beta+252\lambda_i\beta^2-5472\lambda_i)\lambda_j^3 \\
&\quad +(504\lambda_i^2\beta^2+36\lambda_i^2\beta^3-1440\lambda_i^2\beta+576\lambda_i^2)\lambda_j^2 \\
&\quad +(1008\lambda_i^3\beta+1728\lambda_i^3+36\lambda_i^3\beta^3+72\lambda_i^3\beta^2)\lambda_j \\
&\quad +576\lambda_i^4\beta+36\lambda_i^4\beta^2+864\lambda_i^4, \\
\zeta_{212}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2304\beta+21312)\lambda_j^3+(-16416\lambda_i+6696\lambda_i\beta+756\lambda_i\beta^2)\lambda_j^2 \\
&\quad +(1152\lambda_i^2+1008\lambda_i^2\beta^2+72\lambda_i^2\beta^3-2880\lambda_i^2\beta)\lambda_j \\
&\quad +1008\lambda_i^3\beta+1728\lambda_i^3+36\lambda_i^3\beta^3+72\lambda_i^3\beta^2, \\
\zeta_{212}^{(2)}(\lambda_j;\lambda_i;\beta) &= (63936+6912\beta)\lambda_j^2+(1512\lambda_i\beta^2-32832\lambda_i+13392\lambda_i\beta)\lambda_j \\
&\quad +1152\lambda_i^2+1008\lambda_i^2\beta^2+72\lambda_i^2\beta^3-2880\lambda_i^2\beta, \\
\zeta_{212}^{(3)}(\lambda_j;\lambda_i;\beta) &= (13824\beta+127872)\lambda_j+1512\lambda_i\beta^2-32832\lambda_i+13392\lambda_i\beta, \\
\zeta_{212}^{(4)}(\lambda_j;\lambda_i;\beta) &= (13824\beta+127872) \geq 0 \text{ for all } \beta \geq -9.25.
\end{aligned}$$

Now

$$
\begin{aligned}
\zeta_{212}(\lambda_i;\lambda_i;\beta) &= (3024+2952\beta+864\beta^2+72\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2,\\
\zeta_{212}^{(1)}(\lambda_i;\lambda_i;\beta) &= (7128\beta+7776+1836\beta^2+108\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2,\\
\zeta_{212}^{(2)}(\lambda_i;\lambda_i;\beta) &= (32256+17424\beta+2520\beta^2+72\beta^3)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3.18,\\
\zeta_{212}^{(3)}(\lambda_i;\lambda_i;\beta) &= (27216\beta+95040+1512\beta^2)\lambda_i \geq 0 \text{ for all } \beta \geq -4.74.
\end{aligned}
$$

Hence, $\zeta_{21}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{211}(\lambda_j;\lambda_i;\beta) = \zeta_{21}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ζ_{211} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{211} with respect to λ_j gives

$$
\begin{aligned}
\zeta_{211}(\lambda_j;\lambda_i;\beta) &= (2040+240\beta)\lambda_j^4+(144\lambda_i\beta^2-2592\lambda_i+1008\lambda_i\beta)\lambda_j^3\\
&\quad +(720\lambda_i^2-1008\lambda_i^2\beta+36\lambda_i^2\beta^3+252\lambda_i^2\beta^2)\lambda_j^2\\
&\quad +(192\lambda_i^3+48\lambda_i^3\beta^3+432\lambda_i^3\beta)\lambda_j+144\lambda_i^4+252\lambda_i^4\beta+108\lambda_i^4\beta^2,\\
\zeta_{211}^{(1)}(\lambda_j;\lambda_i;\beta) &= (8160+960\beta)\lambda_j^3+(3024\lambda_i\beta+432\lambda_i\beta^2-7776\lambda_i)\lambda_j^2\\
&\quad +(504\lambda_i^2\beta^2+1440\lambda_i^2-2016\lambda_i^2\beta+72\lambda_i^2\beta^3)\lambda_j\\
&\quad +192\lambda_i^3+48\lambda_i^3\beta^3+432\lambda_i^3\beta,\\
\zeta_{211}^{(2)}(\lambda_j;\lambda_i;\beta) &= (2880\beta+24480)\lambda_j^2+(-15552\lambda_i+6048\lambda_i\beta+864\lambda_i\beta^2)\lambda_j\\
&\quad +504\lambda_i^2\beta^2+1440\lambda_i^2-2016\lambda_i^2\beta+72\lambda_i^2\beta^3,\\
\zeta_{211}^{(3)}(\lambda_j;\lambda_i;\beta) &= (48960+5760\beta)\lambda_j-15552\lambda_i+6048\lambda_i\beta+864\lambda_i\beta^2,\\
\zeta_{211}^{(4)}(\lambda_j;\lambda_i;\beta) &= 48960+5760\beta \geq 0 \text{ for all } \beta \geq -8.5.
\end{aligned}
$$

Now

$$
\begin{aligned}
\zeta_{211}(\lambda_i;\lambda_i;\beta) &= (504+924\beta+504\beta^2+84\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -1,\\
\zeta_{211}^{(1)}(\lambda_i;\lambda_i;\beta) &= (2016+2400\beta+936\beta^2+120\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2,\\
\zeta_{211}^{(2)}(\lambda_i;\lambda_i;\beta) &= (6912\beta+10368+1368\beta^2+72\beta^3)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3,\\
\zeta_{211}^{(3)}(\lambda_i;\lambda_i;\beta) &= (33408+11808\beta+864\beta^2)\lambda_i \geq 0 \text{ for all } \beta \geq -4.
\end{aligned}
$$

Hence, $\zeta_{21}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{210}(\lambda_j;\lambda_i;\beta) = \zeta_{21}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ζ_{210} is positive for all

$\lambda_j \geq \lambda_i$. Differentiating ζ_{210} with respect to λ_j gives

$$\begin{aligned}
\zeta_{210}(\lambda_j;\lambda_i;\beta) &= (576+72\beta)\lambda_j^4 + (-900\lambda_i + 54\lambda_i\beta^2 + 342\lambda_i\beta)\lambda_j^3 \\
&\quad +(-450\lambda_i^2\beta + 18\lambda_i^2\beta^3 + 324\lambda_i^2 + 108\lambda_i^2\beta^2)\lambda_j^2 \\
&\quad +(30\lambda_i^3\beta^3 + 312\lambda_i^3\beta + 18\lambda_i^3\beta^2 + 144\lambda_i^3)\lambda_j \\
&\quad +108\lambda_i^4\beta^2 + 252\lambda_i^4\beta + 144\lambda_i^4, \\
\zeta_{210}^{(1)}(\lambda_j;\lambda_i;\beta) &= (288\beta + 2304)\lambda_j^3 + (1026\lambda_i\beta - 2700\lambda_i + 162\lambda_i\beta^2)\lambda_j^2 \\
&\quad +(216\lambda_i^2\beta^2 - 900\lambda_i^2\beta + 36\lambda_i^2\beta^3 + 648\lambda_i^2)\lambda_j \\
&\quad +30\lambda_i^3\beta^3 + 312\lambda_i^3\beta + 18\lambda_i^3\beta^2 + 144\lambda_i^3, \\
\zeta_{210}^{(2)}(\lambda_j;\lambda_i;\beta) &= (6912 + 864\beta)\lambda_j^2 + (324\lambda_i\beta^2 + 2052\lambda_i\beta - 5400\lambda_i)\lambda_j \\
&\quad +216\lambda_i^2\beta^2 - 900\lambda_i^2\beta + 36\lambda_i^2\beta^3 + 648\lambda_i^2, \\
\zeta_{210}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1728\beta + 13824)\lambda_j + 324\lambda_i\beta^2 + 2052\lambda_i\beta - 5400\lambda_i, \\
\zeta_{210}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1728\beta + 13824 \geq 0 \text{ for all } \beta \geq -8.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{210}(\lambda_i;\lambda_i;\beta) &= (288 + 528\beta + 288\beta^2 + 48\beta^3)\lambda_i^4 > 0 \text{ for all } \beta > -1, \\
\zeta_{210}^{(1)}(\lambda_i;\lambda_i;\beta) &= (726\beta + 396 + 396\beta^2 + 66\beta^3)\lambda_i^3 > 0 \text{ for all } \beta > -1, \\
\zeta_{210}^{(2)}(\lambda_i;\lambda_i;\beta) &= (2160 + 2016\beta + 540\beta^2 + 36\beta^3)\lambda_i^2 > 0 \text{ for all } \beta > -2, \\
\zeta_{210}^{(3)}(\lambda_i;\lambda_i;\beta) &= (3780\beta + 8424 + 324\beta^2)\lambda_i > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, $\zeta_{21}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(1)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{22}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{22}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (198\beta + 684)\,\lambda_k^4\left(\lambda_j^2 + \lambda_i^2\right) \\
&\quad +504\,(\beta - 2)\,\lambda_k^4\lambda_j\lambda_i \\
&\quad +(2016 + 36\beta^2 + 360\beta)\lambda_k^3\left(\lambda_j^3 + \lambda_i^3\right) \\
&\quad +(-576 + 324\beta^2 - 360\beta)\lambda_k^3\left(\lambda_j^2\lambda_i + \lambda_j\lambda_i^2\right) \\
&\quad +(432 + 18\beta^2 + 288\beta)\lambda_k^2\left(\lambda_j^4 + \lambda_i^4\right) \\
&\quad +(36\beta^2 + 1080\beta + 36\beta^3 - 2592)(\lambda_k^2\lambda_j^3\lambda_i + \lambda_k^2\lambda_j\lambda_i^3) \\
&\quad +(-756\beta - 108\beta^2 + 1080 + 108\beta^3)\lambda_k^2\lambda_j^2\lambda_i^2 \\
&\quad +(-72\beta + 180\beta^2 - 576)\lambda_k\left(\lambda_j^4\lambda_i + \lambda_j\lambda_i^4\right) \\
&\quad +(72\beta^3 + 252\beta^2 + 288 - 936\beta)(\lambda_k\lambda_j^3\lambda_i^2 + \lambda_k\lambda_j^2\lambda_i^3) \\
&\quad +(126\beta^2 + 576 + 540\beta)\left(\lambda_j^4\lambda_i^2 + \lambda_j^2\lambda_i^4\right) \\
&\quad +(180\beta^2 + 1752\beta + 12\beta^3 + 1440)\lambda_j^3\lambda_i^3,
\end{aligned}$$

$$\begin{aligned}
\zeta_{22}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (792\beta+2736)\lambda_k^3\left(\lambda_j^2+\lambda_i^2\right)+(-4032+2016\beta)\lambda_k^3\lambda_j\lambda_i \\
&\quad +(108\beta^2+6048+1080\beta)\lambda_k^2\left(\lambda_j^3+\lambda_i^3\right) \\
&\quad +(972\beta^2-1080\beta-1728)\lambda_k^2\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right) \\
&\quad +(864+36\beta^2+576\beta)\lambda_k\left(\lambda_j^4+\lambda_i^4\right) \\
&\quad +(-5184+2160\beta+72\beta^2+72\beta^3)\lambda_k\left(\lambda_j^3\lambda_i+\lambda_j\lambda_i^3\right) \\
&\quad +(2160-1512\beta-216\beta^2+216\beta^3)\lambda_k\lambda_j^2\lambda_i^2 \\
&\quad +(-72\beta+180\beta^2-576)\left(\lambda_j^4\lambda_i+\lambda_j\lambda_i^4\right) \\
&\quad +(72\beta^3+252\beta^2+288-936\beta)\left(\lambda_j^3\lambda_i^2+\lambda_j^2\lambda_i^3\right), \\
\zeta_{22}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (2376\beta+8208)\lambda_k^2\left(\lambda_j^2+\lambda_i^2\right)+(-12096+6048\beta)\lambda_k^2\lambda_j\lambda_i \\
&\quad +(216\beta^2+12096+2160\beta)\lambda_k\left(\lambda_j^3+\lambda_i^3\right) \\
&\quad +(-3456-2160\beta+1944\beta^2)\lambda_k\left(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2\right) \\
&\quad +(864+36\beta^2+576\beta)\left(\lambda_j^4+\lambda_i^4\right) \\
&\quad +(-5184+2160\beta+72\beta^2+72\beta^3)\left(\lambda_j^3\lambda_i+\lambda_j\lambda_i^3\right) \\
&\quad +(2160-1512\beta-216\beta^2+216\beta^3)\lambda_j^2\lambda_i^2, \\
\zeta_{22}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (16416+4752\beta)\lambda_k\left(\lambda_j^2+\lambda_i^2\right)+(-24192+12096\beta)\lambda_k\lambda_j\lambda_i \\
&\quad +(216\beta^2+12096+2160\beta)\left(\lambda_j^3+\lambda_i^3\right) \\
&\quad +(-3456-2160\beta+1944\beta^2)\left(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j\right), \\
\zeta_{22}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (16416+4752\beta)\left(\lambda_j^2+\lambda_i^2\right)+(-24192+12096\beta)\lambda_i\lambda_j.
\end{aligned}$$

Define $\zeta_{224}(\lambda_j;\lambda_i;\beta)=\zeta_{22}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{224} is non-negative for all $\lambda_j\geq\lambda_i$. Differentiating ζ_{224} with respect to λ_j gives

$$\begin{aligned}
\zeta_{224}(\lambda_j;\lambda_i;\beta) &= (16416+4752\beta)\left(\lambda_j^2+\lambda_i^2\right)+(-24192+12096\beta)\lambda_i\lambda_j, \\
\zeta_{224}^{(1)}(\lambda_j;\lambda_i;\beta) &= (32832+9504\beta)\lambda_j+(-24192+12096\beta)\lambda_i, \\
\zeta_{224}^{(2)}(\lambda_j;\lambda_i;\beta) &= 32832+9504\beta\geq 0 \text{ for all } \beta\geq-\frac{38}{11}.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{224}(\lambda_i;\lambda_i;\beta) &= (8640+21600\beta)\lambda_i^2\geq 0 \text{ for all } \beta\geq-\frac{2}{5}, \\
\zeta_{224}^{(1)}(\lambda_i;\lambda_i;\beta) &= (8640+21600\beta)\lambda_i\geq 0 \text{ for all } \beta\geq-\frac{2}{5}.
\end{aligned}$$

Hence, $\zeta_{22}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j\geq\lambda_i$ and $\beta\geq-\frac{2}{5}$.

Define $\zeta_{223}(\lambda_j;\lambda_i;\beta)=\zeta_{22}^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{223} is non-negative for

all $\lambda_j \geq \lambda_i$. Differentiating ζ_{223} with respect to λ_j gives

$$\begin{aligned}
\zeta_{223}(\lambda_j;\lambda_i;\beta) &= (28512+216\beta^2+6912\beta)\lambda_j^3+(9936\beta+1944\beta^2-27648)\lambda_i\lambda_j^2 \\
&\quad +(12960+2592\beta+1944\beta^2)\lambda_i^2\lambda_j+(216\beta^2+12096+2160\beta)\lambda_i^3, \\
\zeta_{223}^{(1)}(\lambda_j;\lambda_i;\beta) &= (85536+648\beta^2+20736\beta)\lambda_j^2+(19872\beta+3888\beta^2-55296)\lambda_i\lambda_j \\
&\quad +(12960+2592\beta+1944\beta^2)\lambda_i^2, \\
\zeta_{223}^{(2)}(\lambda_j;\lambda_i;\beta) &= (171072+41472\beta+1296\beta^2)\lambda_j+(19872\beta+3888\beta^2-55296)\lambda_i, \\
\zeta_{223}^{(3)}(\lambda_j;\lambda_i;\beta) &= 171072+41472\beta+1296\beta^2 \geq 0 \text{ for all } \beta \geq -4.86.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{223}(\lambda_i;\lambda_i;\beta) &= (25920+4320\beta^2+21600\beta)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{223}^{(1)}(\lambda_i;\lambda_i;\beta) &= (43200+6480\beta^2+43200\beta)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -1.22, \\
\zeta_{223}^{(2)}(\lambda_i;\lambda_i;\beta) &= (115776+61344\beta+5184\beta^2)\lambda_i \geq 0 \text{ for all } \beta \geq -2.35.
\end{aligned}$$

Hence, $\zeta_{22}^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{222}(\lambda_j;\lambda_i;\beta)=\zeta_{22}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{222} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{222} with respect to λ_j gives

$$\begin{aligned}
\zeta_{222}(\lambda_j;\lambda_i;\beta) &= (5112\beta+252\beta^2+21168)\lambda_j^4+(6048\beta+72\beta^3-20736+2016\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(216\beta^3+1728\beta^2-1296\beta+6912)\lambda_i^2\lambda_j^2 \\
&\quad +(6912+72\beta^3+4320\beta+288\beta^2)\lambda_i^3\lambda_j \\
&\quad +(864+36\beta^2+576\beta)\lambda_i^4, \\
\zeta_{222}^{(1)}(\lambda_j;\lambda_i;\beta) &= (20448\beta+1008\beta^2+84672)\lambda_j^3+(18144\beta+216\beta^3-62208+6048\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(432\beta^3+3456\beta^2-2592\beta+13824)\lambda_i^2\lambda_j+(6912+72\beta^3+4320\beta+288\beta^2)\lambda_i^3, \\
\zeta_{222}^{(2)}(\lambda_j;\lambda_i;\beta) &= (61344\beta+3024\beta^2+254016)\lambda_j^2+(36288\beta+432\beta^3-124416+12096\beta^2)\lambda_i\lambda_j \\
&\quad +(432\beta^3+3456\beta^2-2592\beta+13824)\lambda_i^2, \\
\zeta_{222}^{(3)}(\lambda_j;\lambda_i;\beta) &= (122688\beta+6048\beta^2+508032)\lambda_j+(36288\beta+432\beta^3-124416+12096\beta^2)\lambda_i, \\
\zeta_{222}^{(4)}(\lambda_j;\lambda_i;\beta) &= 122688\beta+6048\beta^2+508032 \geq 0 \text{ for all } \beta \geq -5.79.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{222}(\lambda_i;\lambda_i;\beta) &= (15120+4320\beta^2+14760\beta+360\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{222}^{(1)}(\lambda_i;\lambda_i;\beta) &= (40320\beta+10800\beta^2+43200+720\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{222}^{(2)}(\lambda_i;\lambda_i;\beta) &= (95040\beta+18576\beta^2+143424+864\beta^3)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -14.8, \\
\zeta_{222}^{(3)}(\lambda_i;\lambda_i;\beta) &= (158976\beta+18144\beta^2+383616+432\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -4.34.
\end{aligned}$$

Hence, $\zeta_{22}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{221}(\lambda_j;\lambda_i;\beta) = \zeta_{22}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ζ_{221} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{221} with respect to λ_j gives

$$\begin{aligned}
\zeta_{221}(\lambda_j;\lambda_i;\beta) &= (9648+144\beta^2+2448\beta)\lambda_j^4 + (3024\beta+1224\beta^2+72\beta^3-11520)\lambda_i\lambda_j^3 \\
&\quad +(3456-2736\beta+1008\beta^2+288\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(2304\beta+144\beta^3+1152+432\beta^2)\lambda_i^3\lambda_j + (216\beta^2+504\beta+288)\lambda_i^4, \\
\zeta_{221}^{(1)}(\lambda_j;\lambda_i;\beta) &= (9792\beta+576\beta^2+38592)\lambda_j^3 + (9072\beta-34560+216\beta^3+3672\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(2016\beta^2+576\beta^3-5472\beta+6912)\lambda_i^2\lambda_j \\
&\quad +(2304\beta+144\beta^3+1152+432\beta^2)\lambda_i^3, \\
\zeta_{221}^{(2)}(\lambda_j;\lambda_i;\beta) &= (29376\beta+1728\beta^2+115776)\lambda_j^2 + (18144\beta-69120+432\beta^3+7344\beta^2)\lambda_i\lambda_j \\
&\quad +(2016\beta^2+576\beta^3-5472\beta+6912)\lambda_i^2, \\
\zeta_{221}^{(3)}(\lambda_j;\lambda_i;\beta) &= (58752\beta+3456\beta^2+231552)\lambda_j + (18144\beta-69120+432\beta^3+7344\beta^2)\lambda_i, \\
\zeta_{221}^{(4)}(\lambda_j;\lambda_i;\beta) &= 58752\beta+3456\beta^2+231552 \geq 0 \text{ for all } \beta \geq -6.20.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{221}(\lambda_i;\lambda_i;\beta) &= (3024\beta^2+5544\beta+3024+504\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -1, \\
\zeta_{221}^{(1)}(\lambda_i;\lambda_i;\beta) &= (15696\beta+6696\beta^2+12096+936\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{221}^{(2)}(\lambda_i;\lambda_i;\beta) &= (42048\beta+11088\beta^2+53568+1008\beta^3)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3, \\
\zeta_{221}^{(3)}(\lambda_i;\lambda_i;\beta) &= (76896\beta+10800\beta^2+162432+432\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -4.
\end{aligned}$$

Hence, $\zeta_{22}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{220}(\lambda_j;\lambda_i;\beta) = \zeta_{22}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ζ_{220} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{220} with respect to λ_j gives

$$\begin{aligned}
\zeta_{220}(\lambda_j;\lambda_i;\beta) &= (3132+54\beta^2+846\beta)\lambda_j^4 + (1152\beta-4752+36\beta^3+540\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(594\beta^2+180\beta^3-1314\beta+2052)\lambda_i^2\lambda_j^2 \\
&\quad +(2256\beta+504\beta^2+120\beta^3+1152)\lambda_i^3\lambda_j \\
&\quad +(324\beta^2+756\beta+432)\lambda_i^4, \\
\zeta_{220}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3384\beta+216\beta^2+12528)\lambda_j^3 + (3456\beta-14256+108\beta^3+1620\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(1188\beta^2+360\beta^3-2628\beta+4104)\lambda_i^2\lambda_j \\
&\quad +(2256\beta+504\beta^2+120\beta^3+1152)\lambda_i^3, \\
\zeta_{220}^{(2)}(\lambda_j;\lambda_i;\beta) &= (10152\beta+648\beta^2+37584)\lambda_j^2 + (6912\beta-28512+216\beta^3+3240\beta^2)\lambda_i\lambda_j \\
&\quad +(1188\beta^2+360\beta^3-2628\beta+4104)\lambda_i^2, \\
\zeta_{220}^{(3)}(\lambda_j;\lambda_i;\beta) &= (20304\beta+1296\beta^2+75168)\lambda_j + (6912\beta-28512+216\beta^3+3240\beta^2)\lambda_i, \\
\zeta_{220}^{(4)}(\lambda_j;\lambda_i;\beta) &= 20304\beta+1296\beta^2+75168 \geq 0 \text{ for all } \beta \geq -6.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{220}(\lambda_i;\lambda_i;\beta) &= (2016\beta^2+3696\beta+2016+336\beta^3)\lambda_i^4 > 0 \text{ for all } \beta > -1,\\
\zeta_{220}^{(1)}(\lambda_i;\lambda_i;\beta) &= (6468\beta+3528\beta^2+3528+588\beta^3)\lambda_i^3 > 0 \text{ for all } \beta > -1,\\
\zeta_{220}^{(2)}(\lambda_i;\lambda_i;\beta) &= (14436\beta+5076\beta^2+13176+576\beta^3)\lambda_i^2 > 0 \text{ for all } \beta > -2,\\
\zeta_{220}^{(3)}(\lambda_i;\lambda_i;\beta) &= (27216\beta+4536\beta^2+46656+216\beta^3)\lambda_i > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, $\zeta_{22}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(2)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{23}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{23}(\lambda_k;\lambda_j;\lambda_i;\beta) = {} & (900\beta+360)\,\lambda_k^4\left(\lambda_j+\lambda_i\right)\\
&+(432\beta^2+8208+1512\beta)\lambda_k^3\left(\lambda_j^2+\lambda_i^2\right)\\
&+(-6336+576\beta+1296\beta^2)\lambda_k^3\lambda_j\lambda_i\\
&+(3312\beta+5184+36\beta^3+216\beta^2)\lambda_k^2\lambda_j^3\\
&+(3312\beta+5184+36\beta^3+216\beta^2)\lambda_k^2\lambda_i^3\\
&+(-7344+864\beta^2+324\beta^3+648\beta)\lambda_k^2\lambda_j^2\lambda_i\\
&+(-7344+864\beta^2+324\beta^3+648\beta)\lambda_k^2\lambda_j\lambda_i^2\\
&+(216\beta^2+288+504\beta)\lambda_k\left(\lambda_j^4+\lambda_i^4\right)\\
&+(1296\beta^2+216\beta^3-6912)\lambda_k\left(\lambda_j^3\lambda_i+\lambda_j\lambda_i^3\right)\\
&+(648\beta^3-7128\beta+1296\beta^2+3888)\lambda_k\lambda_j^2\lambda_i^2\\
&+(576+1008\beta+432\beta^2)\left(\lambda_j^4\lambda_i+\lambda_j\lambda_i^4\right)\\
&+(1296\beta^2+6480\beta+6912+108\beta^3)\lambda_j^3\lambda_i^2\\
&+(1296\beta^2+6480\beta+6912+108\beta^3)\lambda_j^2\lambda_i^3,
\end{aligned}$$

$$
\begin{aligned}
\zeta_{23}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (3600\beta+1440)\lambda_k^3(\lambda_j+\lambda_i)+(24624+1296\beta^2+4536\beta)\lambda_k^2(\lambda_j^2+\lambda_i^2)\\
&\quad +(-19008+1728\beta+3888\beta^2)\lambda_k^2\lambda_j\lambda_i\\
&\quad +(1728\beta^2-14688+648\beta^3+1296\beta)\lambda_k(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2)\\
&\quad +(432\beta^2+10368+72\beta^3+6624\beta)\lambda_k(\lambda_j^3+\lambda_i^3)\\
&\quad +(216\beta^2+504\beta+288)(\lambda_j^4+\lambda_i^4)\\
&\quad +(1296\beta^2+216\beta^3-6912)(\lambda_j^3\lambda_i+\lambda_j\lambda_i^3)\\
&\quad +(648\beta^3-7128\beta+1296\beta^2+3888)\lambda_j^2\lambda_i^2,\\
\zeta_{23}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (10800\beta+4320)\lambda_k^2(\lambda_j+\lambda_i)+(7776\beta^2-38016+3456\beta)\lambda_k\lambda_i\lambda_j\\
&\quad +(9072\beta+49248+2592\beta^2)\lambda_k(\lambda_j^2+\lambda_i^2)\\
&\quad +(1728\beta^2-14688+648\beta^3+1296\beta)(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2)\\
&\quad +(432\beta^2+10368+72\beta^3+6624\beta)(\lambda_j^3+\lambda_i^3),\\
\zeta_{23}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (21600\beta+8640)\lambda_k(\lambda_j+\lambda_i)+(7776\beta^2-38016+3456\beta)\lambda_i\lambda_j\\
&\quad +(9072\beta+49248+2592\beta^2)(\lambda_j^2+\lambda_i^2),\\
\zeta_{23}^{(4)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (21600\beta+8640)(\lambda_j+\lambda_i)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}
$$

Define $\zeta_{233}(\lambda_j;\lambda_i;\beta)=\zeta_{23}^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{233} is non-negative for all $\lambda_j\geq\lambda_i$. Differentiating ζ_{233} with respect to λ_j gives

$$
\begin{aligned}
\zeta_{233}(\lambda_j;\lambda_i;\beta) &= (30672\beta+57888+2592\beta^2)\lambda_j^2+(25056\beta-29376+7776\beta^2)\lambda_i\lambda_j\\
&\quad +(9072\beta+49248+2592\beta^2)\lambda_i^2,\\
\zeta_{233}^{(1)}(\lambda_j;\lambda_i;\beta) &= (61344\beta+115776+5184\beta^2)\lambda_j+(25056\beta-29376+7776\beta^2)\lambda_i,\\
\zeta_{233}^{(2)}(\lambda_j;\lambda_i;\beta) &= 61344\beta+115776+5184\beta^2\geq 0 \text{ for all } \beta\geq -2.35.
\end{aligned}
$$

Now

$$
\begin{aligned}
\zeta_{233}(\lambda_i;\lambda_i;\beta) &= (64800\beta+77760+12960\beta^2)\lambda_i^2\geq 0 \text{ for all } \beta\geq -2,\\
\zeta_{233}^{(1)}(\lambda_i;\lambda_i;\beta) &= (86400\beta+86400+12960\beta^2)\lambda_i\geq 0 \text{ for all } \beta\geq -1.22.
\end{aligned}
$$

Hence, $\zeta_{23}^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j\geq\lambda_i$ and $\beta\geq-\frac{2}{5}$.

Define $\zeta_{232}(\lambda_j;\lambda_i;\beta)=\zeta_{23}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{232} is non-negative for

all $\lambda_j \geq \lambda_i$. Differentiating ζ_{232} with respect to λ_j gives

$$\begin{aligned}
\zeta_{232}(\lambda_j;\lambda_i;\beta) &= (63936 + 3024\beta^2 + 26496\beta + 72\beta^3)\lambda_j^3 \\
&\quad +(648\beta^3 + 15552\beta + 9504\beta^2 - 48384)\lambda_i\lambda_j^2 \\
&\quad +(4320\beta^2 + 10368\beta + 648\beta^3 + 34560)\lambda_i^2\lambda_j \\
&\quad +(432\beta^2 + 10368 + 72\beta^3 + 6624\beta)\lambda_i^3, \\
\zeta_{232}^{(1)}(\lambda_j;\lambda_i;\beta) &= (191808 + 9072\beta^2 + 79488\beta + 216\beta^3)\lambda_j^2 \\
&\quad +(1296\beta^3 + 31104\beta + 19008\beta^2 - 96768)\lambda_i\lambda_j \\
&\quad +(4320\beta^2 + 10368\beta + 648\beta^3 + 34560)\lambda_i^2, \\
\zeta_{232}^{(2)}(\lambda_j;\lambda_i;\beta) &= (158976\beta + 18144\beta^2 + 383616 + 432\beta^3)\lambda_j \\
&\quad +(1296\beta^3 + 31104\beta + 19008\beta^2 - 96768)\lambda_i, \\
\zeta_{232}^{(3)}(\lambda_j;\lambda_i;\beta) &= 158976\beta + 18144\beta^2 + 383616 + 432\beta^3 \geq 0 \text{ for all } \beta \geq -4.35.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{232}(\lambda_i;\lambda_i;\beta) &= \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{232}^{(1)}(\lambda_i;\lambda_i;\beta) &= \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{232}^{(2)}(\lambda_i;\lambda_i;\beta) &= \geq 0 \text{ for all } \beta \geq -3.18.
\end{aligned}$$

Hence, $\zeta_{23}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{231}(\lambda_j;\lambda_i;\beta) = \zeta_{23}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{231} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{231} with respect to λ_j gives

$$\begin{aligned}
\zeta_{231}(\lambda_j;\lambda_i;\beta) &= (72\beta^3 + 15264\beta + 1944\beta^2 + 36720)\lambda_j^4 \\
&\quad +(6624\beta - 39168 + 864\beta^3 + 6912\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(4320\beta^2 + 1296\beta^3 - 1296\beta + 13824)\lambda_i^2\lambda_j^2 \\
&\quad +(6624\beta + 1728\beta^2 + 288\beta^3 + 3456)\lambda_i^3\lambda_j + (216\beta^2 + 504\beta + 288)\lambda_i^4, \\
\zeta_{231}^{(1)}(\lambda_j;\lambda_i;\beta) &= (288\beta^3 + 61056\beta + 7776\beta^2 + 146880)\lambda_j^3 \\
&\quad +(19872\beta - 117504 + 2592\beta^3 + 20736\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(8640\beta^2 + 2592\beta^3 - 2592\beta + 27648)\lambda_i^2\lambda_j \\
&\quad +(6624\beta + 1728\beta^2 + 288\beta^3 + 3456)\lambda_i^3, \\
\zeta_{231}^{(2)}(\lambda_j;\lambda_i;\beta) &= (864\beta^3 + 183168\beta + 23328\beta^2 + 440640)\lambda_j^2 \\
&\quad +(39744\beta - 235008 + 5184\beta^3 + 41472\beta^2)\lambda_i\lambda_j \\
&\quad +(8640\beta^2 + 2592\beta^3 - 2592\beta + 27648)\lambda_i^2, \\
\zeta_{231}^{(3)}(\lambda_j;\lambda_i;\beta) &= (881280 + 46656\beta^2 + 366336\beta + 1728\beta^3)\lambda_j \\
&\quad +(39744\beta - 235008 + 5184\beta^3 + 41472\beta^2)\lambda_i, \\
\zeta_{231}^{(4)}(\lambda_j;\lambda_i;\beta) &= 881280 + 46656\beta^2 + 366336\beta + 1728\beta^3 \geq 0 \text{ for all } \beta \geq -5.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{231}(\lambda_i;\lambda_i;\beta) &= (15120\beta^2+27720\beta+15120+2520\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -1,\\
\zeta_{231}^{(1)}(\lambda_i;\lambda_i;\beta) &= (5760\beta^3+84960\beta+38880\beta^2+60480)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2,\\
\zeta_{231}^{(2)}(\lambda_i;\lambda_i;\beta) &= (8640\beta^3+220320\beta+73440\beta^2+233280)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3,\\
\zeta_{231}^{(3)}(\lambda_i;\lambda_i;\beta) &= (646272+88128\beta^2+406080\beta+6912\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -3.70.
\end{aligned}$$

Hence, $\zeta_{23}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{230}(\lambda_j;\lambda_i;\beta) = \zeta_{23}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ζ_{230} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{230} with respect to λ_j gives

$$\begin{aligned}
\zeta_{230}(\lambda_j;\lambda_i;\beta) &= (36\beta^3+864\beta^2+6228\beta+14040)\lambda_j^4\\
&\quad +(540\beta^3-19656+3888\beta^2+3132\beta)\lambda_i\lambda_j^3\\
&\quad +(11664+1080\beta^3+3888\beta^2+1512\beta)\lambda_i^2\lambda_j^2\\
&\quad +(360\beta^3+9792\beta+2808\beta^2+5184)\lambda_i^3\lambda_j\\
&\quad +(864+648\beta^2+1512\beta)\lambda_i^4,
\end{aligned}$$

$$\begin{aligned}
\zeta_{230}^{(1)}(\lambda_j;\lambda_i;\beta) &= (56160+24912\beta+3456\beta^2+144\beta^3)\lambda_j^3\\
&\quad +(1620\beta^3+9396\beta+11664\beta^2-58968)\lambda_i\lambda_j^2\\
&\quad +(23328+7776\beta^2+2160\beta^3+3024\beta)\lambda_i^2\lambda_j\\
&\quad +(360\beta^3+9792\beta+2808\beta^2+5184)\lambda_i^3,\\
\zeta_{230}^{(2)}(\lambda_j;\lambda_i;\beta) &= (168480+74736\beta+10368\beta^2+432\beta^3)\lambda_j^2\\
&\quad +(3240\beta^3+18792\beta+23328\beta^2-117936)\lambda_i\lambda_j\\
&\quad +(23328+7776\beta^2+2160\beta^3+3024\beta)\lambda_i^2,\\
\zeta_{230}^{(3)}(\lambda_j;\lambda_i;\beta) &= (336960+149472\beta+20736\beta^2+864\beta^3)\lambda_j\\
&\quad +(3240\beta^3+18792\beta+23328\beta^2-117936)\lambda_i,\\
\zeta_{230}^{(4)}(\lambda_j;\lambda_i;\beta) &= 336960+149472\beta+20736\beta^2+864\beta^3 \geq 0 \text{ for all } \beta \geq -5.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{230}(\lambda_i;\lambda_i;\beta) &= (12096+12096\beta^2+22176\beta+2016\beta^3)\lambda_i^4 > 0 \text{ for all } \beta > -1,\\
\zeta_{230}^{(1)}(\lambda_i;\lambda_i;\beta) &= (25704+47124\beta+25704\beta^2+4284\beta^3)\lambda_i^3 > 0 \text{ for all } \beta > -1,\\
\zeta_{230}^{(2)}(\lambda_i;\lambda_i;\beta) &= (73872+96552\beta+41472\beta^2+5832\beta^3)\lambda_i^2 > 0 \text{ for all } \beta > -2,\\
\zeta_{230}^{(3)}(\lambda_i;\lambda_i;\beta) &= (219024+168264\beta+44064\beta^2+4104\beta^3)\lambda_i > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, $\zeta_{23}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(1)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{24}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{24}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (1800\beta+720)\lambda_k^4+(7200\beta+2160\beta^2+11520)\lambda_k^3\left(\lambda_j+\lambda_i\right)\\
&\quad +(432\beta^3+2808\beta^2+15120\beta+32832)\lambda_k^2\left(\lambda_j^2+\lambda_i^2\right)\\
&\quad +(-48384+4320\beta+7344\beta^2+1296\beta^3)\lambda_k^2\lambda_j\lambda_i\\
&\quad +(2592\beta^2+4608+8640\beta+288\beta^3)\lambda_k\left(\lambda_j^3+\lambda_i^3\right)\\
&\quad +(8208\beta^2+2592\beta^3-12960\beta-27648)\lambda_k\lambda_j^2\lambda_i\\
&\quad +(8208\beta^2+2592\beta^3-12960\beta-27648)\lambda_k\lambda_j\lambda_i^2,,\\
\zeta_{24}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (2880+7200\beta)\lambda_k^3+(21600\beta+34560+6480\beta^2)\lambda_k^2(\lambda_j+\lambda_i)\\
&\quad +(14688\beta^2+2592\beta^3-96768+8640\beta)\lambda_k\lambda_i\lambda_j\\
&\quad +(30240\beta+864\beta^3+65664+5616\beta^2)\lambda_k(\lambda_j^2+\lambda_i^2)\\
&\quad +(4608+2592\beta^2+8640\beta+288\beta^3)(\lambda_j^3+\lambda_i^3)\\
&\quad +(-27648+8208\beta^2-12960\beta+2592\beta^3)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j),\\
\zeta_{24}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (8640+21600\beta)\lambda_k^2+(43200\beta+69120+12960\beta^2)\lambda_k(\lambda_j+\lambda_i)\\
&\quad +(14688\beta^2+2592\beta^3-96768+8640\beta)\lambda_i\lambda_j\\
&\quad +(30240\beta+864\beta^3+65664+5616\beta^2)(\lambda_j^2+\lambda_i^2),\\
\zeta_{24}^{(3)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (17280+43200\beta)\lambda_k+(43200\beta+69120+12960\beta^2)(\lambda_j+\lambda_i)\\
&\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}$$

Define $\zeta_{242}(\lambda_j;\lambda_i;\beta)=\zeta_{24}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{242} is non-negative for all $\lambda_j\geq\lambda_i$. Differentiating ζ_{242} with respect to λ_j gives

$$\begin{aligned}
\zeta_{242}(\lambda_j;\lambda_i;\beta) &= (143424+95040\beta+18576\beta^2+864\beta^3)\lambda_j^2\\
&\quad +(-27648+27648\beta^2+2592\beta^3+51840\beta)\lambda_i\lambda_j\\
&\quad +(30240\beta+864\beta^3+65664+5616\beta^2)\lambda_i^2,\\
\zeta_{242}^{(1)}(\lambda_j;\lambda_i;\beta) &= (286848+190080\beta+37152\beta^2+1728\beta^3)\lambda_j\\
&\quad +(-27648+27648\beta^2+2592\beta^3+51840\beta)\lambda_i,\\
\zeta_{242}^{(2)}(\lambda_j;\lambda_i;\beta) &= 286848+190080\beta+37152\beta^2+1728\beta^3\geq 0 \text{ for all } \beta\geq -14.8.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{242}(\lambda_i;\lambda_i;\beta) &= (177120\beta+4320\beta^3+181440+51840\beta^2)\lambda_i^2\geq 0 \text{ for all } \beta\geq -2,\\
\zeta_{242}^{(1)}(\lambda_i;\lambda_i;\beta) &= (259200+241920\beta+64800\beta^2+4320\beta^3)\lambda_i\geq 0 \text{ for all } \beta\geq -2.
\end{aligned}$$

Hence, $\zeta_{24}^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j\geq\lambda_i$ and $\beta\geq-\frac{2}{5}$.

Define $\zeta_{241}(\lambda_j;\lambda_i;\beta)=\zeta_{24}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{241} is non-negative for

all $\lambda_j \geq \lambda_i$. Differentiating ζ_{241} with respect to λ_j gives

$$\begin{aligned}
\zeta_{241}(\lambda_j;\lambda_i;\beta) &= (107712+1152\beta^3+14688\beta^2+67680\beta)\lambda_j^3\\
&\quad +(29376\beta^2-89856+5184\beta^3+17280\beta)\lambda_i\lambda_j^2\\
&\quad +(3456\beta^3+38016+17280\beta+13824\beta^2)\lambda_i^2\lambda_j\\
&\quad +(4608+2592\beta^2+8640\beta+288\beta^3)\lambda_i^3,\\
\zeta_{241}^{(1)}(\lambda_j;\lambda_i;\beta) &= (323136+3456\beta^3+44064\beta^2+203040\beta)\lambda_j^2\\
&\quad +(58752\beta^2-179712+10368\beta^3+34560\beta)\lambda_i\lambda_j\\
&\quad +(3456\beta^3+38016+17280\beta+13824\beta^2)\lambda_i^2,\\
\zeta_{241}^{(2)}(\lambda_j;\lambda_i;\beta) &= (646272+88128\beta^2+406080\beta+6912\beta^3)\lambda_j\\
&\quad +(58752\beta^2-179712+10368\beta^3+34560\beta)\lambda_i,\\
\zeta_{241}^{(3)}(\lambda_j;\lambda_i;\beta) &= 646272+88128\beta^2+406080\beta+6912\beta^3 \geq 0 \text{ for all } \beta \geq -3.70.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{241}(\lambda_i;\lambda_i;\beta) &= (60480+60480\beta^2+110880\beta+10080\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -1,\\
\zeta_{241}^{(1)}(\lambda_i;\lambda_i;\beta) &= (181440+17280\beta^3+116640\beta^2+254880\beta)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -2,\\
\zeta_{241}^{(2)}(\lambda_i;\lambda_i;\beta) &= (466560+146880\beta^2+440640\beta+17280\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -3.
\end{aligned}$$

Hence, $\zeta_{24}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{240}(\lambda_j;\lambda_i;\beta) = \zeta_{24}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{240} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{240} with respect to λ_j gives

$$\begin{aligned}
\zeta_{240}(\lambda_j;\lambda_i;\beta) &= (720\beta^3+34272\beta+50544+8208\beta^2)\lambda_j^4\\
&\quad +(15552\beta+23328\beta^2+4320\beta^3-55296)\lambda_i\lambda_j^3\\
&\quad +(33912\beta+20088\beta^2+4320\beta^3+50544)\lambda_i^2\lambda_j^2\\
&\quad +(720\beta^3+13824+8208\beta^2+25632\beta)\lambda_i^3\lambda_j\\
&\quad +(648\beta^2+1512\beta+864)\lambda_i^4,\\
\zeta_{240}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2880\beta^3+137088\beta+202176+32832\beta^2)\lambda_j^3\\
&\quad +(46656\beta+69984\beta^2+12960\beta^3-165888)\lambda_i\lambda_j^2\\
&\quad +(67824\beta+40176\beta^2+8640\beta^3+101088)\lambda_i^2\lambda_j\\
&\quad +(720\beta^3+13824+8208\beta^2+25632\beta)\lambda_i^3,\\
\zeta_{240}^{(2)}(\lambda_j;\lambda_i;\beta) &= (8640\beta^3+411264\beta+606528+98496\beta^2)\lambda_j^2\\
&\quad +(93312\beta+139968\beta^2+25920\beta^3-331776)\lambda_i\lambda_j\\
&\quad +(67824\beta+40176\beta^2+8640\beta^3+101088)\lambda_i^2,\\
\zeta_{240}^{(3)}(\lambda_j;\lambda_i;\beta) &= (17280\beta^3+822528\beta+1213056+196992\beta^2)\lambda_j\\
&\quad +(93312\beta+139968\beta^2+25920\beta^3-331776)\lambda_i,\\
\zeta_{240}^{(4)}(\lambda_j;\lambda_i;\beta) &= 17280\beta^3+822528\beta+1213056+196992\beta^2 \geq 0 \text{ for all } \beta \geq -3.58.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{240}(\lambda_i;\lambda_i;\beta) &= (60480+60480\beta^2+110880\beta+10080\beta^3)\lambda_i^4>0 \text{ for all } \beta>-1,\\
\zeta_{240}^{(1)}(\lambda_i;\lambda_i;\beta) &= (25200\beta^3+277200\beta+151200+151200\beta^2)\lambda_i^3>0 \text{ for all } \beta>-1,\\
\zeta_{240}^{(2)}(\lambda_i;\lambda_i;\beta) &= (43200\beta^3+572400\beta+375840+278640\beta^2)\lambda_i^2>0 \text{ for all } \beta>-2,\\
\zeta_{240}^{(3)}(\lambda_i;\lambda_i;\beta) &= (43200\beta^3+915840\beta+881280+336960\beta^2)\lambda_i>0 \text{ for all } \beta>-3.
\end{aligned}$$

Hence, $\zeta_{24}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(1)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{25}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{25}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (25920+21600\beta+4320\beta^2)\lambda_k^3\\
&\quad +(51840+56160\beta+19440\beta^2+2160\beta^3)\lambda_k^2(\lambda_j+\lambda_i)\\
&\quad +(4320\beta^3+51840+43200\beta+21600\beta^2)\lambda_k(\lambda_j^2+\lambda_i^2)\\
&\quad +(-43200\beta+12960\beta^3-207360+47520\beta^2)\lambda_k\lambda_j\lambda_i\\
&\quad +(720\beta^3+10800\beta^2+17280+31680\beta)(\lambda_j^3+\lambda_i^3)\\
&\quad +(90720+6480\beta^3+101520\beta+43200\beta^2)(\lambda_j^2\lambda_i+\lambda_j\lambda_i^2),
\end{aligned}$$

$$\begin{aligned}
\zeta_{25}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (77760+64800\beta+12960\beta^2)\lambda_k^2\\
&\quad +(112320\beta+103680+38880\beta^2+4320\beta^3)\lambda_k(\lambda_j+\lambda_i)\\
&\quad +(21600\beta^2+43200\beta+51840+4320\beta^3)(\lambda_j^2+\lambda_i^2)\\
&\quad +(12960\beta^3-43200\beta-207360+47520\beta^2)\lambda_i\lambda_j,\\
\zeta_{25}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (155520+129600\beta+25920\beta^2)\lambda_k\\
&\quad +(112320\beta+103680+38880\beta^2+4320\beta^3)(\lambda_j+\lambda_i)\\
&\geq 0 \text{ for all } \beta\geq -2.
\end{aligned}$$

Define $\zeta_{251}(\lambda_j;\lambda_i;\beta)=\zeta_{25}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{251} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{251} with respect to λ_j gives

$$\begin{aligned}
\zeta_{251}(\lambda_j;\lambda_i;\beta) &= (8640\beta^3+220320\beta+73440\beta^2+233280)\lambda_j^2\\
&\quad +(69120\beta-103680+86400\beta^2+17280\beta^3)\lambda_i\lambda_j\\
&\quad +(21600\beta^2+43200\beta+51840+4320\beta^3)\lambda_i^2,\\
\zeta_{251}^{(1)}(\lambda_j;\lambda_i;\beta) &= (466560+146880\beta^2+440640\beta+17280\beta^3)\lambda_j\\
&\quad +(69120\beta-103680+86400\beta^2+17280\beta^3)\lambda_i,\\
\zeta_{251}^{(2)}(\lambda_j;\lambda_i;\beta) &= 466560+146880\beta^2+440640\beta+17280\beta^3\geq 0 \text{ for all } \beta\geq -3.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{251}(\lambda_i;\lambda_i;\beta) &= (181440\beta^2+332640\beta+181440+30240\beta^3)\lambda_i^2\geq 0 \text{ for all } \beta\geq -1,\\
\zeta_{251}^{(1)}(\lambda_i;\lambda_i;\beta) &= (362880+233280\beta^2+509760\beta+34560\beta^3)\lambda_i\geq 0 \text{ for all } \beta\geq -1.75.
\end{aligned}$$

Hence, $\zeta_{25}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{250}(\lambda_j;\lambda_i;\beta) = \zeta_{25}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{250} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{250} with respect to λ_j gives

$$\begin{aligned}
\zeta_{250}(\lambda_j;\lambda_i;\beta) &= (7200\beta^3 + 146880 + 152640\beta + 56160\beta^2)\lambda_j^3 \\
&\quad +(114480\beta + 21600\beta^3 - 64800 + 110160\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(10800\beta^3 + 64800\beta^2 + 144720\beta + 142560)\lambda_i^2\lambda_j \\
&\quad +(10800\beta^2 + 720\beta^3 + 31680\beta + 17280)\lambda_i^3, \\
\zeta_{250}^{(1)}(\lambda_j;\lambda_i;\beta) &= (21600\beta^3 + 440640 + 457920\beta + 168480\beta^2)\lambda_j^2 \\
&\quad +(228960\beta + 43200\beta^3 - 129600 + 220320\beta^2)\lambda_i\lambda_j \\
&\quad +(10800\beta^3 + 64800\beta^2 + 144720\beta + 142560)\lambda_i^2, \\
\zeta_{250}^{(2)}(\lambda_j;\lambda_i;\beta) &= (43200\beta^3 + 915840\beta + 881280 + 336960\beta^2)\lambda_j \\
&\quad +(228960\beta + 43200\beta^3 - 129600 + 220320\beta^2)\lambda_i, \\
\zeta_{250}^{(3)}(\lambda_j;\lambda_i;\beta) &= 43200\beta^3 + 915840\beta + 881280 + 336960\beta^2 \geq 0 \text{ for all } \beta \geq -5.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{250}(\lambda_i;\lambda_i;\beta) &= (241920\beta^2 + 40320\beta^3 + 443520\beta + 241920)\lambda_i^3 > 0 \text{ for all } \beta > -1, \\
\zeta_{250}^{(1)}(\lambda_i;\lambda_i;\beta) &= (75600\beta^3 + 453600 + 831600\beta + 453600\beta^2)\lambda_i^2 > 0 \text{ for all } \beta > -1, \\
\zeta_{250}^{(2)}(\lambda_i;\lambda_i;\beta) &= (86400\beta^3 + 1144800\beta + 751680 + 557280\beta^2)\lambda_i > 0 \text{ for all } \beta > -1.45.
\end{aligned}$$

Hence, $\zeta_{25}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(1)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{26}(\lambda_k;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{26}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (4320\beta^3 + 177120\beta + 181440 + 51840\beta^2)\lambda_k^2 \\
&\quad +(155520\beta + 25920\beta^3 + 129600\beta^2)\lambda_k\lambda_j \\
&\quad +(155520\beta + 25920\beta^3 + 129600\beta^2)\lambda_k\lambda_i \\
&\quad +(239760\beta + 194400 + 12960\beta^3 + 97200\beta^2)\lambda_j^2 \\
&\quad +(239760\beta + 194400 + 12960\beta^3 + 97200\beta^2)\lambda_i^2 \\
&\quad +(220320\beta^2 + 38880\beta^3 + 362880\beta + 155520)\lambda_j\lambda_i, \\
\zeta_{26}^{(1)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= (354240\beta + 8640\beta^3 + 362880 + 103680\beta^2)\lambda_k \\
&\quad +(129600\beta^2 + 25920\beta^3 + 155520\beta)(\lambda_j + \lambda_i), \\
\zeta_{26}^{(2)}(\lambda_k;\lambda_j;\lambda_i;\beta) &= 354240\beta + 8640\beta^3 + 362880 + 103680\beta^2 \geq 0 \text{ for all } \beta \geq -2.
\end{aligned}$$

Define $\zeta_{261}(\lambda_j;\lambda_i;\beta) = \zeta_{26}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{261} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{261} with respect to λ_j gives

$$\begin{aligned}
\zeta_{261}(\lambda_j;\lambda_i;\beta) &= (362880 + 233280\beta^2 + 509760\beta + 34560\beta^3)\lambda_j \\
&\quad +(129600\beta^2 + 25920\beta^3 + 155520\beta)\lambda_i, \\
\zeta_{261}^{(1)}(\lambda_j;\lambda_i;\beta) &= 362880 + 233280\beta^2 + 509760\beta + 34560\beta^3 \geq 0 \text{ for all } \beta \geq -1.75.
\end{aligned}$$

Now

$$\zeta_{261}(\lambda_i;\lambda_i;\beta) = (362880\beta^2+60480\beta^3+665280\beta+362880)\lambda_i \geq 0 \text{ for all } \beta \geq -1.$$

Hence, $\zeta_{26}^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{260}(\lambda_j;\lambda_i;\beta) = \zeta_{26}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ζ_{260} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{260} with respect to λ_j gives

$$\begin{aligned}
\zeta_{260}(\lambda_j;\lambda_i;\beta) &= (43200\beta^3+572400\beta+375840+278640\beta^2)\lambda_j^2 \\
&\quad +(349920\beta^2+64800\beta^3+518400\beta+155520)\lambda_i\lambda_j \\
&\quad +(239760\beta+97200\beta^2+12960\beta^3+194400)\lambda_i^2, \\
\zeta_{260}^{(1)}(\lambda_j;\lambda_i;\beta) &= (86400\beta^3+1144800\beta+751680+557280\beta^2)\lambda_j \\
&\quad +(349920\beta^2+64800\beta^3+518400\beta+155520)\lambda_i \\
&\geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Now

$$\zeta_{260}(\lambda_i;\lambda_i;\beta) = (1330560\beta+725760\beta^2+120960\beta^3+725760)\lambda_i^2 > 0 \text{ for all } \beta > -1.$$

Hence, $\zeta_{26}(\lambda_j;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(6)}(0;\lambda_k;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

$\square$

Lemma B.7 Let

$$\begin{aligned}
\zeta_3(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 60 \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\text{perm}(0,4,4,4)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3}(\lambda+\lambda_l)^{n_4} \\
&\quad + (16\beta-32) \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\text{perm}(1,3,4,4)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3}(\lambda+\lambda_l)^{n_4} \\
&\quad + (18\beta-36) \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\text{perm}(2,2,4,4)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3}(\lambda+\lambda_l)^{n_4} \\
&\quad + \left(6\beta^2-36\beta+48\right) \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\text{perm}(2,3,3,4)}} (\lambda+\lambda_i)^{n_1}(\lambda+\lambda_j)^{n_2}(\lambda+\lambda_k)^{n_3}(\lambda+\lambda_l)^{n_4} \\
&\quad + \left(4\beta^3-48\beta^2+176\beta-192\right)(\lambda+\lambda_k)^3(\lambda+\lambda_i)^3(\lambda+\lambda_j)^3(\lambda+\lambda_l)^3
\end{aligned}$$

and $\lambda_k \geq \max(\lambda_i,\lambda_j,\lambda_l)$. The expression $\zeta_3(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda \in (\lambda_{\text{S}},\infty)$ and $\beta \geq -\frac{2}{5}$.

Proof.

Without loss of generality assume that $\lambda_i \leq \lambda_j \leq \lambda_l$.

Define

$$s(a,b,c,d) = \sum_{\substack{(n_1,n_2,n_3,n_4) \\ \in \mathrm{perm}(a,b,c,d)}} (\lambda+\lambda_i)^{n_1} (\lambda+\lambda_j)^{n_2} (\lambda+\lambda_k)^{n_3} (\lambda+\lambda_l)^{n_4}.$$

Differentiating $\zeta_3(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ we obtain

$$\begin{aligned}
\zeta_3(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 60s(0,4,4,4) + (16\beta-32)\,s(1,3,4,4) \\
&\quad + (18\beta-36)\,s(2,2,4,4) + \big(6\beta^2-36\beta+48\big)\,s(2,3,3,4) \\
&\quad + \big(4\beta^3-48\beta^2+176\beta-192\big)\,s(3,3,3,3),
\end{aligned} \tag{B.22}$$

$$\begin{aligned}
\zeta_3^{(1)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (56\beta+12\beta^2-160)s(1,3,3,4) \\
&\quad +(-144\beta+36\beta^2+144)s(2,2,3,4) \\
&\quad +(208+16\beta)s(0,3,4,4) \\
&\quad +(84\beta-168)s(1,2,4,4) \\
&\quad +(-72\beta^2+96\beta+12\beta^3)s(2,3,3,3)
\end{aligned} \tag{B.23}$$

$$\begin{aligned}
\zeta_3^{(2)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (-1296\beta+324\beta^2+1296)s(2,2,2,4) \\
&\quad +(336\beta-672)s(1,1,4,4) \\
&\quad +(-864+108\beta^2+216\beta)s(1,2,3,4) \\
&\quad +(12\beta^2+1504+184\beta)s(0,3,3,4) \\
&\quad +(132\beta+456)s(0,2,4,4) \\
&\quad +(24\beta^3+864\beta-1920)s(1,3,3,3) \\
&\quad +(1152+72\beta^3-144\beta^2-576\beta)s(2,2,3,3),
\end{aligned} \tag{B.24}$$

$$\begin{aligned}
\zeta_3^{(3)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (648\beta^3+15552-10368\beta)s(2,2,2,3) \\
&\quad +(2208\beta+432\beta^2-6144)s(1,1,3,4) \\
&\quad +(1296\beta^2-1296\beta-2592)s(1,2,2,4) \\
&\quad +(576\beta^2+3168\beta+216\beta^3-10368)s(1,2,3,3) \\
&\quad +(144\beta^2+1296\beta+5472)s(0,2,3,4) \\
&\quad +(240+600\beta)s(0,1,4,4) \\
&\quad +(144\beta^2+3072\beta+16128+24\beta^3)s(0,3,3,3),
\end{aligned} \tag{B.25}$$

$$\begin{aligned}
\zeta_3^{(4)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (1200\beta+480)s(0,0,4,4) \qquad \text{(B.26)}\\
&\quad +(7200\beta+720\beta^2+5760)s(0,1,3,4)\\
&\quad +(30240+6480\beta+2160\beta^2)s(0,2,2,4)\\
&\quad +(-28800+6480\beta^2+1440\beta)s(1,1,2,4)\\
&\quad +(2160\beta^2+288\beta^3+81792+22752\beta)s(0,2,3,3)\\
&\quad +(30336\beta+864\beta^3+5760\beta^2-90624)s(1,1,3,3)\\
&\quad +(2592\beta^3+8640\beta^2-41472-6912\beta)s(1,2,2,3)\\
&\quad +(7776\beta^3+186624-124416\beta)s(2,2,2,2),\\
\zeta_3^{(5)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (8640\beta+38880\beta^2-172800)s(1,1,1,4) \qquad \text{(B.27)}\\
&\quad +(12960\beta^2+48960+36000\beta)s(0,1,2,4)\\
&\quad +(1440\beta^2+19200\beta+13440)s(0,0,3,4)\\
&\quad +(15840\beta^2+119040+133440\beta+1440\beta^3)s(0,1,3,3)\\
&\quad +(12960\beta^3-552960+69120\beta+77760\beta^2)s(1,1,2,3)\\
&\quad +(30240\beta^2+4320\beta^3+570240+155520\beta)s(0,2,2,3)\\
&\quad +(-311040\beta+38880\beta^3+77760\beta^2)s(1,2,2,2),\\
\zeta_3^{(6)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (345600+420480\beta+43200\beta^2+2880\beta^3)s(0,0,3,3) \qquad \text{(B.28)}\\
&\quad +(1140480+924480\beta+237600\beta^2+25920\beta^3)s(0,1,2,3)\\
&\quad +(152640\beta+23040+90720\beta^2)s(0,1,1,4)\\
&\quad +(77760\beta^3-4008960+449280\beta+622080\beta^2)s(1,1,1,3)\\
&\quad +(777600\beta^2-829440\beta+233280\beta^3-3317760)s(1,1,2,2)\\
&\quad +(129600\beta+30240\beta^2+138240)s(0,0,2,4)\\
&\quad +(349920\beta^2+77760\beta^3+5132160+1088640\beta)s(0,2,2,2),\\
\zeta_3^{(7)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 20160(322560+564480\beta+241920\beta^2)s(0,0,1,4) \qquad \text{(B.29)}\\
&\quad +(60480\beta^3+3870720+3628800\beta+725760\beta^2)s(0,0,2,3)\\
&\quad +(1935360\beta^2+181440\beta^3+645120+4757760\beta)s(0,1,1,3)\\
&\quad +(2903040\beta^2+544320\beta^3+13789440+6894720\beta)s(0,1,2,2)\\
&\quad +(-3628800\beta+6531840\beta^2-31933440+1632960\beta^3)s(1,1,1,2),\\
\zeta_3^{(8)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (725760\beta^2+967680+1693440\beta)s(0,0,0,4) \qquad \text{(B.30)}\\
&\quad +(10321920+483840\beta^3+19031040\beta+6289920\beta^2)s(0,0,1,3)\\
&\quad +(38223360\beta+23950080\beta^2+25159680+4354560\beta^3)s(0,1,1,2)\\
&\quad +(50803200+10160640\beta^2+35562240\beta+1451520\beta^3)s(0,0,2,2)\\
&\quad +(-29030400\beta+52254720\beta^2-255467520+13063680\beta^3)s(1,1,1,1),\\
\zeta_3^{(9)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 1451520(15\beta^2+44\beta+\beta^3+24)s(0,0,0,3) \qquad \text{(B.31)}\\
&\quad +4354560(3\beta^3+42+47\beta+20\beta^2)s(0,0,1,2)\\
&\quad +4354560(9\beta^3+45\beta^2-24+46\beta)s(0,1,1,1),
\end{aligned}$$

$$\begin{aligned}
\zeta_3^{(10)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 21772800(2\beta+5)(\beta+3)(\beta+2)s(0,0,0,2) \\
&\quad +43545600(3\beta+2)(\beta+3)(\beta+2)s(0,0,1,1), \\
\zeta_3^{(11)}(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 479001600(\beta+3)(\beta+2)(\beta+1)s(0,0,0,1) \\
&> 0 \text{ for all } \beta > -1.
\end{aligned} \tag{B.32}$$

Consider the case $\lambda_i \leq 0$. Substituting in $\lambda = -\lambda_i$ into (B.22)–(B.32) and defining

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} (\lambda_j-\lambda_i)^{n_1}(\lambda_k-\lambda_i)^{n_2}(\lambda_l-\lambda_i)^{n_3}$$

we obtain

$$\begin{aligned}
\zeta_3(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 60\hat{s}(4,4,4) \geq 0, \\
\zeta_3^{(1)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (208+16\beta)\hat{s}(3,4,4) \geq 0 \text{ for all } \beta > -13, \\
\zeta_3^{(2)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (12\beta^2+1504+184\beta)\hat{s}(3,3,4) \\
&\quad +(132\beta+456)\hat{s}(2,4,4) \\
&\geq 0 \text{ for all } \beta > -\frac{38}{11}, \\
\zeta_3^{(3)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (144\beta^2+1296\beta+5472)\hat{s}(2,3,4) \\
&\quad +(240+600\beta)\hat{s}(1,4,4) \\
&\quad +(144\beta^2+3072\beta+16128+24\beta^3)\hat{s}(3,3,3) \\
&\geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\zeta_3^{(4)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (1200\beta+480)\hat{s}(0,4,4)+(7200\beta+720\beta^2+5760)\hat{s}(1,3,4) \\
&\quad +(30240+6480\beta+2160\beta^2)\hat{s}(2,2,4) \\
&\quad +(2160\beta^2+288\beta^3+81792+22752\beta)\hat{s}(2,3,3) \\
&\geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\zeta_3^{(5)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (12960\beta^2+48960+36000\beta)\hat{s}(1,2,4) \\
&\quad +(1440\beta^2+19200\beta+13440)\hat{s}(0,3,4) \\
&\quad +(15840\beta^2+119040+133440\beta+1440\beta^3)\hat{s}(1,3,3) \\
&\quad +(30240\beta^2+4320\beta^3+570240+155520\beta)\hat{s}(2,2,3) \\
&\geq 0 \text{ for all } \beta \geq -0.74,
\end{aligned}$$

$$\begin{aligned}
\zeta_3^{(6)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 2880(120+146\beta+15\beta^2+\beta^3)\hat{s}(0,3,3)\\
&\quad +4320(264+214\beta+55\beta^2+6\beta^3)\hat{s}(1,2,3)\\
&\quad +1440(106\beta+16+63\beta^2)\hat{s}(1,1,4)\\
&\quad +4320(30\beta+7\beta^2+32)\hat{s}(0,2,4)\\
&\quad +38880(9\beta^2+2\beta^3+132+28\beta)\hat{s}(2,2,2),\\
\zeta_3^{(7)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 80640(4+7\beta+3\beta^2)\hat{s}(0,1,4)\\
&\quad +60480(\beta^3+64+60\beta+12\beta^2)\hat{s}(0,2,3)\\
&\quad +20160(96\beta^2+9\beta^3+32+236\beta)\hat{s}(1,1,3)\\
&\quad +181440(16\beta^2+3\beta^3+76+38\beta)\hat{s}(1,2,2),\\
\zeta_3^{(8)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (725760\beta^2+967680+1693440\beta)\hat{s}(0,0,4)\\
&\quad +(10321920+483840\beta^3+19031040\beta+6289920\beta^2)\hat{s}(0,1,3)\\
&\quad +(38223360\beta+23950080\beta^2+25159680+4354560\beta^3)\hat{s}(1,1,2)\\
&\quad +(50803200+10160640\beta^2+35562240\beta+1451520\beta^3)\hat{s}(0,2,2)\\
&\geq 0 \text{ for all } \beta\geq -0.69,\\
\zeta_3^{(9)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 1451520(15\beta^2+44\beta+\beta^3+24)\hat{s}(0,0,3)\\
&\quad +4354560(3\beta^3+42+47\beta+20\beta^2)\hat{s}(0,1,2)\\
&\quad +4354560(9\beta^3+45\beta^2-24+46\beta)\hat{s}(1,1,1),\\
\zeta_3^{(10)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 21772800(2\beta+5)(\beta+3)(\beta+2)\hat{s}(0,0,2)\\
&\quad +43545600(3\beta+2)(\beta+3)(\beta+2)\hat{s}(0,1,1)\\
&\geq 0 \text{ for all } \beta\geq -\frac{2}{3}.
\end{aligned}$$

From Lemmas B.8—B.10 we deduce that $\zeta_3^{(6)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$, $\zeta_3^{(7)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ and $\zeta_3^{(9)}(-\lambda_i;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ are non-negative for all $\beta\geq-\frac{2}{5}$. Hence, $\zeta_3(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda\geq-\lambda_i$ and $\beta\geq-\frac{2}{5}$.

Consider the case $\lambda_i>0$. From Lemmas B.11-B.21 we deduce that $\zeta_3(0;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$, $\zeta_3^{(1)}(0;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$, $\ldots\zeta_3^{(11)}(0;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ are positive for all $\lambda_k\geq\lambda_l\geq\lambda_j\geq\lambda_i$ and $\beta\geq-\frac{2}{5}$. Hence, $\zeta_3(\lambda;\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda\geq0$ and $\beta\geq-\frac{2}{5}$.

□

Lemma B.8 Let

$$\begin{aligned}\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (106\beta+16+63\beta^2)\hat{s}(1,1,4)\\ &\quad +3(30\beta+7\beta^2+32)\hat{s}(0,2,4), \qquad \text{(B.33)}\end{aligned}$$

where

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} (\lambda_j-\lambda_i)^{n_1}(\lambda_k-\lambda_i)^{n_2}(\lambda_l-\lambda_i)^{n_3}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof. Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 2(90\beta+96+21\beta^2)(\lambda_k-\lambda_i)(\lambda_l-\lambda_i)^4 \qquad \text{(B.34)}\\
&\quad +4(90\beta+96+21\beta^2)(\lambda_k-\lambda_i)^3(\lambda_l-\lambda_i)^2\\
&\quad +4(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2(\lambda_k-\lambda_i)^3\\
&\quad +2(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^4(\lambda_k-\lambda_i)\\
&\quad +(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^4\\
&\quad +4(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_k-\lambda_i)^3(\lambda_l-\lambda_i)\\
&\quad +(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)^4(\lambda_l-\lambda_i),\\
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 2(90\beta+96+21\beta^2)(\lambda_l-\lambda_i)^4 \qquad \text{(B.35)}\\
&\quad +12(90\beta+96+21\beta^2)(\lambda_k-\lambda_i)^2(\lambda_l-\lambda_i)^2\\
&\quad +12(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2(\lambda_k-\lambda_i)^2\\
&\quad +2(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^4\\
&\quad +12(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_k-\lambda_i)^2(\lambda_l-\lambda_i),\\
\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(90\beta+96+21\beta^2)(\lambda_k-\lambda_i)(\lambda_l-\lambda_i)^2 \qquad \text{(B.36)}\\
&\quad +24(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2(\lambda_k-\lambda_i)\\
&\quad +24(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_k-\lambda_i)(\lambda_l-\lambda_i),\\
\xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(90\beta+96+21\beta^2)(\lambda_l-\lambda_i)^2 \qquad \text{(B.37)}\\
&\quad +24(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2\\
&\quad +24(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i),
\end{aligned}$$

Define

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (42\beta^2+192+180\beta)(\lambda_l-\lambda_i)^6\\
&\quad +(42\beta^2+192+180\beta)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i)^4\\
&\quad +(212\beta+32+126\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^5\\
&\quad +(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^4(\lambda_l-\lambda_i)^2,\\
\xi_0^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(42\beta^2+192+180\beta)(\lambda_l-\lambda_i)^5\\
&\quad +4(42\beta^2+192+180\beta)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i)^3\\
&\quad +5(212\beta+32+126\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^4\\
&\quad +2(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^4(\lambda_l-\lambda_i),\\
\xi_0^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 30(42\beta^2+192+180\beta)(\lambda_l-\lambda_i)^4\\
&\quad +12(42\beta^2+192+180\beta)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i)^2\\
&\quad +20(212\beta+32+126\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^3\\
&\quad +2(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^4,\\
\xi_0^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 120(42\beta^2+192+180\beta)(\lambda_l-\lambda_i)^3\\
&\quad +24(42\beta^2+192+180\beta)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i)\\
&\quad +60(212\beta+32+126\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^2,\\
\xi_0^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 360(42\beta^2+192+180\beta)(\lambda_l-\lambda_i)^2\\
&\quad +24(42\beta^2+192+180\beta)(\lambda_j-\lambda_i)^2\\
&\quad +120(212\beta+32+126\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i),\\
\xi_0^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 720(42\beta^2+192+180\beta)(\lambda_l-\lambda_i)\\
&\quad +120(212\beta+32+126\beta^2)(\lambda_j-\lambda_i),\\
\xi_0^{(6)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 129600\beta+138240+30240\beta^2\\
&> 0 \text{ for all } \beta > -2.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_0(\lambda_j;\lambda_j;\lambda_i;\beta) &= 3(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^6 \geq 0 \text{ for all } \beta,\\
\xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 12(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^5 \geq 0 \text{ for all } \beta,\\
\xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 6(\lambda_j-\lambda_i)^4(749\beta^2+2062\beta+1520) \geq 0 \text{ for all } \beta,\\
\xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 168(\lambda_j-\lambda_i)^3(81\beta^2+176+230\beta) \geq 0 \text{ for all } \beta,\\
\xi_0^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 48(\lambda_j-\lambda_i)^2(1970\beta+1616+651\beta^2) \geq 0 \text{ for all } \beta,\\
\xi_0^{(5)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 240(189\beta^2+592+646\beta)(\lambda_j-\lambda_i) \geq 0 \text{ for all } \beta,
\end{aligned}$$

and, hence, $\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

Differentiating $\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (540\beta+576+126\beta^2)(\lambda_l-\lambda_i)^5\\
&\quad +(360\beta+384+84\beta^2)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i)^3\\
&\quad +(530\beta+80+315\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^4\\
&\quad +(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^4(\lambda_l-\lambda_i),\\
\xi_1^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 5(540\beta+576+126\beta^2)(\lambda_l-\lambda_i)^4\\
&\quad +3(360\beta+384+84\beta^2)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i)^2\\
&\quad +4(530\beta+80+315\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^3\\
&\quad +(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^4,\\
\xi_1^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 20(540\beta+576+126\beta^2)(\lambda_l-\lambda_i)^3\\
&\quad +6(360\beta+384+84\beta^2)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i)\\
&\quad +12(530\beta+80+315\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^2,\\
\xi_1^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 60(540\beta+576+126\beta^2)(\lambda_l-\lambda_i)^2\\
&\quad +6(360\beta+384+84\beta^2)(\lambda_j-\lambda_i)^2\\
&\quad +24(530\beta+80+315\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i),\\
\xi_1^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 120(540\beta+576+126\beta^2)(\lambda_l-\lambda_i)\\
&\quad +24(530\beta+80+315\beta^2)(\lambda_j-\lambda_i),\\
\xi_1^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 64800\beta+69120+15120\beta^2\\
&> 0 \text{ for all } \beta > -2.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_1(\lambda_j;\lambda_j;\lambda_i;\beta) &= 6(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^5 \geq 0 \text{ for all } \beta,\\
\xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 3(749\beta^2+2062\beta+1520)(\lambda_j-\lambda_i)^4 \geq 0 \text{ for all } \beta,\\
\xi_1^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 84(81\beta^2+176+230\beta)(\lambda_j-\lambda_i)^3 \geq 0 \text{ for all } \beta,\\
\xi_1^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 24(1970\beta+1616+651\beta^2)(\lambda_j-\lambda_i)^2 \geq 0 \text{ for all } \beta,\\
\xi_1^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 120(189\beta^2+592+646\beta)(\lambda_j-\lambda_i) \geq 0 \text{ for all } \beta,
\end{aligned}$$

and, hence, $\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

Differentiating $\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= 14(90\beta+96+21\beta^2)(\lambda_l-\lambda_i)^4 \\
&\quad +12(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i)^2 \\
&\quad +2(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^4 \\
&\quad +12(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^3,
\end{aligned}$$

$$\begin{aligned}
\xi_2^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 56(90\beta+96+21\beta^2)(\lambda_l-\lambda_i)^3 \\
&\quad +24(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i) \\
&\quad +36(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^2, \\
\xi_2^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 168(90\beta+96+21\beta^2)(\lambda_l-\lambda_i)^2 \\
&\quad +24(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2 \\
&\quad +72(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i), \\
\xi_2^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 336(90\beta+96+21\beta^2)(\lambda_l-\lambda_i) \\
&\quad +72(106\beta+16+63\beta^2)(\lambda_j-\lambda_i), \\
\xi_2^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 30240\beta+32256+7056\beta^2 \\
&> 0 \text{ for all } \beta>-2.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_2(\lambda_j;\lambda_j;\lambda_i;\beta) &= 48(79\beta+28\beta^2+60)(\lambda_j-\lambda_i)^4 \geq 0 \text{ for all } \beta, \\
\xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 12(918\beta+688+329\beta^2)(\lambda_j-\lambda_i)^3 \geq 0 \text{ for all } \beta, \\
\xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 72(119\beta^2+272+346\beta)(\lambda_j-\lambda_i)^2 \geq 0 \text{ for all } \beta, \\
\xi_2^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 72(526\beta+464+161\beta^2)(\lambda_j-\lambda_i) \geq 0 \text{ for all } \beta,
\end{aligned}$$

and, hence, $\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

Differentiating $\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(90\beta+96+21\beta^2)(\lambda_l-\lambda_i)^3 \\
&\quad +24(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2(\lambda_l-\lambda_i) \\
&\quad +24(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i)^2, \\
\xi_3^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 72(90\beta+96+21\beta^2)(\lambda_l-\lambda_i)^2 \\
&\quad +24(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2 \\
&\quad +48(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i), \\
\xi_3^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 144(90\beta+96+21\beta^2)(\lambda_l-\lambda_i) \\
&\quad +48(106\beta+16+63\beta^2)(\lambda_j-\lambda_i), \\
\xi_3^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 12960\beta+13824+3024\beta^2 \\
&> 0 \text{ for all } \beta>-2.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_3(\lambda_j;\lambda_j;\lambda_i;\beta) &= 24(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^3 \geq 0 \text{ for all } \beta,\\
\xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 48(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^2 \geq 0 \text{ for all } \beta,\\
\xi_3^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 96(188\beta+152+63\beta^2)(\lambda_j-\lambda_i) \geq 0 \text{ for all } \beta
\end{aligned}$$

and, hence, $\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

Differentiating $\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(90\beta+96+21\beta^2)(\lambda_l-\lambda_i)^2\\
&\quad +24(90\beta+96+21\beta^2)(\lambda_j-\lambda_i)^2\\
&\quad +24(106\beta+16+63\beta^2)(\lambda_j-\lambda_i)(\lambda_l-\lambda_i),\\
\xi_4^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 48(90\beta+96+21\beta^2)(\lambda_l-\lambda_i)\\
&\quad +24(106\beta+16+63\beta^2)(\lambda_j-\lambda_i),\\
\xi_4^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 4320\beta+4608+1008\beta^2\\
&> 0 \text{ for all } \beta > -2.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_4(\lambda_j;\lambda_j;\lambda_i;\beta) &= 24(105\beta^2+208+286\beta)(\lambda_j-\lambda_i)^2 \geq 0 \text{ for all } \beta,\\
\xi_4^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 24(105\beta^2+208+286\beta)(\lambda_j-\lambda_i) \geq 0 \text{ for all } \beta
\end{aligned}$$

and, hence, $\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

This completes the proof.

$\square$

Lemma B.9 Let

$$\begin{aligned}
\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 3(\beta^3+64+60\beta+12\beta^2)\hat{s}(0,2,3)\\
&\quad +(96\beta^2+9\beta^3+32+236\beta)\hat{s}(1,1,3),
\end{aligned}$$

where

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} (\lambda_j-\lambda_i)^{n_1}(\lambda_k-\lambda_i)^{n_2}(\lambda_l-\lambda_i)^{n_3}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof. Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)^3 \\
&\quad +(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)^3(\lambda_j-\lambda_i) \\
&\quad +3(96\beta^2+9\beta^3+32+236\beta)(\lambda_k-\lambda_i)^2(\lambda_l-\lambda_i)(\lambda_j-\lambda_i) \\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_k-\lambda_i)(\lambda_j-\lambda_i)^3 \\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_k-\lambda_i)(\lambda_l-\lambda_i)^3 \\
&\quad +3(36\beta^2+3\beta^3+192+180\beta)(\lambda_k-\lambda_i)^2(\lambda_j-\lambda_i)^2 \\
&\quad +3(36\beta^2+3\beta^3+192+180\beta)(\lambda_k-\lambda_i)^2(\lambda_l-\lambda_i)^2, \\
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(96\beta^2+9\beta^3+32+236\beta)(\lambda_k-\lambda_i)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i) \\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_j-\lambda_i)^3 \\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^3 \\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_k-\lambda_i)(\lambda_j-\lambda_i)^2 \\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_k-\lambda_i)(\lambda_l-\lambda_i)^2, \\
\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i) \\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_j-\lambda_i)^2 \\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^2,
\end{aligned}$$

Define

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta), \\
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta), \\
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta), \\
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \ge \lambda_l \ge \lambda_j \ge \lambda_i$ and $\beta \ge -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \ge \lambda_l \ge \lambda_j \ge \lambda_i > 0$ and $\beta \ge -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)^2(\lambda_j-\lambda_i)^3\\
&\quad +2(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)^4(\lambda_j-\lambda_i)\\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^2(\lambda_j-\lambda_i)^3\\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^3(\lambda_j-\lambda_i)^2\\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^5,\\
\xi_0^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 2(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)^3\\
&\quad +8(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)^3(\lambda_j-\lambda_i)\\
&\quad +4(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)^3\\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^2(\lambda_j-\lambda_i)^2\\
&\quad +10(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^4,\\
\xi_0^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 2(96\beta^2+9\beta^3+32+236\beta)(\lambda_j-\lambda_i)^3\\
&\quad +24(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)^2(\lambda_j-\lambda_i)\\
&\quad +4(36\beta^2+3\beta^3+192+180\beta)(\lambda_j-\lambda_i)^3\\
&\quad +12(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)^2\\
&\quad +40(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^3,\\
\xi_0^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 48(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)\\
&\quad +12(36\beta^2+3\beta^3+192+180\beta)(\lambda_j-\lambda_i)^2\\
&\quad +120(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^2,\\
\xi_0^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 48(96\beta^2+9\beta^3+32+236\beta)(\lambda_j-\lambda_i)\\
&\quad +240(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i),\\
\xi_0^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 8640\beta^2+720\beta^3+46080+43200\beta\\
&> 0 \text{ for all } \beta > -1.4242.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_0(\lambda_j;\lambda_j;\lambda_i;\beta) &= 3(\lambda_j-\lambda_i)^5(168\beta^2+15\beta^3+416+596\beta) \geq 0 \text{ for all } \beta \geq -0.9145,\\
\xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 10(\lambda_j-\lambda_i)^4(168\beta^2+15\beta^3+416+596\beta) \geq 0 \text{ for all } \beta \geq -0.9145,\\
\xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 2(\lambda_j-\lambda_i)^3(201\beta^3+8108\beta+5792+2256\beta^2) \geq 0 \text{ for all } \beta \geq -0.9393,\\
\xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 12(\lambda_j-\lambda_i)^2(780\beta^2+2924\beta+2240+69\beta^3) \geq 0 \text{ for all } \beta \geq -1.0173,\\
\xi_0^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 192(69\beta^2+6\beta^3+248+284\beta)(\lambda_j-\lambda_i) \geq 0 \text{ for all } \beta \geq -1.1738
\end{aligned}$$

and, hence, $\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

Differentiating $\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)^3 \\
&\quad +4(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)^3(\lambda_j-\lambda_i) \\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)^3 \\
&\quad +5(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^4 \\
&\quad +3(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^2(\lambda_j-\lambda_i)^2, \\
\xi_1^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (96\beta^2+9\beta^3+32+236\beta)(\lambda_j-\lambda_i)^3+12(96\beta^2 \\
&\quad +9\beta^3+32+236\beta)(\lambda_l-\lambda_i)^2(\lambda_j-\lambda_i) \\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_j-\lambda_i)^3 \\
&\quad +20(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^3 \\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)^2, \\
\xi_1^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i) \\
&\quad +60(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^2 \\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_j-\lambda_i)^2, \\
\xi_1^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(96\beta^2+9\beta^3+32+236\beta)(\lambda_j-\lambda_i) \\
&\quad +120(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i), \\
\xi_1^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 4320\beta^2+360\beta^3+23040+21600\beta \\
&> 0 \text{ for all } \beta > -1.4242.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_1(\lambda_j;\lambda_j;\lambda_i;\beta) &= 5(\lambda_j-\lambda_i)^4(168\beta^2+15\beta^3+416+596\beta) \geq 0 \text{ for all } \beta \geq -0.9145, \\
\xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= (\lambda_j-\lambda_i)^3(201\beta^3+8108\beta+5792+2256\beta^2) \geq 0 \text{ for all } \beta \geq -0.9393, \\
\xi_1^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 6(\lambda_j-\lambda_i)^2(780\beta^2+2924\beta+2240+69\beta^3) \geq 0 \text{ for all } \beta \geq -1.0173, \\
\xi_1^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 96(69\beta^2+6\beta^3+248+284\beta)(\lambda_j-\lambda_i) \geq 0 \text{ for all } \beta \geq -1.1738,
\end{aligned}$$

and, hence, $\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

Differentiating $\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)^2(\lambda_j-\lambda_i)\\
&\quad +2(36\beta^2+3\beta^3+192+180\beta)(\lambda_j-\lambda_i)^3\\
&\quad +8(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^3\\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)^2,\\
\xi_2^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 12(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)\\
&\quad +24(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^2\\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_j-\lambda_i)^2,\\
\xi_2^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 12(96\beta^2+9\beta^3+32+236\beta)(\lambda_j-\lambda_i)\\
&\quad +48(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i),\\
\xi_2^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 1728\beta^2+144\beta^3+9216+8640\beta\\
&> 0 \text{ for all } \beta > -1.4242.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_2(\lambda_j;\lambda_j;\lambda_i;\beta) &= 6(\lambda_j-\lambda_i)^3(192\beta^2+17\beta^3+544+716\beta)\geq 0 \text{ for all } \beta\geq -1.0079,\\
\xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 6(\lambda_j-\lambda_i)^2(372\beta^2+1372\beta+1024+33\beta^3)\geq 0 \text{ for all } \beta\geq -0.9877,\\
\xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 12(240\beta^2+21\beta^3+800+956\beta)(\lambda_j-\lambda_i)\geq 0 \text{ for all } \beta\geq -1.1217
\end{aligned}$$

and, hence, $\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

Differentiating $\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(96\beta^2+9\beta^3+32+236\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)\\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_j-\lambda_i)^2\\
&\quad +6(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i)^2,\\
\xi_3^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(96\beta^2+9\beta^3+32+236\beta)(\lambda_j-\lambda_i)\\
&\quad +12(36\beta^2+3\beta^3+192+180\beta)(\lambda_l-\lambda_i),\\
\xi_3^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 432\beta^2+36\beta^3+2304+2160\beta\\
&> 0 \text{ for all } \beta > -1.4242.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_3(\lambda_j;\lambda_j;\lambda_i;\beta) &= 6(\lambda_j-\lambda_i)^2(168\beta^2+15\beta^3+416+596\beta)\geq 0 \text{ for all } \beta\geq -0.9145,\\
\xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 6(168\beta^2+15\beta^3+416+596\beta)(\lambda_j-\lambda_i)\geq 0 \text{ for all } \beta\geq -0.9145
\end{aligned}$$

and, hence, $\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

This completes the proof.

$\square$

Lemma B.10 Let

$$\begin{aligned}\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (15\beta^2+44\beta+\beta^3+24)\hat{s}(0,0,3)\\ &\quad +3(3\beta^3+42+47\beta+20\beta^2)\hat{s}(0,1,2)\\ &\quad +3(9\beta^3+45\beta^2-24+46\beta)\hat{s}(1,1,1),\end{aligned}$$

where

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} (\lambda_j-\lambda_i)^{n_1}(\lambda_k-\lambda_i)^{n_2}(\lambda_l-\lambda_i)^{n_3}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof. Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 3(15\beta^2+\beta^3+24+44\beta)(\lambda_k-\lambda_i)^2\\ &\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_j-\lambda_i)^2\\ &\quad +(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)^2\\ &\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_k-\lambda_i)(\lambda_j-\lambda_i)\\ &\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_k-\lambda_i)(\lambda_l-\lambda_i)\\ &\quad +(135\beta^2+27\beta^3-72+138\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i),\\ \xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(15\beta^2+\beta^3+24+44\beta)(\lambda_k-\lambda_i)\\ &\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_j-\lambda_i)\\ &\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i),\\ \xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 90\beta^2+6\beta^3+144+264\beta\\ &> 0 \text{ for all } \beta > -0.7085,\end{aligned}$$

Define

$$\begin{aligned}\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta).\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (15\beta^2+\beta^3+24+44\beta)(\lambda_j-\lambda_i)^3\\
&\quad +2(15\beta^2+\beta^3+24+44\beta)(\lambda_l-\lambda_i)^3\\
&\quad +3(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)^2\\
&\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)^3\\
&\quad +(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)^2(\lambda_j-\lambda_i)\\
&\quad +(135\beta^2+27\beta^3-72+138\beta)(\lambda_l-\lambda_i)^2(\lambda_j-\lambda_i),\\
\xi_0^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(15\beta^2+\beta^3+24+44\beta)(\lambda_l-\lambda_i)^2\\
&\quad +3(60\beta^2+9\beta^3+126+141\beta)(\lambda_j-\lambda_i)^2\\
&\quad +6(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)^2\\
&\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)\\
&\quad +2(135\beta^2+27\beta^3-72+138\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i),\\
\xi_0^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 12(15\beta^2+\beta^3+24+44\beta)(\lambda_l-\lambda_i)\\
&\quad +12(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)\\
&\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_j-\lambda_i)\\
&\quad +2(135\beta^2+27\beta^3-72+138\beta)(\lambda_j-\lambda_i),\\
\xi_0^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 900\beta^2+120\beta^3+1800+2220\beta\\
&> 0 \text{ for all } \beta>-2.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_0(\lambda_j;\lambda_j;\lambda_i;\beta) &= 12(\lambda_j-\lambda_i)^3(\beta+3)(7\beta^2+24\beta+21)\geq 0 \text{ for all } \beta\geq -3,\\
\xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 3(\lambda_j-\lambda_i)^2(\beta+3)(53\beta^2+181\beta+154)\geq 0 \text{ for all } \beta\geq -1.6073,\\
\xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 6(\beta+3)(32\beta^2+119\beta+106)(\lambda_j-\lambda_i)\geq 0 \text{ for all } \beta\geq -1.4789
\end{aligned}$$

and, hence, $\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

Differentiating $\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= 3(15\beta^2+\beta^3+24+44\beta)(\lambda_l-\lambda_i)^2\\
&\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_j-\lambda_i)^2\\
&\quad +3(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)^2\\
&\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i)\\
&\quad +(135\beta^2+27\beta^3-72+138\beta)(\lambda_l-\lambda_i)(\lambda_j-\lambda_i),\\
\xi_1^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(15\beta^2+\beta^3+24+44\beta)(\lambda_l-\lambda_i)\\
&\quad +6(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i)\\
&\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_j-\lambda_i)\\
&\quad +(135\beta^2+27\beta^3-72+138\beta)(\lambda_j-\lambda_i),
\end{aligned}$$

$$\begin{aligned}\xi_1^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 450\beta^2+60\beta^3+900+1110\beta \\ &> 0 \text{ for all } \beta>-2.\end{aligned}$$

Now

$$\begin{aligned}\xi_1(\lambda_j;\lambda_j;\lambda_i;\beta) &= 3(\lambda_j-\lambda_i)^2(\beta+3)(31\beta^2+107\beta+98)\geq 0 \text{ for all } \beta\geq -3, \\ \xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta) &= 15(\beta+3)(\beta+2)(7\beta+12)(\lambda_j-\lambda_i)\geq 0 \text{ for all } \beta\geq -1.7143\end{aligned}$$

and, hence, $\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

Differentiating $\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(15\beta^2+\beta^3+24+44\beta)(\lambda_l-\lambda_i) \\ &\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_j-\lambda_i) \\ &\quad +2(60\beta^2+9\beta^3+126+141\beta)(\lambda_l-\lambda_i), \\ \xi_2^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 210\beta^2+24\beta^3+396+546\beta \\ &> 0 \text{ for all } \beta>-1.2120.\end{aligned}$$

Now

$$\xi_2(\lambda_j;\lambda_j;\lambda_i;\beta) = 6(\beta+3)(7\beta^2+34\beta+36)(\lambda_j-\lambda_i)\geq 0 \text{ for all } \beta\geq -1.5596$$

and, hence, $\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta)$ is non-negative.

This completes the proof.

$\square$

Lemma B.11 Let

$$\begin{aligned}\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 30t(0,4,4,4)+(8\beta-16)\,t(1,3,4,4) \\ &\quad +(9\beta-18)\,t(2,2,4,4)+\left(3\beta^2-18\beta+24\right)t(2,3,3,4) \\ &\quad +\left(2\beta^3-24\beta^2+88\beta-96\right)t(3,3,3,3),\end{aligned}$$

where

$$t(a,b,c,d)=\sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\mathrm{perm}(a,b,c,d)}}\lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_k^{n_4}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k\geq\lambda_l\geq\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Proof. Let

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 120\lambda_k^3\hat{s}(0,4,4)\\
&\quad +2(18\beta-36)\lambda_k^3\hat{s}(2,2,4)\\
&\quad +2(6\beta^2-36\beta+48)\lambda_k^3\hat{s}(2,3,3)\\
&\quad +3(2\beta^3-24\beta^2+88\beta-96)\lambda_k^2\hat{s}(3,3,3)\\
&\quad +3(3\beta^2-18\beta+24)\lambda_k^2\hat{s}(2,3,4)\\
&\quad +3(8\beta-16)\lambda_k^2\hat{s}(1,4,4)\\
&\quad +2(16\beta-32)\lambda_k^3\hat{s}(1,3,4)\\
&\quad +(18\beta-36)\lambda_k\hat{s}(2,3,4)\\
&\quad +(6\beta^2-36\beta+48)\lambda_k\hat{s}(3,3,4)\\
&\quad +(8\beta-16)\hat{s}(3,4,4),\\
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 360\lambda_k^2\hat{s}(0,4,4)\\
&\quad +6(18\beta-36)\lambda_k^2\hat{s}(2,2,4)\\
&\quad +6(6\beta^2-36\beta+48)\lambda_k^2\hat{s}(2,3,3)\\
&\quad +6(16\beta-32)\lambda_k^2\hat{s}(1,3,4)\\
&\quad +3(16\beta-32)\lambda_k\hat{s}(1,4,4)\\
&\quad +3(4\beta^3-48\beta^2+176\beta-192)\lambda_k\hat{s}(3,3,3)\\
&\quad +3(6\beta^2-36\beta+48)\lambda_k\hat{s}(2,3,4)\\
&\quad +(6\beta^2-36\beta+48)\hat{s}(3,3,4)\\
&\quad +(18\beta-36)\hat{s}(2,4,4),\\
\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 720\lambda_k\hat{s}(0,4,4)\\
&\quad +12(18\beta-36)\lambda_k\hat{s}(2,2,4)\\
&\quad +12(16\beta-32)\lambda_k\hat{s}(1,3,4)\\
&\quad +12(6\beta^2-36\beta+48)\lambda_k\hat{s}(2,3,3)\\
&\quad +(48\beta-96)\hat{s}(1,4,4)\\
&\quad +(12\beta^3-144\beta^2+528\beta-576)\hat{s}(3,3,3)\\
&\quad +(18\beta^2-108\beta+144)\hat{s}(2,3,4),\\
\xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 720\hat{s}(0,4,4)\\
&\quad +12(18\beta-36)\hat{s}(2,2,4)\\
&\quad +12(16\beta-32)\hat{s}(1,3,4)\\
&\quad +12(6\beta^2-36\beta+48)\hat{s}(2,3,3).
\end{aligned}$$

Define

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l^4,\\
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l^3,\\
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l^2,\\
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l,\\
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= 30\lambda_j^4\lambda_l^4 + 30\lambda_i^4\lambda_l^4\\
&\quad +(9\beta-18)\lambda_i^2\lambda_j^2\lambda_l^4\\
&\quad +(8\beta-16)\lambda_l^4(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\\
&\quad +(16\beta-32)\lambda_l^3(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\\
&\quad +(6\beta^2-36\beta+48)\lambda_l^3(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\\
&\quad +(3\beta^2-12)\lambda_l^2(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\\
&\quad +(-48-18\beta^2+52\beta+2\beta^3)\lambda_l^2\lambda_i^3\lambda_j^3\\
&\quad +(-20\beta+6\beta^2+16)\lambda_l(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\\
&\quad +(10+25\beta)\lambda_i^4\lambda_j^4,\\
\xi_0^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 120\lambda_l^3(\lambda_j^4+\lambda_i^4)\\
&\quad +(-72+36\beta)\lambda_i^2\lambda_j^2\lambda_l^3\\
&\quad +(-64+32\beta)\lambda_l^3(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\\
&\quad +(48\beta-96)\lambda_l^2(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\\
&\quad +(18\beta^2-108\beta+144)\lambda_l^2(\lambda_i^3\lambda_j^2+\lambda_i^2\lambda_j^3)\\
&\quad +(6\beta^2-24)\lambda_l(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\\
&\quad +(-36\beta^2+4\beta^3-96+104\beta)\lambda_i^3\lambda_j^3\lambda_l\\
&\quad +(-20\beta+6\beta^2+16)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3),\\
\xi_0^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 360\lambda_l^2(\lambda_i^4+\lambda_j^4)\\
&\quad +(-216+108\beta)\lambda_l^2\lambda_i^2\lambda_j^2\\
&\quad +(-192+96\beta)\lambda_l^2(\lambda_i^3\lambda_j+\lambda_i\lambda_j^3)\\
&\quad +(36\beta^2+288-216\beta)\lambda_l(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\\
&\quad +(-192+96\beta)\lambda_l(\lambda_i^4\lambda_j+\lambda_i\lambda_j^4)\\
&\quad +(6\beta^2-24)(\lambda_i^4\lambda_j^2+\lambda_i^2\lambda_j^4)\\
&\quad +(-36\beta^2+4\beta^3-96+104\beta)\lambda_i^3\lambda_j^3,
\end{aligned}$$

$$
\begin{aligned}
\xi_0^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 720\lambda_l(\lambda_i^4+\lambda_j^4) \\
&\quad +(192\beta-384)\lambda_l(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j) \\
&\quad +(-432+216\beta)\lambda_l\lambda_i^2\lambda_j^2 \\
&\quad (-192+96\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j) \\
&\quad +(36\beta^2+288-216\beta)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2), \\
\xi_0^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 720(\lambda_j^4+\lambda_i^4)+(-432+216\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(192\beta-384)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j).
\end{aligned}
$$

Define $\xi_{04}(\lambda_j;\lambda_i;\beta)=\xi_0^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{04} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{04} with respect to λ_j gives

$$
\begin{aligned}
\xi_{04}(\lambda_j;\lambda_i;\beta) &= 720(\lambda_j^4+\lambda_i^4)+(-432+216\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(192\beta-384)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j), \\
\xi_{04}^{(1)}(\lambda_j;\lambda_i;\beta) &= 2880\lambda_j^3+3(-384+192\beta)\lambda_i\lambda_j^2 \\
&\quad +2(-432+216\beta)\lambda_i^2\lambda_j+(-384+192\beta)\lambda_i^3, \\
\xi_{04}^{(2)}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^2+6(-384+192\beta)\lambda_i\lambda_j \\
&\quad +2(-432+216\beta)\lambda_i^2, \\
\xi_{04}^{(3)}(\lambda_j;\lambda_i;\beta) &= 17280\lambda_j+6(-384+192\beta)\lambda_i, \\
\xi_{04}^{(4)}(\lambda_j;\lambda_i;\beta) &= 17280>0.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{04}(\lambda_i;\lambda_i;\beta) &= 120\lambda_i^4(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}, \\
\xi_{04}^{(1)}(\lambda_i;\lambda_i;\beta) &= 240\lambda_i^3(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}, \\
\xi_{04}^{(2)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^2(38+11\beta)\geq 0 \text{ for all } \beta\geq -3.4545, \\
\xi_{04}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1152\lambda_i(13+\beta)\geq 0 \text{ for all } \beta\geq -13.
\end{aligned}
$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{03}(\lambda_j;\lambda_i;\beta)=\xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{03} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}
\xi_{03}(\lambda_j;\lambda_i;\beta) &= 720\lambda_j^4 + (-576+288\beta)\lambda_i\lambda_j^3 \\
&\quad +(36\beta^2-144)\lambda_i^2\lambda_j^2 + (36\beta^2-96-24\beta)\lambda_i^3\lambda_j \\
&\quad +(528+96\beta)\lambda_i^4, \\
\xi_{03}^{(1)}(\lambda_j;\lambda_i;\beta) &= 2880\lambda_j^3 + 3(-576+288\beta)\lambda_i\lambda_j^2 \\
&\quad +2(36\beta^2-144)\lambda_i^2\lambda_j + (36\beta^2-96-24\beta)\lambda_i^3, \\
\xi_{03}^{(2)}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^2 + 6(-576+288\beta)\lambda_i\lambda_j \\
&\quad +2(36\beta^2-144)\lambda_i^2, \\
\xi_{03}^{(3)}(\lambda_j;\lambda_i;\beta) &= 17280\lambda_j + 6(-576+288\beta)\lambda_i, \\
\xi_{03}^{(4)}(\lambda_j;\lambda_i;\beta) &= 17280 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{03}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i^4(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{03}^{(1)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^3(64+70\beta+9\beta^2) \geq 0 \text{ for all } \beta \geq -1.0583, \\
\xi_{03}^{(2)}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i^2(68+24\beta+\beta^2) \geq 0 \text{ for all } \beta \geq -3.2822, \\
\xi_{03}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i(8+\beta) \geq 0 \text{ for all } \beta \geq -8.
\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j;\lambda_i;\beta) = \xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned}
\xi_{02}(\lambda_j;\lambda_i;\beta) &= 360\lambda_j^4 + (-384+192\beta)\lambda_i\lambda_j^3 \\
&\quad +(42\beta^2-108\beta+48)\lambda_i^2\lambda_j^2 + (-16\beta+4\beta^3)\lambda_i^3\lambda_j \\
&\quad +(144+6\beta^2+96\beta)\lambda_i^4, \\
\xi_{02}^{(1)}(\lambda_j;\lambda_i;\beta) &= 1440\lambda_j^3 + 3(-384+192\beta)\lambda_i\lambda_j^2 \\
&\quad +2(42\beta^2-108\beta+48)\lambda_i^2\lambda_j + (-16\beta+4\beta^3)\lambda_i^3, \\
\xi_{02}^{(2)}(\lambda_j;\lambda_i;\beta) &= 4320\lambda_j^2 + 6(-384+192\beta)\lambda_i\lambda_j \\
&\quad +2(42\beta^2-108\beta+48)\lambda_i^2, \\
\xi_{02}^{(3)}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j + 6(-384+192\beta)\lambda_i, \\
\xi_{02}^{(4)}(\lambda_j;\lambda_i;\beta) &= 8640 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{02}(\lambda_i;\lambda_i;\beta) &= 4\lambda_i^4(\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(1)}(\lambda_i;\lambda_i;\beta) &= 4\lambda_i^3(\beta+3)(\beta+2)(\beta+16) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(2)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^2(8+\beta)(7\beta+22) \geq 0 \text{ for all } \beta \geq -\frac{22}{7}, \\
\xi_{02}^{(3)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i(11+2\beta) \geq 0 \text{ for all } \beta \geq -\frac{11}{2}.
\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j;\lambda_i;\beta) &= 120\lambda_j^4 + (-160+80\beta)\lambda_i\lambda_j^3 \\
&\quad +(-72\beta+48+24\beta^2)\lambda_i^2\lambda_j^2 + (8\beta+4\beta^3-12\beta^2)\lambda_i^3\lambda_j \\
&\quad +(28\beta+16+12\beta^2)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= 480\lambda_j^3 + 3(-160+80\beta)\lambda_i\lambda_j^2 \\
&\quad +2(-72\beta+48+24\beta^2)\lambda_i^2\lambda_j + (8\beta+4\beta^3-12\beta^2)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= 1440\lambda_j^2 + 6(-160+80\beta)\lambda_i\lambda_j \\
&\quad +2(-72\beta+48+24\beta^2)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j;\lambda_i;\beta) &= 2880\lambda_j + 6(-160+80\beta)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2880 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 4\lambda_i^4(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 4\lambda_i^3(\beta+4)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{01}^{(2)}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i^2(\beta+4)(\beta+3) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i;\lambda_i;\beta) &= 480\lambda_i(\beta+4) \geq 0 \text{ for all } \beta \geq -4.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{00} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= 30\lambda_j^4 + (24\beta-48)\lambda_i\lambda_j^3 \\
&\quad +(-27\beta+9\beta^2+18)\lambda_i^2\lambda_j^2 + (-6\beta^2+2\beta^3+4\beta)\lambda_i^3\lambda_j \\
&\quad +(9\beta^2+12+21\beta)\lambda_i^4, \\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= 120\lambda_j^3 + 3(24\beta-48)\lambda_i\lambda_j^2 \\
&\quad +2(-27\beta+9\beta^2+18)\lambda_i^2\lambda_j + (-6\beta^2+2\beta^3+4\beta)\lambda_i^3, \\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= 360\lambda_j^2 + 6(24\beta-48)\lambda_i\lambda_j \\
&\quad +2(-27\beta+9\beta^2+18)\lambda_i^2, \\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= 720\lambda_j + 6(24\beta-48)\lambda_i, \\
\xi_{00}^{(4)}(\lambda_j;\lambda_i;\beta) &= 720 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 2\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 2\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 18\lambda_i^2(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -2, \\
\xi_{00}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(\beta+3) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= 120\lambda_l^4(\lambda_j^4+\lambda_i^4) + (-72+36\beta)\lambda_l^4\lambda_i^2\lambda_j^2 \\
&\quad +(32\beta-64)\lambda_l^4(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j) + (-112+56\beta)\lambda_l^3(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j) \\
&\quad +(21\beta^2-126\beta+168)\lambda_l^3(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2) \\
&\quad +(-144+156\beta+6\beta^3-54\beta^2)\lambda_i^3\lambda_j^3\lambda_l^2 \\
&\quad +(-36+9\beta^2)\lambda_l^2(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2) \\
&\quad +(15\beta^2+40-50\beta)\lambda_l(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3) \\
&\quad +(50\beta+20)\lambda_i^4\lambda_j^4, \\
\xi_1^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 2(-36+9\beta^2)\lambda_l\lambda_i^2\lambda_j^4 + (15\beta^2+40-50\beta)\lambda_i^3\lambda_j^4 \\
&\quad +3(-112+56\beta)\lambda_l^2\lambda_i + 480\lambda_l^3\lambda_j^4 + (15\beta^2+40-50\beta)\lambda_i^4\lambda_j^3 \\
&\quad +4(32\beta-64)\lambda_l^3\lambda_i\lambda_j^3 + 2(-144+156\beta+6\beta^3-54\beta^2)\lambda_l\lambda_i^3\lambda_j^3 \\
&\quad +3(21\beta^2-126\beta+168)\lambda_l^2\lambda_i^2\lambda_j^3 + 2(-36+9\beta^2)\lambda_l\lambda_i^4\lambda_j^2 \\
&\quad +3(21\beta^2-126\beta+168)\lambda_l^2\lambda_i^3\lambda_j^2 + 4(-72+36\beta)\lambda_l^3\lambda_i^2\lambda_j^2 \\
&\quad +3(-112+56\beta)\lambda_l^2\lambda_i^4\lambda_j + 4(32\beta-64)\lambda_l^3\lambda_i^3\lambda_j \\
&\quad +480\lambda_i^4\lambda_l^3, \\
\xi_1^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 2(-36+9\beta^2)\lambda_i^2\lambda_j^4 + 6(-112+56\beta)\lambda_l\lambda_i\lambda_j^4 \\
&\quad +1440\lambda_l^2\lambda_j^4 + 12(32\beta-64)\lambda_l^2\lambda_i\lambda_j^3 + 2(-144+156\beta+6\beta^3-54\beta^2)\lambda_i^3\lambda_j^3 \\
&\quad +6(21\beta^2-126\beta+168)\lambda_l\lambda_i^2\lambda_j^3 + 2(-36+9\beta^2)\lambda_i^4\lambda_j^2 \\
&\quad +6(21\beta^2-126\beta+168)\lambda_l\lambda_i^3\lambda_j^2 + 12(-72+36\beta)\lambda_l^2\lambda_i^2\lambda_j^2 \\
&\quad +6(-112+56\beta)\lambda_l\lambda_i^4\lambda_j + 12(32\beta-64)\lambda_l^2\lambda_i^3\lambda_j \\
&\quad +1440\lambda_i^4\lambda_l^2, \\
\xi_1^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(-112+56\beta)\lambda_i\lambda_j^4 + 2880\lambda_l\lambda_j^4 \\
&\quad +24(32\beta-64)\lambda_l\lambda_i\lambda_j^3 + 6(21\beta^2-126\beta+168)\lambda_i^2\lambda_j^3 \\
&\quad +6(21\beta^2-126\beta+168)\lambda_i^3\lambda_j^2 + 24(-72+36\beta)\lambda_l\lambda_i^2\lambda_j^2 \\
&\quad +6(-112+56\beta)\lambda_i^4\lambda_j + 24(32\beta-64)\lambda_l\lambda_i^3\lambda_j \\
&\quad +2880\lambda_i^4\lambda_l, \\
\xi_1^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 2880\lambda_j^4 + 24(32\beta-64)\lambda_i\lambda_j^3 \\
&\quad +24(-72+36\beta)\lambda_i^2\lambda_j^2 + 24(32\beta-64)\lambda_i^3\lambda_j \\
&\quad +2880\lambda_i^4.
\end{aligned}$$

Define $\xi_{14}(\lambda_j; \lambda_i; \beta) = \xi_1^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{14} is non-negative for all

$\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned}
\xi_{14}(\lambda_j;\lambda_i;\beta) &= 2880\lambda_j^4 + 24(32\beta-64)\lambda_i\lambda_j^3 \\
&\quad +24(-72+36\beta)\lambda_i^2\lambda_j^2 + 24(32\beta-64)\lambda_i^3\lambda_j \\
&\quad +2880\lambda_i^4, \\
\xi_{14}^{(1)}(\lambda_j;\lambda_i;\beta) &= 11520\lambda_j^3 + 72(32\beta-64)\lambda_i\lambda_j^2 \\
&\quad +48(-72+36\beta)\lambda_i^2\lambda_j + 24(32\beta-64)\lambda_i^3, \\
\xi_{14}^{(2)}(\lambda_j;\lambda_i;\beta) &= 34560\lambda_j^2 + 144(32\beta-64)\lambda_i\lambda_j \\
&\quad +48(-72+36\beta)\lambda_i^2, \\
\xi_{14}^{(3)}(\lambda_j;\lambda_i;\beta) &= 69120\lambda_j + 144(32\beta-64)\lambda_i, \\
\xi_{14}^{(4)}(\lambda_j;\lambda_i;\beta) &= 69120 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{14}(\lambda_i;\lambda_i;\beta) &= 480\lambda_i^4(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{14}^{(1)}(\lambda_i;\lambda_i;\beta) &= 960\lambda_i^3(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{14}^{(2)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i^2(38+11\beta) \geq 0 \text{ for all } \beta \geq -3.4545, \\
\xi_{14}^{(3)}(\lambda_i;\lambda_i;\beta) &= 4608\lambda_i(13+\beta) \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j;\lambda_i;\beta) = \xi_1^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}
\xi_{13}(\lambda_j;\lambda_i;\beta) &= (2208+336\beta)\lambda_i^4 + (-528+12\beta+126\beta^2)\lambda_j\lambda_i^3 \\
&\quad +(108\beta-720+126\beta^2)\lambda_j^2\lambda_i^2 + (-2208+1104\beta)\lambda_j^3\lambda_i \\
&\quad +2880\lambda_j^4, \\
\xi_{13}^{(1)}(\lambda_j;\lambda_i;\beta) &= (-528+12\beta+126\beta^2)\lambda_i^3 + 2(108\beta-720+126\beta^2)\lambda_j\lambda_i^2 \\
&\quad +3(-2208+1104\beta)\lambda_j^2\lambda_i + 11520\lambda_j^3, \\
\xi_{13}^{(2)}(\lambda_j;\lambda_i;\beta) &= 2(108\beta-720+126\beta^2)\lambda_i^2 + 6(-2208+1104\beta)\lambda_j\lambda_i \\
&\quad +34560\lambda_j^2, \\
\xi_{13}^{(3)}(\lambda_j;\lambda_i;\beta) &= 6(-2208+1104\beta)\lambda_i + 69120\lambda_j, \\
\xi_{13}^{(4)}(\lambda_j;\lambda_i;\beta) &= 69120 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{13}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^4(7\beta+34)(3\beta+4) \geq 0 \text{ for all } \beta \geq -\frac{4}{3}, \\
\xi_{13}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(488+590\beta+63\beta^2) \geq 0 \text{ for all } \beta \geq -0.9169, \\
\xi_{13}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(190\beta+552+7\beta^2) \geq 0 \text{ for all } \beta \geq -3.3086, \\
\xi_{13}^{(3)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i(194+23\beta) \geq 0 \text{ for all } \beta \geq -8.4348.
\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j; \lambda_i; \beta) = \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{12} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned}
\xi_{12}(\lambda_j; \lambda_i; \beta) &= 1440\lambda_j^4 + (720\beta - 1440)\lambda_i\lambda_j^3 \\
&\quad +(72 - 324\beta + 144\beta^2)\lambda_i^2\lambda_j^2 + (12\beta^3 - 48 + 18\beta^2 - 60\beta)\lambda_i^3\lambda_j \\
&\quad +(696 + 336\beta + 18\beta^2)\lambda_i^4, \\
\xi_{12}^{(1)}(\lambda_j; \lambda_i; \beta) &= 5760\lambda_j^3 + 3(720\beta - 1440)\lambda_i\lambda_j^2 \\
&\quad +2(72 - 324\beta + 144\beta^2)\lambda_i^2\lambda_j + (12\beta^3 - 48 + 18\beta^2 - 60\beta)\lambda_i^3, \\
\xi_{12}^{(2)}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j^2 + 6(720\beta - 1440)\lambda_i\lambda_j \\
&\quad +2(72 - 324\beta + 144\beta^2)\lambda_i^2, \\
\xi_{12}^{(3)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j + 6(720\beta - 1440)\lambda_i, \\
\xi_{12}^{(4)}(\lambda_j; \lambda_i; \beta) &= 34560 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{12}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(\beta + 10)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(256 + 242\beta + 51\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.5098, \\
\xi_{12}^{(2)}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^2(122 + 51\beta + 4\beta^2) \geq 0 \text{ for all } \beta \geq -3.1906, \\
\xi_{12}^{(3)}(\lambda_i; \lambda_i; \beta) &= 4320\lambda_i(6 + \beta) \geq 0 \text{ for all } \beta \geq -6.
\end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j; \lambda_i; \beta) &= 480\lambda_j^4 + (296\beta - 592)\lambda_i\lambda_j^3 \\
&\quad +(144 - 234\beta + 81\beta^2)\lambda_i^2\lambda_j^2 + (12\beta - 30\beta^2 + 12\beta^3)\lambda_i^3\lambda_j \\
&\quad +(112 + 118\beta + 33\beta^2)\lambda_i^4, \\
\xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= 1920\lambda_j^3 + 3(296\beta - 592)\lambda_i\lambda_j^2 + 2(144 - 234\beta + 81\beta^2)\lambda_i^2\lambda_j \\
&\quad +(12\beta - 30\beta^2 + 12\beta^3)\lambda_i^3, \\
\xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= 5760\lambda_j^2 + 6(296\beta - 592)\lambda_i\lambda_j + 2(144 - 234\beta + 81\beta^2)\lambda_i^2, \\
\xi_{11}^{(3)}(\lambda_j; \lambda_i; \beta) &= 11520\lambda_j + 6(296\beta - 592)\lambda_i, \\
\xi_{11}^{(4)}(\lambda_j; \lambda_i; \beta) &= 11520 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(\beta + 3)(\beta + 2)^2 \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^3(\beta + 3)(\beta + 2)(6 + \beta) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^2(416 + 218\beta + 27\beta^2) \geq 0 \text{ for all } \beta \geq -3.0935, \\
\xi_{11}^{(3)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i(166 + 37\beta) \geq 0 \text{ for all } \beta \geq -\frac{166}{37}.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j;\lambda_i;\beta) = \xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= 120\lambda_j^4 + (88\beta - 176)\lambda_i\lambda_j^3 \\
&\quad + (60 - 90\beta + 30\beta^2)\lambda_i^2\lambda_j^2 + (12\beta - 18\beta^2 + 6\beta^3)\lambda_i^3\lambda_j \\
&\quad + (32 + 56\beta + 24\beta^2)\lambda_i^4, \\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= 480\lambda_j^3 + 3(88\beta - 176)\lambda_i\lambda_j^2 + 2(60 - 90\beta + 30\beta^2)\lambda_i^2\lambda_j \\
&\quad + (12\beta - 18\beta^2 + 6\beta^3)\lambda_i^3, \\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= 1440\lambda_j^2 + 6(88\beta - 176)\lambda_i\lambda_j \\
&\quad + 2(60 - 90\beta + 30\beta^2)\lambda_i^2, \\
\xi_{10}^{(3)}(\lambda_j;\lambda_i;\beta) &= 2880\lambda_j + 6(88\beta - 176)\lambda_i, \\
\xi_{10}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2880 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(\beta+3)(\beta+2)^2 > 0 \text{ for all } \beta > -2, \\
\xi_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^2(\beta+3)(5\beta+14) > 0 \text{ for all } \beta > -\frac{14}{5}, \\
\xi_{10}^{(3)}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i(38 + 11\beta) > 0 \text{ for all } \beta > -\frac{38}{11}.
\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= 360\lambda_j^4\lambda_l^4 + (-216 + 108\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&\quad + (96\beta - 192)\lambda_i\lambda_j^3\lambda_l^4 + (96\beta - 192)\lambda_i^3\lambda_j\lambda_l^4 \\
&\quad + 360\lambda_i^4\lambda_l^4 + (-288 + 144\beta)\lambda_i\lambda_j^4\lambda_l^3 \\
&\quad + (-288 + 144\beta)\lambda_i^4\lambda_j\lambda_l^3 + (-324\beta + 54\beta^2 + 432)\lambda_i^2\lambda_j^3\lambda_l^3 \\
&\quad + (-324\beta + 54\beta^2 + 432)\lambda_i^3\lambda_j^2\lambda_l^3 + (18\beta^2 + 18\beta - 108)\lambda_i^2\lambda_j^4\lambda_l^2 \\
&\quad + (-102\beta^2 - 240 + 276\beta + 12\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 + (18\beta^2 + 18\beta - 108)\lambda_i^4\lambda_j^2\lambda_l^2 \\
&\quad + (-48\beta + 24\beta^2)\lambda_i^3\lambda_j^4\lambda_l + (-48\beta + 24\beta^2)\lambda_i^4\lambda_j^3\lambda_l \\
&\quad + (228 + 66\beta)\lambda_i^4\lambda_j^4,
\end{aligned}$$

$$
\begin{aligned}
\xi_2^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 4(360\lambda_j^4+(-216+108\beta)\lambda_i^2\lambda_j^2\\
&\quad +(96\beta-192)\lambda_i\lambda_j^3+(96\beta-192)\lambda_i^3\lambda_j\\
&\quad +360\lambda_i^4)\lambda_l^3+3((-288+144\beta)\lambda_i\lambda_j^4\\
&\quad +(-288+144\beta)\lambda_i^4\lambda_j+(-324\beta+54\beta^2+432)\lambda_i^2\lambda_j^3\\
&\quad +(-324\beta+54\beta^2+432)\lambda_i^3\lambda_j^2)\lambda_l^2+2((18\beta^2+18\beta-108)\lambda_i^2\lambda_j^4\\
&\quad +(-102\beta^2-240+276\beta+12\beta^3)\lambda_i^3\lambda_j^3+(18\beta^2+18\beta-108)\lambda_i^4\lambda_j^2)\lambda_l\\
&\quad +(-48\beta+24\beta^2)\lambda_i^3\lambda_j^4+(-48\beta+24\beta^2)\lambda_i^4\lambda_j^3,\\
\xi_2^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 4320\lambda_j^4\lambda_l^2+12(-216+108\beta)\lambda_i^2\lambda_j^2\lambda_l^2\\
&\quad +12(96\beta-192)\lambda_i\lambda_j^3\lambda_l^2+12(96\beta-192)\lambda_i^3\lambda_j\lambda_l^2\\
&\quad +4320\lambda_i^4\lambda_l^2+6(-288+144\beta)\lambda_i\lambda_j^4\lambda_l\\
&\quad +6(-288+144\beta)\lambda_i^4\lambda_j\lambda_l+6(-324\beta+54\beta^2+432)\lambda_i^2\lambda_j^3\lambda_l\\
&\quad +6(-324\beta+54\beta^2+432)\lambda_i^3\lambda_j^2\lambda_l+2(18\beta^2+18\beta-108)\lambda_i^2\lambda_j^4\\
&\quad +2(-102\beta^2-240+276\beta+12\beta^3)\lambda_i^3\lambda_j^3+2(18\beta^2+18\beta-108)\lambda_i^4\lambda_j^2,\\
\xi_2^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^4\lambda_l+24(-216+108\beta)\lambda_i^2\lambda_j^2\lambda_l\\
&\quad +24(96\beta-192)\lambda_i\lambda_j^3\lambda_l+24(96\beta-192)\lambda_i^3\lambda_j\lambda_l\\
&\quad +8640\lambda_i^4\lambda_l+6(-288+144\beta)\lambda_i\lambda_j^4\\
&\quad +6(-288+144\beta)\lambda_i^4\lambda_j+6(-324\beta+54\beta^2+432)\lambda_i^2\lambda_j^3\\
&\quad +6(-324\beta+54\beta^2+432)\lambda_i^3\lambda_j^2,\\
\xi_2^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^4+24(-216+108\beta)\lambda_i^2\lambda_j^2\\
&\quad +24(96\beta-192)\lambda_i\lambda_j^3+24(96\beta-192)\lambda_i^3\lambda_j\\
&\quad +8640\lambda_i^4.
\end{aligned}
$$

Define $\xi_{24}(\lambda_j;\lambda_i;\beta)=\xi_2^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{24} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{24} with respect to λ_j gives

$$
\begin{aligned}
\xi_{24}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^4+24(-216+108\beta)\lambda_i^2\lambda_j^2\\
&\quad +24(96\beta-192)\lambda_i\lambda_j^3+24(96\beta-192)\lambda_i^3\lambda_j+8640\lambda_i^4,\\
\xi_{24}^{(1)}(\lambda_j;\lambda_i;\beta) &= 34560\lambda_j^3+48(-216+108\beta)\lambda_i^2\lambda_j+72(96\beta-192)\lambda_i\lambda_j^2+24(96\beta-192)\lambda_i^3,\\
\xi_{24}^{(2)}(\lambda_j;\lambda_i;\beta) &= 103680\lambda_j^2+48(-216+108\beta)\lambda_i^2+144(96\beta-192)\lambda_i\lambda_j,\\
\xi_{24}^{(3)}(\lambda_j;\lambda_i;\beta) &= 207360\lambda_j+144(96\beta-192)\lambda_i,\\
\xi_{24}^{(4)}(\lambda_j;\lambda_i;\beta) &= 207360>0.
\end{aligned}
$$

Now

$$\begin{aligned}
\xi_{24}(\lambda_i;\lambda_i;\beta) &= 1440\lambda_i^4(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5},\\
\xi_{24}^{(1)}(\lambda_i;\lambda_i;\beta) &= 2880\lambda_i^3(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5},\\
\xi_{24}^{(2)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i^2(38+11\beta) \geq 0 \text{ for all } \beta \geq -3.4545,\\
\xi_{24}^{(3)}(\lambda_i;\lambda_i;\beta) &= 13824\lambda_i(13+\beta) \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{23}(\lambda_j;\lambda_i;\beta) = \xi_2^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{23} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned}
\xi_{23}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^4 + (3168\beta - 6336)\lambda_i\lambda_j^3\\
&\quad +(-2592+648\beta+324\beta^2)\lambda_i^2\lambda_j^2 + (-2016+360\beta+324\beta^2)\lambda_i^3\lambda_j\\
&\quad +(6912+864\beta)\lambda_i^4,\\
\xi_{23}^{(1)}(\lambda_j;\lambda_i;\beta) &= 34560\lambda_j^3 + 3(3168\beta-6336)\lambda_i\lambda_j^2\\
&\quad +2(-2592+648\beta+324\beta^2)\lambda_i^2\lambda_j + (-2016+360\beta+324\beta^2)\lambda_i^3,\\
\xi_{23}^{(2)}(\lambda_j;\lambda_i;\beta) &= 103680\lambda_j^2 + 6(3168\beta-6336)\lambda_i\lambda_j\\
&\quad +2(-2592+648\beta+324\beta^2)\lambda_i^2,\\
\xi_{23}^{(3)}(\lambda_j;\lambda_i;\beta) &= 207360\lambda_j + 6(3168\beta-6336)\lambda_i,\\
\xi_{23}^{(4)}(\lambda_j;\lambda_i;\beta) &= 207360 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{23}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i^4(64+70\beta+9\beta^2) \geq 0 \text{ for all } \beta \geq -1.0583,\\
\xi_{23}^{(1)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^3(232+310\beta+27\beta^2) \geq 0 \text{ for all } \beta \geq -0.8048,\\
\xi_{23}^{(2)}(\lambda_i;\lambda_i;\beta) &= 216\lambda_i^2(\beta+28)(3\beta+10) \geq 0 \text{ for all } \beta \geq -\frac{10}{3},\\
\xi_{23}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i(98+11\beta) \geq 0 \text{ for all } \beta \geq -\frac{98}{11}.
\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j;\lambda_i;\beta) = \xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{22} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}
\xi_{22}(\lambda_j;\lambda_i;\beta) &= 4320\lambda_j^4 + (2016\beta - 4032)\lambda_i\lambda_j^3 \\
&\quad +(-216 - 612\beta + 360\beta^2)\lambda_i^2\lambda_j^2 + (24\beta^3 - 192 + 120\beta^2 - 240\beta)\lambda_i^3\lambda_j \\
&\quad +(2376 + 900\beta + 36\beta^2)\lambda_i^4, \\
\xi_{22}^{(1)}(\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^3 + 3(2016\beta - 4032)\lambda_i\lambda_j^2 \\
&\quad +2(-216 - 612\beta + 360\beta^2)\lambda_i^2\lambda_j + (24\beta^3 - 192 + 120\beta^2 - 240\beta)\lambda_i^3, \\
\xi_{22}^{(2)}(\lambda_j;\lambda_i;\beta) &= 51840\lambda_j^2 + 6(2016\beta - 4032)\lambda_i\lambda_j \\
&\quad +2(-216 - 612\beta + 360\beta^2)\lambda_i^2, \\
\xi_{22}^{(3)}(\lambda_j;\lambda_i;\beta) &= 103680\lambda_j + 6(2016\beta - 4032)\lambda_i, \\
\xi_{22}^{(4)}(\lambda_j;\lambda_i;\beta) &= 103680 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{22}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^4(\beta+2)(2\beta^2 + 39\beta + 94) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{22}^{(1)}(\lambda_i;\lambda_i;\beta) &= 24\lambda_i^3(190 + 191\beta + 35\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -1.2872, \\
\xi_{22}^{(2)}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i^2(378 + 151\beta + 10\beta^2) \geq 0 \text{ for all } \beta \geq -3.1679, \\
\xi_{22}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i(46 + 7\beta) \geq 0 \text{ for all } \beta \geq -\frac{46}{7}.
\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j;\lambda_i;\beta) = \xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{21} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}
\xi_{21}(\lambda_j;\lambda_i;\beta) &= 1440\lambda_j^4 + (816\beta - 1632)\lambda_i\lambda_j^3 \\
&\quad +(216 - 504\beta + 198\beta^2)\lambda_i^2\lambda_j^2 + (24\beta^3 + 48 - 18\beta^2 - 84\beta)\lambda_i^3\lambda_j \\
&\quad +(360 + 420\beta + 60\beta^2)\lambda_i^4, \\
\xi_{21}^{(1)}(\lambda_j;\lambda_i;\beta) &= 5760\lambda_j^3 + 3(816\beta - 1632)\lambda_i\lambda_j^2 \\
&\quad +2(216 - 504\beta + 198\beta^2)\lambda_i^2\lambda_j + (24\beta^3 + 48 - 18\beta^2 - 84\beta)\lambda_i^3, \\
\xi_{21}^{(2)}(\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^2 + 6(816\beta - 1632)\lambda_i\lambda_j \\
&\quad +2(216 - 504\beta + 198\beta^2)\lambda_i^2, \\
\xi_{21}^{(3)}(\lambda_j;\lambda_i;\beta) &= 34560\lambda_j + 6(816\beta - 1632)\lambda_i, \\
\xi_{21}^{(4)}(\lambda_j;\lambda_i;\beta) &= 34560 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{21}(\lambda_i;\lambda_i;\beta) &= 24\lambda_i^4(6+\beta)(\beta+3)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{21}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(4\beta+7)(\beta^2 + 14\beta + 32) \geq 0 \text{ for all } \beta \geq -\frac{7}{4}, \\
\xi_{21}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(220 + 108\beta + 11\beta^2) \geq 0 \text{ for all } \beta \geq -2.8844, \\
\xi_{21}^{(3)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i(86 + 17\beta) \geq 0 \text{ for all } \beta \geq -\frac{86}{17}.
\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j;\lambda_i;\beta) = \xi_2(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{20} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}
\xi_{20}(\lambda_j;\lambda_i;\beta) &= 360\lambda_j^4 + (240\beta - 480)\lambda_i\lambda_j^3 \\
&\quad +(108 - 198\beta + 72\beta^2)\lambda_i^2\lambda_j^2 + (-24\beta^2 + 12\beta^3)\lambda_i^3\lambda_j \\
&\quad +(192 + 180\beta + 42\beta^2)\lambda_i^4, \\
\xi_{20}^{(1)}(\lambda_j;\lambda_i;\beta) &= 1440\lambda_j^3 + 3(240\beta - 480)\lambda_i\lambda_j^2 \\
&\quad +2(108 - 198\beta + 72\beta^2)\lambda_i^2\lambda_j + (-24\beta^2 + 12\beta^3)\lambda_i^3, \\
\xi_{20}^{(2)}(\lambda_j;\lambda_i;\beta) &= 4320\lambda_j^2 + 6(240\beta - 480)\lambda_i\lambda_j \\
&\quad +2(108 - 198\beta + 72\beta^2)\lambda_i^2, \\
\xi_{20}^{(3)}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j + 6(240\beta - 480)\lambda_i, \\
\xi_{20}^{(4)}(\lambda_j;\lambda_i;\beta) &= 8640 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{20}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\
\xi_{20}^{(1)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^3(6 + \beta)(\beta + 3)(\beta + 1) > 0 \text{ for all } \beta > -1, \\
\xi_{20}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(46 + 29\beta + 4\beta^2) > 0 \text{ for all } \beta > -2.3441, \\
\xi_{20}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1440\lambda_i(\beta + 4) > 0 \text{ for all } \beta > -4.
\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &= 720\lambda_j^4\lambda_l^4 + (-432 + 216\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&\quad +(-384 + 192\beta)\lambda_i\lambda_j^3\lambda_l^4 + (-384 + 192\beta)\lambda_i^3\lambda_j\lambda_l^4 \\
&\quad +720\lambda_i^4\lambda_l^4 + (240\beta - 480)\lambda_i\lambda_j^4\lambda_l^3 \\
&\quad +(240\beta - 480)\lambda_i^4\lambda_j\lambda_l^3 + (90\beta^2 + 720 - 540\beta)\lambda_i^2\lambda_j^3\lambda_l^3 \\
&\quad +(90\beta^2 + 720 - 540\beta)\lambda_i^3\lambda_j^2\lambda_l^3 + (-288 + 108\beta + 18\beta^2)\lambda_i^2\lambda_j^4\lambda_l^2 \\
&\quad +(96\beta - 72\beta^2 + 12\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 + (-288 + 108\beta + 18\beta^2)\lambda_i^4\lambda_j^2\lambda_l^2 \\
&\quad +(84\beta - 240 + 18\beta^2)\lambda_i^3\lambda_j^4\lambda_l + (84\beta - 240 + 18\beta^2)\lambda_i^4\lambda_j^3\lambda_l \\
&\quad +(624 + 48\beta)\lambda_i^4\lambda_j^4,
\end{aligned}$$

$$
\begin{aligned}
\xi_3^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 2880\lambda_j^4\lambda_l^3 + 4(-432+216\beta)\lambda_i^2\lambda_j^2\lambda_l^3 \\
&\quad +4(-384+192\beta)\lambda_i\lambda_j^3\lambda_l^3 + 4(-384+192\beta)\lambda_i^3\lambda_j\lambda_l^3 \\
&\quad +2880\lambda_i^4\lambda_l^3 + 3(240\beta-480)\lambda_i\lambda_j^4\lambda_l^2 \\
&\quad +3(240\beta-480)\lambda_i^4\lambda_j\lambda_l^2 + 3(90\beta^2+720-540\beta)\lambda_i^2\lambda_j^3\lambda_l^2 \\
&\quad +3(90\beta^2+720-540\beta)\lambda_i^3\lambda_j^2\lambda_l^2 + 2(-288+108\beta+18\beta^2)\lambda_i^2\lambda_j^4\lambda_l \\
&\quad +2(96\beta-72\beta^2+12\beta^3)\lambda_i^3\lambda_j^3\lambda_l + 2(-288+108\beta+18\beta^2)\lambda_i^4\lambda_j^2\lambda_l \\
&\quad +(84\beta-240+18\beta^2)\lambda_i^3\lambda_j^4 + (84\beta-240+18\beta^2)\lambda_i^4\lambda_j^3, \\
\xi_3^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^4\lambda_l^2 + 12(-432+216\beta)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&\quad +12(-384+192\beta)\lambda_i\lambda_j^3\lambda_l^2 + 12(-384+192\beta)\lambda_i^3\lambda_j\lambda_l^2 \\
&\quad +8640\lambda_i^4\lambda_l^2 + 6(240\beta-480)\lambda_i\lambda_j^4\lambda_l \\
&\quad +6(240\beta-480)\lambda_i^4\lambda_j\lambda_l + 6(90\beta^2+720-540\beta)\lambda_i^2\lambda_j^3\lambda_l \\
&\quad +6(90\beta^2+720-540\beta)\lambda_i^3\lambda_j^2\lambda_l + 2(-288+108\beta+18\beta^2)\lambda_i^2\lambda_j^4 \\
&\quad +2(96\beta-72\beta^2+12\beta^3)\lambda_i^3\lambda_j^3 + 2(-288+108\beta+18\beta^2)\lambda_i^4\lambda_j^2, \\
\xi_3^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^4\lambda_l + 24(-432+216\beta)\lambda_i^2\lambda_j^2\lambda_l \\
&\quad +24(-384+192\beta)\lambda_i\lambda_j^3\lambda_l + 24(-384+192\beta)\lambda_i^3\lambda_j\lambda_l \\
&\quad +17280\lambda_i^4\lambda_l + 6(240\beta-480)\lambda_i\lambda_j^4 \\
&\quad +6(240\beta-480)\lambda_i^4\lambda_j + 6(90\beta^2+720-540\beta)\lambda_i^2\lambda_j^3 \\
&\quad +6(90\beta^2+720-540\beta)\lambda_i^3\lambda_j^2, \\
\xi_3^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^4 + 24(-432+216\beta)\lambda_i^2\lambda_j^2 \\
&\quad +24(-384+192\beta)\lambda_i\lambda_j^3 + 24(-384+192\beta)\lambda_i^3\lambda_j \\
&\quad +17280\lambda_i^4.
\end{aligned}
$$

Define $\xi_{34}(\lambda_j;\lambda_i;\beta) = \xi_3^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{34} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{34} with respect to λ_j gives

$$
\begin{aligned}
\xi_{34}(\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^4 + 24(-432+216\beta)\lambda_i^2\lambda_j^2 \\
&\quad +24(-384+192\beta)\lambda_i\lambda_j^3 + 24(-384+192\beta)\lambda_i^3\lambda_j + 17280\lambda_i^4, \\
\xi_{34}^{(1)}(\lambda_j;\lambda_i;\beta) &= 69120\lambda_j^3 + 48(-432+216\beta)\lambda_i^2\lambda_j \\
&\quad +72(-384+192\beta)\lambda_i\lambda_j^2 + 24(-384+192\beta)\lambda_i^3, \\
\xi_{34}^{(2)}(\lambda_j;\lambda_i;\beta) &= 207360\lambda_j^2 + 48(-432+216\beta)\lambda_i^2 \\
&\quad +144(-384+192\beta)\lambda_i\lambda_j, \\
\xi_{34}^{(3)}(\lambda_j;\lambda_i;\beta) &= 414720\lambda_j + 144(-384+192\beta)\lambda_i, \\
\xi_{34}^{(4)}(\lambda_j;\lambda_i;\beta) &= 414720 > 0.
\end{aligned}
$$

Now

$$\begin{aligned}
\xi_{34}(\lambda_i;\lambda_i;\beta) &= 2880\lambda_i^4(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{34}^{(1)}(\lambda_i;\lambda_i;\beta) &= 5760\lambda_i^3(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{34}^{(2)}(\lambda_i;\lambda_i;\beta) &= 3456\lambda_i^2(38+11\beta) \geq 0 \text{ for all } \beta \geq -\frac{38}{11}, \\
\xi_{34}^{(3)}(\lambda_i;\lambda_i;\beta) &= 27648\lambda_i(13+\beta) \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{33}(\lambda_j;\lambda_i;\beta) = \xi_3^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{33} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{33} with respect to λ_j gives

$$\begin{aligned}
\xi_{33}(\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^4 + (-12096+6048\beta)\lambda_i\lambda_j^3 \\
&\quad +(-6048+1944\beta+540\beta^2)\lambda_i^2\lambda_j^2 + (1368\beta-4896+540\beta^2)\lambda_i^3\lambda_j \\
&\quad +(14400+1440\beta)\lambda_i^4, \\
\xi_{33}^{(1)}(\lambda_j;\lambda_i;\beta) &= 69120\lambda_j^3 + 3(-12096+6048\beta)\lambda_i\lambda_j^2 \\
&\quad +2(-6048+1944\beta+540\beta^2)\lambda_i^2\lambda_j + (1368\beta-4896+540\beta^2)\lambda_i^3, \\
\xi_{33}^{(2)}(\lambda_j;\lambda_i;\beta) &= 207360\lambda_j^2 + 6(-12096+6048\beta)\lambda_i\lambda_j \\
&\quad +2(-6048+1944\beta+540\beta^2)\lambda_i^2, \\
\xi_{33}^{(3)}(\lambda_j;\lambda_i;\beta) &= 414720\lambda_j + 6(-12096+6048\beta)\lambda_i, \\
\xi_{33}^{(4)}(\lambda_j;\lambda_i;\beta) &= 414720 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{33}(\lambda_i;\lambda_i;\beta) &= 1080\lambda_i^4(8+10\beta+\beta^2) \geq 0 \text{ for all } \beta \geq -0.8769, \\
\xi_{33}^{(1)}(\lambda_i;\lambda_i;\beta) &= 180\lambda_i^3(88+130\beta+9\beta^2) \geq 0 \text{ for all } \beta \geq -0.7120, \\
\xi_{33}^{(2)}(\lambda_i;\lambda_i;\beta) &= 216\lambda_i^2(568+186\beta+5\beta^2) \geq 0 \text{ for all } \beta \geq -3.3566, \\
\xi_{33}^{(3)}(\lambda_i;\lambda_i;\beta) &= 5184\lambda_i(66+7\beta) \geq 0 \text{ for all } \beta \geq -\frac{66}{7}.
\end{aligned}$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{32}(\lambda_j;\lambda_i;\beta) = \xi_3^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{32} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}
\xi_{32}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^4 + (3744\beta - 7488)\lambda_i\lambda_j^3 \\
&\quad +(-1440 - 432\beta + 576\beta^2)\lambda_i^2\lambda_j^2 + (24\beta^3 - 288 + 396\beta^2 - 744\beta)\lambda_i^3\lambda_j \\
&\quad +(5184 + 1656\beta + 36\beta^2)\lambda_i^4, \\
\xi_{32}^{(1)}(\lambda_j;\lambda_i;\beta) &= 34560\lambda_j^3 + 3(3744\beta - 7488)\lambda_i\lambda_j^2 \\
&\quad +2(-1440 - 432\beta + 576\beta^2)\lambda_i^2\lambda_j + (24\beta^3 - 288 + 396\beta^2 - 744\beta)\lambda_i^3, \\
\xi_{32}^{(2)}(\lambda_j;\lambda_i;\beta) &= 103680\lambda_j^2 + 6(3744\beta - 7488)\lambda_i\lambda_j \\
&\quad +2(-1440 - 432\beta + 576\beta^2)\lambda_i^2, \\
\xi_{32}^{(3)}(\lambda_j;\lambda_i;\beta) &= 207360\lambda_j + 6(3744\beta - 7488)\lambda_i, \\
\xi_{32}^{(4)}(\lambda_j;\lambda_i;\beta) &= 207360 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i;\lambda_i;\beta) &= 24\lambda_i^4(\beta+2)(\beta^2 + 40\beta + 96) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{32}^{(1)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^3(744 + 802\beta + 129\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1292, \\
\xi_{32}^{(2)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i^2(194 + 75\beta + 4\beta^2) \geq 0 \text{ for all } \beta \geq -3.0988, \\
\xi_{32}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i(94 + 13\beta) \geq 0 \text{ for all } \beta \geq -\frac{94}{13}.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j;\lambda_i;\beta) = \xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{31} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j;\lambda_i;\beta) &= 2880\lambda_j^4 + (1488\beta - 2976)\lambda_i\lambda_j^3 \\
&\quad +(-144 - 540\beta + 306\beta^2)\lambda_i^2\lambda_j^2 + (24\beta^3 + 384 + 144\beta^2 - 576\beta)\lambda_i^3\lambda_j \\
&\quad +(624 + 1020\beta + 54\beta^2)\lambda_i^4, \\
\xi_{31}^{(1)}(\lambda_j;\lambda_i;\beta) &= 11520\lambda_j^3 + 3(1488\beta - 2976)\lambda_i\lambda_j^2 \\
&\quad +2(-144 - 540\beta + 306\beta^2)\lambda_i^2\lambda_j + (24\beta^3 + 384 + 144\beta^2 - 576\beta)\lambda_i^3, \\
\xi_{31}^{(2)}(\lambda_j;\lambda_i;\beta) &= 34560\lambda_j^2 + 6(1488\beta - 2976)\lambda_i\lambda_j \\
&\quad +2(-144 - 540\beta + 306\beta^2)\lambda_i^2, \\
\xi_{31}^{(3)}(\lambda_j;\lambda_i;\beta) &= 69120\lambda_j + 6(1488\beta - 2976)\lambda_i, \\
\xi_{31}^{(4)}(\lambda_j;\lambda_i;\beta) &= 69120 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{31}(\lambda_i;\lambda_i;\beta) &= 24\lambda_i^4(32 + 58\beta + 21\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -0.7461, \\
\xi_{31}^{(1)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^3(224 + 234\beta + 63\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.6603, \\
\xi_{31}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(456 + 218\beta + 17\beta^2) \geq 0 \text{ for all } \beta \geq -2.6319, \\
\xi_{31}^{(3)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i(178 + 31\beta) \geq 0 \text{ for all } \beta \geq -\frac{178}{31}.
\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j;\lambda_i;\beta) = \xi_3(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}
\xi_{30}(\lambda_j;\lambda_i;\beta) &= 720\lambda_j^4 + (-864+432\beta)\lambda_i\lambda_j^3 \\
&\quad +(-216\beta+108\beta^2)\lambda_i^2\lambda_j^2 + (96-168\beta+36\beta^2+12\beta^3)\lambda_i^3\lambda_j \\
&\quad +(336+480\beta+36\beta^2)\lambda_i^4, \\
\xi_{30}^{(1)}(\lambda_j;\lambda_i;\beta) &= 2880\lambda_j^3 + 3(-864+432\beta)\lambda_i\lambda_j^2 \\
&\quad +2(-216\beta+108\beta^2)\lambda_i^2\lambda_j + (96-168\beta+36\beta^2+12\beta^3)\lambda_i^3, \\
\xi_{30}^{(2)}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^2 + 6(-864+432\beta)\lambda_i\lambda_j \\
&\quad +2(-216\beta+108\beta^2)\lambda_i^2, \\
\xi_{30}^{(3)}(\lambda_j;\lambda_i;\beta) &= 17280\lambda_j + 6(-864+432\beta)\lambda_i, \\
\xi_{30}^{(4)}(\lambda_j;\lambda_i;\beta) &= 17280 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{30}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^4(\beta+3)(\beta^2+12\beta+8) > 0 \text{ for all } \beta > -0.7085, \\
\xi_{30}^{(1)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^3(32+58\beta+21\beta^2+\beta^3) > 0 \text{ for all } \beta > -0.7461, \\
\xi_{30}^{(2)}(\lambda_i;\lambda_i;\beta) &= 216\lambda_i^2(8+\beta)(\beta+2) > 0 \text{ for all } \beta > -2, \\
\xi_{30}^{(3)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i(14+3\beta) > 0 \text{ for all } \beta > -\frac{14}{3}.
\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &= 720\lambda_j^4\lambda_l^4 + (-432+216\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&\quad +(-384+192\beta)\lambda_i\lambda_j^3\lambda_l^4 + (-384+192\beta)\lambda_i^3\lambda_j\lambda_l^4 \\
&\quad +720\lambda_i^4\lambda_l^4 + (-384+192\beta)\lambda_i\lambda_j^4\lambda_l^3 \\
&\quad +(-384+192\beta)\lambda_i^4\lambda_j\lambda_l^3 + (576-432\beta+72\beta^2)\lambda_i^2\lambda_j^3\lambda_l^3 \\
&\quad +(576-432\beta+72\beta^2)\lambda_i^3\lambda_j^2\lambda_l^3 + (-432+216\beta)\lambda_i^2\lambda_j^4\lambda_l^2 \\
&\quad +(576-432\beta+72\beta^2)\lambda_i^3\lambda_j^3\lambda_l^2 + (-432+216\beta)\lambda_i^4\lambda_j^2\lambda_l^2 \\
&\quad +(-384+192\beta)\lambda_i^3\lambda_j^4\lambda_l + (-384+192\beta)\lambda_i^4\lambda_j^3\lambda_l \\
&\quad +720\lambda_i^4\lambda_j^4,
\end{aligned}$$

$$
\begin{aligned}
\xi_4^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 2880\lambda_j^4\lambda_l^3 + 4(-432+216\beta)\lambda_i^2\lambda_j^2\lambda_l^3 \\
&\quad +4(-384+192\beta)\lambda_i\lambda_j^3\lambda_l^3 + 4(-384+192\beta)\lambda_i^3\lambda_j\lambda_l^3 \\
&\quad +2880\lambda_i^4\lambda_l^3 + 3(-384+192\beta)\lambda_i\lambda_j^4\lambda_l^2 \\
&\quad +3(-384+192\beta)\lambda_i^4\lambda_j\lambda_l^2 + 3(576-432\beta+72\beta^2)\lambda_i^2\lambda_j^3\lambda_l^2 \\
&\quad +3(576-432\beta+72\beta^2)\lambda_i^3\lambda_j^2\lambda_l^2 + 2(-432+216\beta)\lambda_i^2\lambda_j^4\lambda \\
&\quad +2(576-432\beta+72\beta^2)\lambda_i^3\lambda_j^3\lambda + 2(-432+216\beta)\lambda_i^4\lambda_j^2\lambda \\
&\quad +(-384+192\beta)\lambda_i^3\lambda_j^4 + (-384+192\beta)\lambda_i^4\lambda_j^3, \\
\xi_4^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^4\lambda_l^2 + 12(-432+216\beta)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&\quad +12(-384+192\beta)\lambda_i\lambda_j^3\lambda_l^2 + 12(-384+192\beta)\lambda_i^3\lambda_j\lambda_l^2 \\
&\quad +8640\lambda_i^4\lambda_l^2 + 6(-384+192\beta)\lambda_i\lambda_j^4\lambda_l \\
&\quad +6(-384+192\beta)\lambda_i^4\lambda_j\lambda_l + 6(576-432\beta+72\beta^2)\lambda_i^2\lambda_j^3\lambda_l \\
&\quad +6(576-432\beta+72\beta^2)\lambda_i^3\lambda_j^2\lambda_l + 2(-432+216\beta)\lambda_i^2\lambda_j^4 \\
&\quad +2(576-432\beta+72\beta^2)\lambda_i^3\lambda_j^3 + 2(-432+216\beta)\lambda_i^4\lambda_j^2, \\
\xi_4^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^4\lambda_l + 24(-432+216\beta)\lambda_i^2\lambda_j^2\lambda_l \\
&\quad +24(-384+192\beta)\lambda_i\lambda_j^3\lambda_l + 24(-384+192\beta)\lambda_i^3\lambda_j\lambda_l \\
&\quad +17280\lambda_i^4\lambda_l + 6(-384+192\beta)\lambda_i\lambda_j^4 \\
&\quad +6(-384+192\beta)\lambda_i^4\lambda_j + 6(576-432\beta+72\beta^2)\lambda_i^2\lambda_j^3 \\
&\quad +6(576-432\beta+72\beta^2)\lambda_i^3\lambda_j^2, \\
\xi_4^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^4 + 24(-432+216\beta)\lambda_i^2\lambda_j^2 \\
&\quad +24(-384+192\beta)\lambda_i\lambda_j^3 + 24(-384+192\beta)\lambda_i^3\lambda_j \\
&\quad +17280\lambda_i^4.
\end{aligned}
$$

Define $\xi_{44}(\lambda_j;\lambda_i;\beta) = \xi_4^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{44} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{44} with respect to λ_j gives

$$
\begin{aligned}
\xi_{44}(\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^4 + (-9216+4608\beta)\lambda_i\lambda_j^3 \\
&\quad +(-10368+5184\beta)\lambda_i^2\lambda_j^2 + (-9216+4608\beta)\lambda_i^3\lambda_j \\
&\quad +17280\lambda_i^4, \\
\xi_{44}^{(1)}(\lambda_j;\lambda_i;\beta) &= 69120\lambda_j^3 + 3(-9216+4608\beta)\lambda_i\lambda_j^2 \\
&\quad +2(-10368+5184\beta)\lambda_i^2\lambda_j + (-9216+4608\beta)\lambda_i^3, \\
\xi_{44}^{(2)}(\lambda_j;\lambda_i;\beta) &= 207360\lambda_j^2 + 6(-9216+4608\beta)\lambda_i\lambda_j \\
&\quad +2(-10368+5184\beta)\lambda_i^2, \\
\xi_{44}^{(3)}(\lambda_j;\lambda_i;\beta) &= 414720\lambda_j + 6(-9216+4608\beta)\lambda_i, \\
\xi_{44}^{(4)}(\lambda_j;\lambda_i;\beta) &= 414720 > 0.
\end{aligned}
$$

Now

$$\begin{aligned}
\xi_{44}(\lambda_i;\lambda_i;\beta) &= 2880\lambda_i^4(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5},\\
\xi_{44}^{(1)}(\lambda_i;\lambda_i;\beta) &= 5760\lambda_i^3(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5},\\
\xi_{44}^{(2)}(\lambda_i;\lambda_i;\beta) &= 3456\lambda_i^2(38+11\beta) \geq 0 \text{ for all } \beta \geq -\frac{38}{11},\\
\xi_{44}^{(3)}(\lambda_i;\lambda_i;\beta) &= 27648\lambda_i(13+\beta) \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Hence, $\xi_4^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{43}(\lambda_j;\lambda_i;\beta) = \xi_4^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{43} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{43} with respect to λ_j gives

$$\begin{aligned}
\xi_{43}(\lambda_j;\lambda_i;\beta) &= 17280\lambda_j^4 + (-11520+5760\beta)\lambda_i\lambda_j^3\\
&\quad +(-6912+2592\beta+432\beta^2)\lambda_i^2\lambda_j^2 + (2016\beta-5760+432\beta^2)\lambda_i^3\lambda_j\\
&\quad +(14976+1152\beta)\lambda_i^4,\\
\xi_{43}^{(1)}(\lambda_j;\lambda_i;\beta) &= 69120\lambda_j^3 + 3(-11520+5760\beta)\lambda_i\lambda_j^2\\
&\quad +2(-6912+2592\beta+432\beta^2)\lambda_i^2\lambda_j + (2016\beta-5760+432\beta^2)\lambda_i^3,\\
\xi_{43}^{(2)}(\lambda_j;\lambda_i;\beta) &= 207360\lambda_j^2 + 6(-11520+5760\beta)\lambda_i\lambda_j\\
&\quad +2(-6912+2592\beta+432\beta^2)\lambda_i^2,\\
\xi_{43}^{(3)}(\lambda_j;\lambda_i;\beta) &= 414720\lambda_j + 6(-11520+5760\beta)\lambda_i,\\
\xi_{43}^{(4)}(\lambda_j;\lambda_i;\beta) &= 414720 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{43}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i^4(28+40\beta+3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412,\\
\xi_{43}^{(1)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^3(104+170\beta+9\beta^2) \geq 0 \text{ for all } \beta \geq -0.6330,\\
\xi_{43}^{(2)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^2(144+46\beta+\beta^2) \geq 0 \text{ for all } \beta \geq -3.3786,\\
\xi_{43}^{(3)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i(\beta+10) \geq 0 \text{ for all } \beta \geq -10.
\end{aligned}$$

Hence, $\xi_4^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{42}(\lambda_j;\lambda_i;\beta) = \xi_4^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{42} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned}
\xi_{42}(\lambda_j;\lambda_i;\beta) &= 8640\lambda_j^4 + (-6912+3456\beta)\lambda_i\lambda_j^3 \\
&\quad +(-2592+432\beta+432\beta^2)\lambda_i^2\lambda_j^2 + (-1152\beta+576\beta^2)\lambda_i^3\lambda_j \\
&\quad +(5472+1584\beta)\lambda_i^4, \\
\xi_{42}^{(1)}(\lambda_j;\lambda_i;\beta) &= 34560\lambda_j^3 + 3(-6912+3456\beta)\lambda_i\lambda_j^2 + 2(-2592+432\beta \\
&\quad +432\beta^2)\lambda_i^2\lambda_j + (-1152\beta+576\beta^2)\lambda_i^3, \\
\xi_{42}^{(2)}(\lambda_j;\lambda_i;\beta) &= 103680\lambda_j^2 + 6(-6912+3456\beta)\lambda_i\lambda_j + 2(-2592+432\beta \\
&\quad +432\beta^2)\lambda_i^2, \\
\xi_{42}^{(3)}(\lambda_j;\lambda_i;\beta) &= 207360\lambda_j + 6(-6912+3456\beta)\lambda_i, \\
\xi_{42}^{(4)}(\lambda_j;\lambda_i;\beta) &= 207360 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{42}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^4(7\beta+16)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{42}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1440\lambda_i^3(6+\beta)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{42}^{(2)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^2(\beta+22)(\beta+3) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{42}^{(3)}(\lambda_i;\lambda_i;\beta) &= 20736\lambda_i(8+\beta) \geq 0 \text{ for all } \beta \geq -8.
\end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{41}(\lambda_j;\lambda_i;\beta) = \xi_4^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}
\xi_{41}(\lambda_j;\lambda_i;\beta) &= 2880\lambda_j^4 + (-2688+1344\beta)\lambda_i\lambda_j^3 \\
&\quad +(-864+216\beta^2)\lambda_i^2\lambda_j^2 + (-1200\beta+960+360\beta^2)\lambda_i^3\lambda_j \\
&\quad +(480+1200\beta)\lambda_i^4, \\
\xi_{41}^{(1)}(\lambda_j;\lambda_i;\beta) &= 11520\lambda_j^3 + 3(-2688+1344\beta)\lambda_i\lambda_j^2 \\
&\quad +2(-864+216\beta^2)\lambda_i^2\lambda_j + (-1200\beta+960+360\beta^2)\lambda_i^3, \\
\xi_{41}^{(2)}(\lambda_j;\lambda_i;\beta) &= 34560\lambda_j^2 + 6(-2688+1344\beta)\lambda_i\lambda_j \\
&\quad +2(-864+216\beta^2)\lambda_i^2, \\
\xi_{41}^{(3)}(\lambda_j;\lambda_i;\beta) &= 69120\lambda_j + 6(-2688+1344\beta)\lambda_i, \\
\xi_{41}^{(4)}(\lambda_j;\lambda_i;\beta) &= 69120 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{41}(\lambda_i;\lambda_i;\beta) &= 192\lambda_i^4(3\beta+4)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{41}^{(1)}(\lambda_i;\lambda_i;\beta) &= 24\lambda_i^3(112+118\beta+33\beta^2) \geq 0 \text{ for all } \beta, \\
\xi_{41}^{(2)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^2(116+56\beta+3\beta^2) \geq 0 \text{ for all } \beta \geq -2.3731, \\
\xi_{41}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1152\lambda_i(46+7\beta) \geq 0 \text{ for all } \beta \geq -\frac{46}{7}.
\end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j; \lambda_i; \beta) = \xi_4(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}
\xi_{40}(\lambda_j; \lambda_i; \beta) &= 720\lambda_j^4 + (-768 + 384\beta)\lambda_i\lambda_j^3 \\
&\quad +(-288 + 72\beta^2)\lambda_i^2\lambda_j^2 + (-480\beta + 384 + 144\beta^2)\lambda_i^3\lambda_j \\
&\quad +(240 + 600\beta)\lambda_i^4, \\
\xi_{40}^{(1)}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j^3 + 3(-768 + 384\beta)\lambda_i\lambda_j^2 \\
&\quad +2(-288 + 72\beta^2)\lambda_i^2\lambda_j + (-480\beta + 384 + 144\beta^2)\lambda_i^3, \\
\xi_{40}^{(2)}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j^2 + 6(-768 + 384\beta)\lambda_i\lambda_j \\
&\quad +2(-288 + 72\beta^2)\lambda_i^2, \\
\xi_{40}^{(3)}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j + 6(-768 + 384\beta)\lambda_i, \\
\xi_{40}^{(4)}(\lambda_j; \lambda_i; \beta) &= 17280 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{40}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^4(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\
\xi_{40}^{(1)}(\lambda_i; \lambda_i; \beta) &= 96\lambda_i^3(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\
\xi_{40}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(24 + 16\beta + \beta^2) > 0 \text{ for all } \beta > -1.6754, \\
\xi_{40}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1152\lambda_i(11 + 2\beta) > 0 \text{ for all } \beta > -\frac{11}{2}.
\end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.12 Let

$$\begin{aligned}
\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (56\beta + 12\beta^2 - 160)t(1, 3, 3, 4) \\
&\quad +(-144\beta + 36\beta^2 + 144)t(2, 2, 3, 4) \\
&\quad +(208 + 16\beta)t(0, 3, 4, 4) \\
&\quad +(84\beta - 168)t(1, 2, 4, 4) \\
&\quad +(-72\beta^2 + 96\beta + 12\beta^3)s(2, 3, 3, 3),
\end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1,n_2,n_3,n_4) \\ \in \mathrm{perm}(a,b,c,d)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (208+16\beta)\lambda_k^3\hat{s}(0,3,4)\\
&\quad +(56\beta+12\beta^2-160)\lambda_k^3\hat{s}(1,3,3)\\
&\quad +(-144\beta+36\beta^2+144)\lambda_k^3\hat{s}(2,2,3)\\
&\quad +(-168+84\beta)\lambda_k^3\hat{s}(1,2,4)\\
&\quad +(156+12\beta)\lambda_k^2\hat{s}(0,4,4)\\
&\quad +(9\beta^2+42\beta-120)\lambda_k^2\hat{s}(1,3,4)\\
&\quad +(72\beta+9\beta^3-54\beta^2)\lambda_k^2\hat{s}(2,3,3)\\
&\quad +(-108\beta+27\beta^2+108)\lambda_k^2\hat{s}(2,2,4)\\
&\quad +(-36\beta^2+48\beta+6\beta^3)\lambda_k\lambda_k\hat{s}(3,3,3)\\
&\quad +(18\beta^2-72\beta+72)\lambda_k\hat{s}(2,3,4)\\
&\quad +(-84+42\beta)\lambda_k\hat{s}(1,4,4)\\
&\quad +(14\beta+3\beta^2-40)\hat{s}(3,3,4)\\
&\quad +(-42+21\beta)\hat{s}(2,4,4),\\
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (-480+36\beta^2+168\beta)\lambda_k^2\hat{s}(1,3,3)\\
&\quad +(432+108\beta^2-432\beta)\lambda_k^2\hat{s}(2,2,3)\\
&\quad +(252\beta-504)\lambda_k^2\hat{s}(1,2,4)\\
&\quad +(624+48\beta)\lambda_k^2\hat{s}(0,3,4)\\
&\quad +(-240+84\beta+18\beta^2)\lambda_k\hat{s}(1,3,4)\\
&\quad +(-216\beta+216+54\beta^2)\lambda_k\hat{s}(2,2,4)\\
&\quad +(-108\beta^2+18\beta^3+144\beta)\lambda_k\hat{s}(2,3,3)\\
&\quad +(24\beta+312)\lambda_k\hat{s}(0,4,4)\\
&\quad +(-36\beta^2+48\beta+6\beta^3)\hat{s}(3,3,3)\\
&\quad +(18\beta^2-72\beta+72)\hat{s}(2,3,4)\\
&\quad +(-84+42\beta)\hat{s}(1,4,4),
\end{aligned}$$

$$\begin{aligned}\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (72\beta^2-960+336\beta)\lambda_k\hat{s}(1,3,3)\\ &\quad +(864-864\beta+216\beta^2)\lambda_k\hat{s}(2,2,3)\\ &\quad +(-1008+504\beta)\lambda_k\hat{s}(1,2,4)\\ &\quad +(96\beta+1248)\lambda_k\hat{s}(0,3,4)\\ &\quad +(-240+84\beta+18\beta^2)\hat{s}(1,3,4)\\ &\quad +(-216\beta+216+54\beta^2)\hat{s}(2,2,4)\\ &\quad +(-108\beta^2+18\beta^3+144\beta)\hat{s}(2,3,3)\\ &\quad +(24\beta+312)\hat{s}(0,4,4),\\ \xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (864-864\beta+216\beta^2)\hat{s}(2,2,3)\\ &\quad +(72\beta^2-960+336\beta)\hat{s}(1,3,3)\\ &\quad +(-1008+504\beta)\hat{s}(1,2,4)\\ &\quad +(96\beta+1248)\hat{s}(0,3,4).\end{aligned}$$

Define

$$\begin{aligned}\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l^3,\\ \xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l^2,\\ \xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l,\\ \xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta).\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (4\beta+52)\lambda_l^5(\lambda_j^3+\lambda_i^3)\\
&+(-42+21\beta)\lambda_l^5(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\\
&+(28\beta-80+6\beta^2)\lambda_l^4(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\\
&+(8\beta+104)\lambda_l^4(\lambda_j^4+\lambda_i^4)\\
&+(-72\beta+72+18\beta^2)\lambda_i^2\lambda_j^2\lambda_l^4\\
&+(-124+56\beta+3\beta^2)\lambda_l^3(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\\
&+(-48\beta+72+3\beta^3)\lambda_l^3(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\\
&+(-80-30\beta^2+76\beta+6\beta^3)\lambda_l^2\lambda_i^3\lambda_j^3\\
&+(18\beta^2-12-30\beta)\lambda_l^2(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\\
&+(60+15\beta^2)\lambda_l(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\\
&+(20+50\beta)\lambda_i^4\lambda_j^4,\\
\xi_0^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (105\beta-210)\lambda_l^4(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\\
&+(20\beta+260)\lambda_l^4(\lambda_j^3+\lambda_i^3)\\
&+(24\beta^2-320+112\beta)\lambda_l^3(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\\
&+(416+32\beta)\lambda_l^3(\lambda_j^4+\lambda_i^4)\\
&+(288+72\beta^2-288\beta)\lambda_l^3\lambda_i^2\lambda_j^2\\
&+(216-144\beta+9\beta^3)\lambda_l^2(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\\
&+(9\beta^2-372+168\beta)\lambda_l^2(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\\
&+(-60\beta-24+36\beta^2)\lambda_l(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\\
&+(-160+152\beta-60\beta^2+12\beta^3)\lambda_l\lambda_i^3\lambda_j^3\\
&+(60+15\beta^2)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3),\\
\xi_0^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (420\beta-840)\lambda_l^3(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\\
&+(1040+80\beta)\lambda_l^3(\lambda_j^3+\lambda_i^3)\\
&+(72\beta^2-960+336\beta)\lambda_l^2(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\\
&+(96\beta+1248)(\lambda_i^4+\lambda_j^4)\lambda_l^2\\
&+(864-864\beta+216\beta^2)\lambda_l^2\lambda_i^2\lambda_j^2\\
&+(432-288\beta+18\beta^3)\lambda_l(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\\
&+(336\beta-744+18\beta^2)\lambda_l(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\\
&+(-160+152\beta-60\beta^2+12\beta^3)\lambda_i^3\lambda_j^3\\
&+(-60\beta-24+36\beta^2)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2),
\end{aligned}$$

$$
\begin{aligned}
\xi_0^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-2520+1260\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^2 \\
&\quad +(240\beta+3120)(\lambda_j^3+\lambda_i^3)\lambda_l^2 \\
&\quad +(-1920+144\beta^2+672\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l \\
&\quad +(192\beta+2496)(\lambda_j^4+\lambda_i^4)\lambda_l \\
&\quad +(-1728\beta+1728+432\beta^2)\lambda_l\lambda_i^2\lambda_j^2 \\
&\quad +(432-288\beta+18\beta^3)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2) \\
&\quad +(336\beta-744+18\beta^2)(\lambda_i^4\lambda_j+\lambda_i\lambda_j^4), \\
\xi_0^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-5040+2520\beta)\lambda_l(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j) \\
&\quad +(480\beta+6240)(\lambda_j^3+\lambda_i^3)\lambda_l \\
&\quad +(-1728\beta+1728+432\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(192\beta+2496)(\lambda_j^4+\lambda_i^4) \\
&\quad +(-1920+144\beta^2+672\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j), \\
\xi_0^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (480\beta+6240)(\lambda_j^3+\lambda_i^3) \\
&\quad +(-5040+2520\beta)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j).
\end{aligned}
$$

Define $\xi_{05}(\lambda_j;\lambda_i;\beta)=\xi_0^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{05} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{05} with respect to λ_j gives

$$
\begin{aligned}
\xi_{05}(\lambda_j;\lambda_i;\beta) &= (480\beta+6240)\lambda_j^3+(-5040+2520\beta)\lambda_i\lambda_j^2 \\
&\quad +(-5040+2520\beta)\lambda_i^2\lambda_j+(480\beta+6240)\lambda_i^3, \\
\xi_{05}^{(1)}(\lambda_j;\lambda_i;\beta) &= (-5040+2520\beta)\lambda_j^2+2(-5040+2520\beta)\lambda_i\lambda_j \\
&\quad +3(480\beta+6240)\lambda_i^2, \\
\xi_{05}^{(2)}(\lambda_j;\lambda_i;\beta) &= 2(-5040+2520\beta)\lambda_j+6(480\beta+6240)\lambda_i, \\
\xi_{05}^{(3)}(\lambda_j;\lambda_i;\beta) &= 2(-5040+2520\beta)\geq 0 \text{ for all } \beta\geq -13.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{05}(\lambda_i;\lambda_i;\beta) &= 1200\lambda_i^3(5\beta+2)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}, \\
\xi_{05}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1800\lambda_i^2(5\beta+2)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}, \\
\xi_{05}^{(2)}(\lambda_i;\lambda_i;\beta) &= 720\lambda_i(38+11\beta)\geq 0 \text{ for all } \beta\geq -3.4545.
\end{aligned}
$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{04}(\lambda_j;\lambda_i;\beta)=\xi_0^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{04} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned}
\xi_{04}(\lambda_j;\lambda_i;\beta) &= (672\beta+8736)\lambda_j^4 + (-6960+144\beta^2+3192\beta)\lambda_i\lambda_j^3 \\
&\quad +(-3312+792\beta+432\beta^2)\lambda_i^2\lambda_j^2 + (1152\beta+4320+144\beta^2)\lambda_i^3\lambda_j \\
&\quad +(192\beta+2496)\lambda_i^4, \\
\xi_{04}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2688\beta+34944)\lambda_j^3 + (-20880+432\beta^2+9576\beta)\lambda_i\lambda_j^2 \\
&\quad +(-6624+1584\beta+864\beta^2)\lambda_i^2\lambda_j + (1152\beta+4320+144\beta^2)\lambda_i^3, \\
\xi_{04}^{(2)}(\lambda_j;\lambda_i;\beta) &= (8064\beta+104832)\lambda_j^2 + (-41760+864\beta^2+19152\beta)\lambda_i\lambda_j \\
&\quad +(-6624+1584\beta+864\beta^2)\lambda_i^2, \\
\xi_{04}^{(3)}(\lambda_j;\lambda_i;\beta) &= (16128\beta+209664)\lambda_j + (-41760+864\beta^2+19152\beta)\lambda_i, \\
\xi_{04}^{(4)}(\lambda_j;\lambda_i;\beta) &= 16128\beta+209664 \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{04}(\lambda_i;\lambda_i;\beta) &= 240\lambda_i^4(3\beta+22)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{04}^{(1)}(\lambda_i;\lambda_i;\beta) &= 120\lambda_i^3(125\beta+98+12\beta^2) \geq 0 \text{ for all } \beta \geq -0.8540, \\
\xi_{04}^{(2)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i^2(50\beta+98+3\beta^2) \geq 0 \text{ for all } \beta \geq -2.2689, \\
\xi_{04}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(3\beta+106)(2\beta+11) \geq 0 \text{ for all } \beta \geq -5.5.
\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j;\lambda_i;\beta) = \xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{03} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}
\xi_{03}(\lambda_j;\lambda_i;\beta) &= (432\beta+5616)\lambda_j^4 + (-5184+2268\beta+162\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(-360-756\beta+432\beta^2+18\beta^3)\lambda_i^2\lambda_j^2 + (624\beta+1632+144\beta^2+18\beta^3)\lambda_i^3\lambda_j \\
&\quad +(528\beta+1752+18\beta^2)\lambda_i^4, \\
\xi_{03}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1728\beta+22464)\lambda_j^3 + (-15552+6804\beta+486\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(-720-1512\beta+864\beta^2+36\beta^3)\lambda_i^2\lambda_j + (624\beta+1632+144\beta^2+18\beta^3)\lambda_i^3, \\
\xi_{03}^{(2)}(\lambda_j;\lambda_i;\beta) &= (5184\beta+67392)\lambda_j^2 + (-31104+13608\beta+972\beta^2)\lambda_i\lambda_j \\
&\quad +(-720-1512\beta+864\beta^2+36\beta^3)\lambda_i^2, \\
\xi_{03}^{(3)}(\lambda_j;\lambda_i;\beta) &= (10368\beta+134784)\lambda_j + (-31104+13608\beta+972\beta^2)\lambda_i, \\
\xi_{03}^{(4)}(\lambda_j;\lambda_i;\beta) &= 10368\beta+134784 \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{03}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^4(\beta+3)(\beta+2)(\beta+16) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{03}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(1274\beta+1304+249\beta^2+9\beta^3) \geq 0 \text{ for all } \beta \geq -1.3744, \\
\xi_{03}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(480\beta+988+51\beta^2+\beta^3) \geq 0 \text{ for all } \beta \geq -2.9025, \\
\xi_{03}^{(3)}(\lambda_i;\lambda_i;\beta) &= 324\lambda_i(74\beta+320+3\beta^2) \geq 0 \text{ for all } \beta \geq -5.5921.
\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j;\lambda_i;\beta) = \xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned}
\xi_{02}(\lambda_j;\lambda_i;\beta) &= (2288+176\beta)\lambda_j^4 + (-2544+1092\beta+90\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(432-792\beta+252\beta^2+18\beta^3)\lambda_i^2\lambda_j^2 + (352+280\beta+12\beta^2+30\beta^3)\lambda_i^3\lambda_j \\
&\quad +(372\beta+480+54\beta^2)\lambda_i^4, \\
\xi_{02}^{(1)}(\lambda_j;\lambda_i;\beta) &= (9152+704\beta)\lambda_j^3 + (-7632+3276\beta+270\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(864-1584\beta+504\beta^2+36\beta^3)\lambda_i^2\lambda_j + (352+280\beta+12\beta^2+30\beta^3)\lambda_i^3, \\
\xi_{02}^{(2)}(\lambda_j;\lambda_i;\beta) &= (27456+2112\beta)\lambda_j^2 + (-15264+6552\beta+540\beta^2)\lambda_i\lambda_j \\
&\quad +(864-1584\beta+504\beta^2+36\beta^3)\lambda_i^2, \\
\xi_{02}^{(3)}(\lambda_j;\lambda_i;\beta) &= (54912+4224\beta)\lambda_j + (-15264+6552\beta+540\beta^2)\lambda_i, \\
\xi_{02}^{(4)}(\lambda_j;\lambda_i;\beta) &= (54912+4224\beta) \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{02}(\lambda_i;\lambda_i;\beta) &= 24\lambda_i^4(\beta+3)(\beta+2)(2\beta+7) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(\beta+3)(\beta+2)(11\beta+76) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(2)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^2(1088+590\beta+87\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -3.1423, \\
\xi_{02}^{(3)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i(3304+898\beta+45\beta^2) \geq 0 \text{ for all } \beta \geq -4.8657.
\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j;\lambda_i;\beta) &= (676+52\beta)\lambda_j^4 + (385\beta-902+33\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(270+108\beta^2-387\beta+9\beta^3)\lambda_i^2\lambda_j^2 + (56-21\beta^2+21\beta^3+140\beta)\lambda_i^3\lambda_j \\
&\quad +(80+140\beta+60\beta^2)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2704+208\beta)\lambda_j^3 + (1155\beta-2706+99\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(540+216\beta^2-774\beta+18\beta^3)\lambda_i^2\lambda_j + (56-21\beta^2+21\beta^3+140\beta)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= (8112+624\beta)\lambda_j^2 + (2310\beta-5412+198\beta^2)\lambda_i\lambda_j \\
&\quad +(540+216\beta^2-774\beta+18\beta^3)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j;\lambda_i;\beta) &= (16224+1248\beta)\lambda_j + (2310\beta-5412+198\beta^2)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j;\lambda_i;\beta) &= 16224+1248\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 30\lambda_i^4(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 3\lambda_i^3(\beta+3)(\beta+2)(13\beta+33) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{01}^{(2)}(\lambda_i;\lambda_i;\beta) &= 18\lambda_i^2(\beta+3)(\beta^2+20\beta+60) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i(1802+593\beta+33\beta^2) \geq 0 \text{ for all } \beta \geq -4.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{00} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= (156+12\beta)\lambda_j^4 + (105\beta - 246 + 9\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(90+36\beta^2-129\beta+3\beta^3)\lambda_i^2\lambda_j^2 + (24-9\beta^2+9\beta^3+60\beta)\lambda_i^3\lambda_j \\
&\quad +(48+84\beta+36\beta^2)\lambda_i^4, \\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= (624+48\beta)\lambda_j^3 + (315\beta-738+27\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(180+72\beta^2-258\beta+6\beta^3)\lambda_i^2\lambda_j + (24-9\beta^2+9\beta^3+60\beta)\lambda_i^3, \\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1872+144\beta)\lambda_j^2 + (630\beta-1476+54\beta^2)\lambda_i\lambda_j \\
&\quad +(180+72\beta^2-258\beta+6\beta^3)\lambda_i^2, \\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= (3744+288\beta)\lambda_j + (630\beta-1476+54\beta^2)\lambda_i, \\
\xi_{00}^{(4)}(\lambda_j;\lambda_i;\beta) &= 3744+288\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 15\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^2(\beta+3)(\beta+2)(\beta+16) > 0 \text{ for all } \beta > -2, \\
\xi_{00}^{(3)}(\lambda_i;\lambda_i;\beta) &= 54\lambda_i(\beta+14)(\beta+3) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-168+84\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^5 \\
&\quad +(16\beta+208)(\lambda_j^3+\lambda_i^3)\lambda_l^5 \\
&\quad +(21\beta^2+98\beta-280)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^4 \\
&\quad +(364+28\beta)(\lambda_j^4+\lambda_i^4)\lambda_l^4 \\
&\quad +(63\beta^2-252\beta+252)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&\quad +(216-144\beta+9\beta^3)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^3 \\
&\quad +(9\beta^2+168\beta-372)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^3 \\
&\quad +(45\beta^2-75\beta-30)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l^2 \\
&\quad +(190\beta+15\beta^3-75\beta^2-200)\lambda_i^3\lambda_j^3\lambda_l^2 \\
&\quad +(30\beta^2+120)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\lambda_l \\
&\quad +(30+75\beta)\lambda_i^4\lambda_j^4,
\end{aligned}$$

$$
\begin{aligned}
\xi_1^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (420\beta-840)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^4\\
&\quad+(1040+80\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^4\\
&\quad+(392\beta-1120+84\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^3\\
&\quad+(1456+112\beta)(\lambda_j^4+\lambda_i^4)\lambda_l^3\\
&\quad+(1008-1008\beta+252\beta^2)\lambda_i^2\lambda_j^2\lambda_l^3\\
&\quad+(648-432\beta+27\beta^3)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^2\\
&\quad+(504\beta-1116+27\beta^2)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^2\\
&\quad+(90\beta^2-60-150\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l\\
&\quad+(-400+380\beta+30\beta^3-150\beta^2)\lambda_i^3\lambda_j^3\lambda_l\\
&\quad+(30\beta^2+120)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3),\\
\xi_1^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (1680\beta-3360)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^3\\
&\quad+(320\beta+4160)(\lambda_j^3+\lambda_i^3)\lambda_l^3\\
&\quad+(-3360+252\beta^2+1176\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^2\\
&\quad+(4368+336\beta)(\lambda_j^4+\lambda_i^4)\lambda_l^2\\
&\quad+(3024+756\beta^2-3024\beta)\lambda_i^2\lambda_j^2\lambda_l^2\\
&\quad+(54\beta^3-864\beta+1296)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l\\
&\quad+(54\beta^2-2232+1008\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l\\
&\quad+(-400+380\beta+30\beta^3-150\beta^2)\lambda_i^3\lambda_j^3\\
&\quad+(90\beta^2-60-150\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2),\\
\xi_1^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-10080+5040\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^2\\
&\quad+(12480+960\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^2\\
&\quad+(672\beta+8736)(\lambda_j^4+\lambda_i^4)\lambda_l\\
&\quad+(2352\beta+504\beta^2-6720)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l\\
&\quad+(6048-6048\beta+1512\beta^2)\lambda_i^2\lambda_j^2\lambda_l\\
&\quad+(54\beta^2-2232+1008\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\\
&\quad+(54\beta^3-864\beta+1296)(\lambda_i^3\lambda_j^2+\lambda_i^2\lambda_j^3),\\
\xi_1^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-20160+10080\beta)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l\\
&\quad+(24960+1920\beta)(\lambda_j^3+\lambda_i^3)\lambda_l\\
&\quad+(6048-6048\beta+1512\beta^2)\lambda_i^2\lambda_j^2\\
&\quad+(672\beta+8736)(\lambda_j^4+\lambda_i^4)\\
&\quad+(2352\beta+504\beta^2-6720)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j),\\
\xi_1^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (24960+1920\beta)(\lambda_j^3+\lambda_i^3)\\
&\quad+(-20160+10080\beta)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j).
\end{aligned}
$$

Define $\xi_{15}(\lambda_j;\lambda_i;\beta)=\xi_1^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{15} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{15} with respect to λ_j gives

$$\begin{aligned}
\xi_{15}(\lambda_j;\lambda_i;\beta) &= (24960+1920\beta)\lambda_j^3+(-20160+10080\beta)\lambda_i\lambda_j^2\\
&\quad +(-20160+10080\beta)\lambda_i^2\lambda_j+(24960+1920\beta)\lambda_i^3,\\
\xi_{15}^{(1)}(\lambda_j;\lambda_i;\beta) &= (74880+5760\beta)\lambda_j^2+(-40320+20160\beta)\lambda_i\lambda_j\\
&\quad +(-20160+10080\beta)\lambda_i^2,\\
\xi_{15}^{(2)}(\lambda_j;\lambda_i;\beta) &= (149760+11520\beta)\lambda_j+(-40320+20160\beta)\lambda_i,\\
\xi_{15}^{(3)}(\lambda_j;\lambda_i;\beta) &= 149760+11520\beta\geq 0 \text{ for all } \beta\geq-\frac{2}{5}.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{15}(\lambda_i;\lambda_i;\beta) &= 4800\lambda_i^3(2+5\beta)\geq 0 \text{ for all } \beta\geq-\frac{2}{5},\\
\xi_{15}^{(1)}(\lambda_i;\lambda_i;\beta) &= 7200\lambda_i^2(2+5\beta)\geq 0 \text{ for all } \beta\geq-\frac{2}{5},\\
\xi_{15}^{(2)}(\lambda_i;\lambda_i;\beta) &= 149760+11520\beta\geq 0 \text{ for all } \beta\geq-13.
\end{aligned}$$

Hence, $\xi_1^{(5)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{14}(\lambda_j;\lambda_i;\beta)=\xi_1^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{14} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned}
\xi_{14}(\lambda_j;\lambda_i;\beta) &= (33696+2592\beta)\lambda_j^4+(12432\beta+504\beta^2-26880)\lambda_i\lambda_j^3\\
&\quad +(-14112+4032\beta+1512\beta^2)\lambda_i^2\lambda_j^2+(18240+4272\beta+504\beta^2)\lambda_i^3\lambda_j\\
&\quad +(672\beta+8736)\lambda_i^4,\\
\xi_{14}^{(1)}(\lambda_j;\lambda_i;\beta) &= (134784+10368\beta)\lambda_j^3+(37296\beta+1512\beta^2-80640)\lambda_i\lambda_j^2\\
&\quad +(-28224+8064\beta+3024\beta^2)\lambda_i^2\lambda_j+(18240+4272\beta+504\beta^2)\lambda_i^3,\\
\xi_{14}^{(2)}(\lambda_j;\lambda_i;\beta) &= (404352+31104\beta)\lambda_j^2+(74592\beta+3024\beta^2-161280)\lambda_i\lambda_j\\
&\quad +(-28224+8064\beta+3024\beta^2)\lambda_i^2,\\
\xi_{14}^{(3)}(\lambda_j;\lambda_i;\beta) &= (808704+62208\beta)\lambda_j+(74592\beta+3024\beta^2-161280)\lambda_i,\\
\xi_{14}^{(4)}(\lambda_j;\lambda_i;\beta) &= 808704+62208\beta\geq 0 \text{ for all } \beta\geq-13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{14}(\lambda_i;\lambda_i;\beta) &= 120\lambda_i^4(164+200\beta+21\beta^2)\geq 0 \text{ for all } \beta\geq-0.9062,\\
\xi_{14}^{(1)}(\lambda_i;\lambda_i;\beta) &= 240\lambda_i^3(184+250\beta+21\beta^2)\geq 0 \text{ for all } \beta\geq-0.7882,\\
\xi_{14}^{(2)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i^2(746+395\beta+21\beta^2)\geq 0 \text{ for all } \beta\geq-2.1298,\\
\xi_{14}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(4496+950\beta+21\beta^2)\geq 0 \text{ for all } \beta\geq-5.3701.
\end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{13}(\lambda_j;\lambda_i;\beta) = \xi_1^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}
\xi_{13}(\lambda_j;\lambda_i;\beta) &= (21216+1632\beta)\lambda_j^4 + (-19032+8400\beta+558\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(-2736-1872\beta+1512\beta^2+54\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(7056+2448\beta+504\beta^2+54\beta^3)\lambda_i^3\lambda_j \\
&\quad +(1680\beta+6504+54\beta^2)\lambda_i^4, \\
\xi_{13}^{(1)}(\lambda_j;\lambda_i;\beta) &= (84864+6528\beta)\lambda_j^3 + (-57096+25200\beta+1674\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(-5472-3744\beta+3024\beta^2+108\beta^3)\lambda_i^2\lambda_j \\
&\quad +(7056+2448\beta+504\beta^2+54\beta^3)\lambda_i^3, \\
\xi_{13}^{(2)}(\lambda_j;\lambda_i;\beta) &= (254592+19584\beta)\lambda_j^2 + (-114192+50400\beta+3348\beta^2)\lambda_i\lambda_j \\
&\quad +(-5472-3744\beta+3024\beta^2+108\beta^3)\lambda_i^2, \\
\xi_{13}^{(3)}(\lambda_j;\lambda_i;\beta) &= (509184+39168\beta)\lambda_j + (-114192+50400\beta+3348\beta^2)\lambda_i, \\
\xi_{13}^{(4)}(\lambda_j;\lambda_i;\beta) &= 509184+39168\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{13}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^4(1084+1024\beta+219\beta^2+9\beta^3) \geq 0 \text{ for all } \beta \geq -1.5245, \\
\xi_{13}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(4892+5072\beta+867\beta^2+27\beta^3) \geq 0 \text{ for all } \beta \geq -1.2024, \\
\xi_{13}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(3748+1840\beta+177\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -2.7119, \\
\xi_{13}^{(3)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i(10972+2488\beta+93\beta^2) \geq 0 \text{ for all } \beta \geq -5.5694.
\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j;\lambda_i;\beta) = \xi_1^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{12} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned}
\xi_{12}(\lambda_j;\lambda_i;\beta) &= (8528+656\beta)\lambda_j^4 + (3864\beta+306\beta^2-8952)\lambda_i\lambda_j^3 \\
&\quad +(900-2358\beta+846\beta^2+54\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(1696+1012\beta+102\beta^2+84\beta^3)\lambda_i^3\lambda_j \\
&\quad +(1194\beta+2076+144\beta^2)\lambda_i^4, \\
\xi_{12}^{(1)}(\lambda_j;\lambda_i;\beta) &= (34112+2624\beta)\lambda_j^3 + (11592\beta+918\beta^2-26856)\lambda_i\lambda_j^2 \\
&\quad +(1800-4716\beta+1692\beta^2+108\beta^3)\lambda_i^2\lambda_j \\
&\quad +(1696+1012\beta+102\beta^2+84\beta^3)\lambda_i^3, \\
\xi_{12}^{(2)}(\lambda_j;\lambda_i;\beta) &= (102336+7872\beta)\lambda_j^2 + (23184\beta+1836\beta^2-53712)\lambda_i\lambda_j \\
&\quad +(1800-4716\beta+1692\beta^2+108\beta^3)\lambda_i^2, \\
\xi_{12}^{(3)}(\lambda_j;\lambda_i;\beta) &= (204672+15744\beta)\lambda_j + (23184\beta+1836\beta^2-53712)\lambda_i, \\
\xi_{12}^{(4)}(\lambda_j;\lambda_i;\beta) &= 204672+15744\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{12}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^4(\beta+3)(\beta+2)(23\beta+118) \geq 0 \text{ for all } \beta \geq -2,\\
\xi_{12}^{(1)}(\lambda_i;\lambda_i;\beta) &= 24\lambda_i^3(448+438\beta+113\beta^2+8\beta^3) \geq 0 \text{ for all } \beta \geq -1.6273,\\
\xi_{12}^{(2)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^2(4202+2195\beta+294\beta^2+9\beta^3) \geq 0 \text{ for all } \beta \geq -3.0300,\\
\xi_{12}^{(3)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i(12580+3244\beta+153\beta^2) \geq 0 \text{ for all } \beta \geq -5.1090.
\end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j;\lambda_i;\beta) = \xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j;\lambda_i;\beta) &= (2496+192\beta)\lambda_j^4 + (111\beta^2-3076+1316\beta)\lambda_i\lambda_j^3\\
&\quad +(756-1170\beta+342\beta^2+27\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(288+57\beta^3+420\beta-36\beta^2)\lambda_i^3\lambda_j\\
&\quad +(466\beta+400+147\beta^2)\lambda_i^4,\\
\xi_{11}^{(1)}(\lambda_j;\lambda_i;\beta) &= (9984+768\beta)\lambda_j^3 + (333\beta^2-9228+3948\beta)\lambda_i\lambda_j^2\\
&\quad +(1512-2340\beta+684\beta^2+54\beta^3)\lambda_i^2\lambda_j\\
&\quad +(288+57\beta^3+420\beta-36\beta^2)\lambda_i^3,\\
\xi_{11}^{(2)}(\lambda_j;\lambda_i;\beta) &= (29952+2304\beta)\lambda_j^2 + (666\beta^2-18456+7896\beta)\lambda_i\lambda_j\\
&\quad +(1512-2340\beta+684\beta^2+54\beta^3)\lambda_i^2,\\
\xi_{11}^{(3)}(\lambda_j;\lambda_i;\beta) &= (59904+4608\beta)\lambda_j + (666\beta^2-18456+7896\beta)\lambda_i,\\
\xi_{11}^{(4)}(\lambda_j;\lambda_i;\beta) &= 59904+4608\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^4(\beta+3)(\beta+2)(7\beta+12) \geq 0 \text{ for all } \beta \geq -2,\\
\xi_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= 3\lambda_i^3(\beta+3)(\beta+2)(37\beta+142) \geq 0 \text{ for all } \beta \geq -2,\\
\xi_{11}^{(2)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^2(2168+1310\beta+225\beta^2+9\beta^3) \geq 0 \text{ for all } \beta \geq -3.1065,\\
\xi_{11}^{(3)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i(6908+2084\beta+111\beta^2) \geq 0 \text{ for all } \beta \geq -4.2993.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j;\lambda_i;\beta) = \xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{10} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= (572+44\beta)\lambda_j^4 + (30\beta^2 - 820 + 350\beta)\lambda_i\lambda_j^3 \\
&\quad +(270 - 387\beta + 108\beta^2 + 9\beta^3)\lambda_i^2\lambda_j^2 + (64 + 24\beta^3 + 160\beta - 24\beta^2)\lambda_i^3\lambda_j \\
&\quad +(196\beta + 112 + 84\beta^2)\lambda_i^4, \\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2288 + 176\beta)\lambda_j^3 + (90\beta^2 - 2460 + 1050\beta)\lambda_i\lambda_j^2 \\
&\quad +(540 - 774\beta + 216\beta^2 + 18\beta^3)\lambda_i^2\lambda_j + (64 + 24\beta^3 + 160\beta - 24\beta^2)\lambda_i^3, \\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= (6864 + 528\beta)\lambda_j^2 + (180\beta^2 - 4920 + 2100\beta)\lambda_i\lambda_j \\
&\quad +(540 - 774\beta + 216\beta^2 + 18\beta^3)\lambda_i^2, \\
\xi_{10}^{(3)}(\lambda_j;\lambda_i;\beta) &= (13728 + 1056\beta)\lambda_j + (180\beta^2 - 4920 + 2100\beta)\lambda_i, \\
\xi_{10}^{(4)}(\lambda_j;\lambda_i;\beta) &= 13728 + 1056\beta > 0 \text{ for all } \beta > -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 33\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(\beta+3)(\beta+2)(7\beta+12) > 0 \text{ for all } \beta > -1.7143, \\
\xi_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= 18\lambda_i^2(\beta+3)(\beta^2+19\beta+46) > 0 \text{ for all } \beta > -2.8479, \\
\xi_{10}^{(3)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i(734 + 263\beta + 15\beta^2) > 0 \text{ for all } \beta > -3.4826.
\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= (624 + 48\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^5 \\
&\quad +(252\beta - 504)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^5 \\
&\quad +(252\beta + 54\beta^2 - 720)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^4 \\
&\quad +(72\beta + 936)(\lambda_j^4 + \lambda_i^4)\lambda_l^4 \\
&\quad +(162\beta^2 - 648\beta + 648)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&\quad +(18\beta^3 + 504 - 360\beta + 18\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^3 \\
&\quad +(18\beta^2 - 828 + 378\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^3 \\
&\quad +(-36\beta - 216 + 72\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l^2 \\
&\quad +(360\beta - 108\beta^2 + 24\beta^3 - 480)\lambda_i^3\lambda_j^3\lambda_l^2 \\
&\quad +(60\beta + 456 + 36\beta^2)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_l \\
&\quad +(66\beta + 228)\lambda_i^4\lambda_j^4,
\end{aligned}$$

$$
\begin{aligned}
\xi_2^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (1260\beta-2520)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^4\\
&\quad +(240\beta+3120)(\lambda_j^3+\lambda_i^3)\lambda_l^4\\
&\quad +(1008\beta+216\beta^2-2880)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^3\\
&\quad +(288\beta+3744)(\lambda_j^4+\lambda_i^4)\lambda_l^3\\
&\quad +(-2592\beta+648\beta^2+2592)\lambda_i^2\lambda_j^2\lambda_l^3\\
&\quad +(1134\beta+54\beta^2-2484)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^2\\
&\quad +(1512-1080\beta+54\beta^3+54\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^2\\
&\quad +(144\beta^2-72\beta-432)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l\\
&\quad +(720\beta-216\beta^2-960+48\beta^3)\lambda_i^3\lambda_j^3\lambda_l\\
&\quad +(60\beta+456+36\beta^2)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3),\\
\xi_2^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-10080+5040\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^3\\
&\quad +(12480+960\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^3\\
&\quad +(3024\beta-8640+648\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^2\\
&\quad +(864\beta+11232)(\lambda_j^4+\lambda_i^4)\lambda_l^2\\
&\quad +(7776+1944\beta^2-7776\beta)\lambda_i^2\lambda_j^2\lambda_l^2\\
&\quad +(108\beta^3-2160\beta+3024+108\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l\\
&\quad +(2268\beta+108\beta^2-4968)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l\\
&\quad +(720\beta-216\beta^2-960+48\beta^3)\lambda_i^3\lambda_j^3\\
&\quad +(144\beta^2-72\beta-432)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2),\\
\xi_2^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (15120\beta-30240)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^2\\
&\quad +(37440+2880\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^2\\
&\quad +(1728\beta+22464)(\lambda_j^4+\lambda_i^4)\lambda_l\\
&\quad +(1296\beta^2-17280+6048\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l\\
&\quad +(-15552\beta+15552+3888\beta^2)\lambda_i^2\lambda_j^2\\
&\quad +(108\beta^3-2160\beta+3024+108\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\\
&\quad +(2268\beta+108\beta^2-4968)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j),\\
\xi_2^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (30240\beta-60480)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l\\
&\quad +(74880+5760\beta)(\lambda_j^3+\lambda_i^3)\lambda_l\\
&\quad +(-15552\beta+15552+3888\beta^2)\lambda_i^2\lambda_j^2\\
&\quad +(1728\beta+22464)(\lambda_j^4+\lambda_i^4)\\
&\quad +(1296\beta^2-17280+6048\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j),\\
\xi_2^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (30240\beta-60480)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\\
&\quad +(74880+5760\beta)(\lambda_j^3+\lambda_i^3).
\end{aligned}
$$

Define $\xi_{25}(\lambda_j;\lambda_i;\beta)=\xi_2^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{25} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{25} with respect to λ_j gives

$$\begin{aligned}
\xi_{25}(\lambda_j;\lambda_i;\beta) &= (74880+5760\beta)\lambda_j^3+(30240\beta-60480)\lambda_i\lambda_j^2\\
&\quad+(30240\beta-60480)\lambda_i^2\lambda_j+(74880+5760\beta)\lambda_i^3,\\
\xi_{25}^{(1)}(\lambda_j;\lambda_i;\beta) &= (224640+17280\beta)\lambda_j^2+(60480\beta-120960)\lambda_i\lambda_j\\
&\quad+(30240\beta-60480)\lambda_i^2,\\
\xi_{25}^{(2)}(\lambda_j;\lambda_i;\beta) &= (449280+34560\beta)\lambda_j+(60480\beta-120960)\lambda_i,\\
\xi_{25}^{(3)}(\lambda_j;\lambda_i;\beta) &= 449280+34560\beta\geq 0 \text{ for all } \beta\geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{25}(\lambda_i;\lambda_i;\beta) &= 14400\lambda_i^3(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5},\\
\xi_{25}^{(1)}(\lambda_i;\lambda_i;\beta) &= 21600\lambda_i^2(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5},\\
\xi_{25}^{(2)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i(38+11\beta)\geq 0 \text{ for all } \beta\geq -3.4545.
\end{aligned}$$

Hence, $\xi_2^{(5)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{24}(\lambda_j;\lambda_i;\beta)=\xi_2^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{24} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{24} with respect to λ_j gives

$$\begin{aligned}
\xi_{24}(\lambda_j;\lambda_i;\beta) &= (97344+7488\beta)\lambda_j^4+(1296\beta^2-77760+36288\beta)\lambda_i\lambda_j^3\\
&\quad+(14688\beta-44928+3888\beta^2)\lambda_i^2\lambda_j^2+(57600+11808\beta+1296\beta^2)\lambda_i^3\lambda_j\\
&\quad+(1728\beta+22464)\lambda_i^4,\\
\xi_{24}^{(1)}(\lambda_j;\lambda_i;\beta) &= (389376+29952\beta)\lambda_j^3+(3888\beta^2-233280+108864\beta)\lambda_i\lambda_j^2\\
&\quad+(29376\beta-89856+7776\beta^2)\lambda_i^2\lambda_j+(57600+11808\beta+1296\beta^2)\lambda_i^3,\\
\xi_{24}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1168128+89856\beta)\lambda_j^2+(7776\beta^2-466560+217728\beta)\lambda_i\lambda_j\\
&\quad+(29376\beta-89856+7776\beta^2)\lambda_i^2,\\
\xi_{24}^{(3)}(\lambda_j;\lambda_i;\beta) &= (2336256+179712\beta)\lambda_j+(7776\beta^2-466560+217728\beta)\lambda_i,\\
\xi_{24}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2336256+179712\beta\geq 0 \text{ for all } \beta\geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{24}(\lambda_i;\lambda_i;\beta) &= 720\lambda_i^4(76+100\beta+9\beta^2)\geq 0 \text{ for all } \beta\geq -0.8206,\\
\xi_{24}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1440\lambda_i^3(86+125\beta+9\beta^2)\geq 0 \text{ for all } \beta\geq -0.7259,\\
\xi_{24}^{(2)}(\lambda_i;\lambda_i;\beta) &= 5184\lambda_i^2(3\beta+59)(\beta+2)\geq 0 \text{ for all } \beta\geq -2,\\
\xi_{24}^{(3)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i(2164+460\beta+9\beta^2)\geq 0 \text{ for all } \beta\geq -5.2420.
\end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{23}(\lambda_j;\lambda_i;\beta) = \xi_2^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{23} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned}
\xi_{23}(\lambda_j;\lambda_i;\beta) &= (59904+4608\beta)\lambda_j^4 + (23436\beta - 52488 + 1404\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(-2592\beta - 11664 + 3996\beta^2 + 108\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(23184 + 6768\beta + 1404\beta^2 + 108\beta^3)\lambda_i^3\lambda_j \\
&\quad +(3996\beta + 17496 + 108\beta^2)\lambda_i^4, \\
\xi_{23}^{(1)}(\lambda_j;\lambda_i;\beta) &= (239616 + 18432\beta)\lambda_j^3 + (70308\beta - 157464 + 4212\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(-5184\beta - 23328 + 7992\beta^2 + 216\beta^3)\lambda_i^2\lambda_j \\
&\quad +(23184 + 6768\beta + 1404\beta^2 + 108\beta^3)\lambda_i^3, \\
\xi_{23}^{(2)}(\lambda_j;\lambda_i;\beta) &= (718848 + 55296\beta)\lambda_j^2 + (140616\beta - 314928 + 8424\beta^2)\lambda_i\lambda_j \\
&\quad +(-5184\beta - 23328 + 7992\beta^2 + 216\beta^3)\lambda_i^2, \\
\xi_{23}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1437696 + 110592\beta)\lambda_j + (140616\beta - 314928 + 8424\beta^2)\lambda_i, \\
\xi_{23}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1437696 + 110592\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{23}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i^4(506 + 503\beta + 96\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.3291, \\
\xi_{23}^{(1)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^3(2278 + 2509\beta + 378\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -1.0787, \\
\xi_{23}^{(2)}(\lambda_i;\lambda_i;\beta) &= 216\lambda_i^2(1762 + 883\beta + 76\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -2.5267, \\
\xi_{23}^{(3)}(\lambda_i;\lambda_i;\beta) &= 216\lambda_i(5198 + 1163\beta + 39\beta^2) \geq 0 \text{ for all } \beta \geq -5.4745.
\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j;\lambda_i;\beta) = \xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}
\xi_{22}(\lambda_j;\lambda_i;\beta) &= (23712 + 1824\beta)\lambda_j^4 + (10332\beta + 756\beta^2 - 23688)\lambda_i\lambda_j^3 \\
&\quad +(288 - 4968\beta + 2196\beta^2 + 108\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(5904 + 2544\beta + 540\beta^2 + 156\beta^3)\lambda_i^3\lambda_j \\
&\quad +(3060\beta + 5832 + 252\beta^2)\lambda_i^4, \\
\xi_{22}^{(1)}(\lambda_j;\lambda_i;\beta) &= (94848 + 7296\beta)\lambda_j^3 + (30996\beta + 2268\beta^2 - 71064)\lambda_i\lambda_j^2 \\
&\quad +(576 - 9936\beta + 4392\beta^2 + 216\beta^3)\lambda_i^2\lambda_j \\
&\quad +(5904 + 2544\beta + 540\beta^2 + 156\beta^3)\lambda_i^3, \\
\xi_{22}^{(2)}(\lambda_j;\lambda_i;\beta) &= (284544 + 21888\beta)\lambda_j^2 + (61992\beta + 4536\beta^2 - 142128)\lambda_i\lambda_j \\
&\quad +(576 - 9936\beta + 4392\beta^2 + 216\beta^3)\lambda_i^2, \\
\xi_{22}^{(3)}(\lambda_j;\lambda_i;\beta) &= (569088 + 43776\beta)\lambda_j + (61992\beta + 4536\beta^2 - 142128)\lambda_i, \\
\xi_{22}^{(4)}(\lambda_j;\lambda_i;\beta) &= 569088 + 43776\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{22}(\lambda_i;\lambda_i;\beta) &= 24\lambda_i^4(502+533\beta+156\beta^2+11\beta^3) \geq 0 \text{ for all } \beta \geq -1.6304,\\
\xi_{22}^{(1)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^3(2522+2575\beta+600\beta^2+31\beta^3) \geq 0 \text{ for all } \beta \geq -1.4074,\\
\xi_{22}^{(2)}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i^2(1986+1027\beta+124\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -2.8419,\\
\xi_{22}^{(3)}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i(5930+1469\beta+63\beta^2) \geq 0 \text{ for all } \beta \geq -5.1935.
\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j;\lambda_i;\beta) = \xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{21} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}
\xi_{21}(\lambda_j;\lambda_i;\beta) &= (6864+528\beta)\lambda_j^4+(3402\beta+270\beta^2-7884)\lambda_i\lambda_j^3\\
&\quad +(1152-2484\beta+846\beta^2+54\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(1248+948\beta+90\beta^2+102\beta^3)\lambda_i^3\lambda_j\\
&\quad +(1410\beta+1284+234\beta^2)\lambda_i^4,\\
\xi_{21}^{(1)}(\lambda_j;\lambda_i;\beta) &= (27456+2112\beta)\lambda_j^3+(10206\beta+810\beta^2-23652)\lambda_i\lambda_j^2\\
&\quad +(2304-4968\beta+1692\beta^2+108\beta^3)\lambda_i^2\lambda_j\\
&\quad +(1248+948\beta+90\beta^2+102\beta^3)\lambda_i^3,\\
\xi_{21}^{(2)}(\lambda_j;\lambda_i;\beta) &= (82368+6336\beta)\lambda_j^2+(20412\beta+1620\beta^2-47304)\lambda_i\lambda_j\\
&\quad +(2304-4968\beta+1692\beta^2+108\beta^3)\lambda_i^2,\\
\xi_{21}^{(3)}(\lambda_j;\lambda_i;\beta) &= (164736+12672\beta)\lambda_j+(20412\beta+1620\beta^2-47304)\lambda_i,\\
\xi_{21}^{(4)}(\lambda_j;\lambda_i;\beta) &= 164736+12672\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{21}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^4(\beta+3)(13\beta^2+81\beta+74) \geq 0 \text{ for all } \beta \geq -1.1121,\\
\xi_{21}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(1226+1383\beta+432\beta^2+35\beta^3) \geq 0 \text{ for all } \beta \geq -1.5169,\\
\xi_{21}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(1038+605\beta+92\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -2.7964,\\
\xi_{21}^{(3)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i(3262+919\beta+45\beta^2) \geq 0 \text{ for all } \beta \geq -4.5739.
\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j;\lambda_i;\beta) = \xi_2(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{20} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}
\xi_{20}(\lambda_j;\lambda_i;\beta) &= (1560+120\beta)\lambda_j^4 + (882\beta+72\beta^2-2052)\lambda_i\lambda_j^3 \\
&\quad +(432+18\beta^3-792\beta+252\beta^2)\lambda_i^2\lambda_j^2 + (384+360\beta+42\beta^3)\lambda_i^3\lambda_j \\
&\quad +(540\beta+126\beta^2+576)\lambda_i^4, \\
\xi_{20}^{(1)}(\lambda_j;\lambda_i;\beta) &= (6240+480\beta)\lambda_j^3 + (2646\beta+216\beta^2-6156)\lambda_i\lambda_j^2 \\
&\quad +(864+36\beta^3-1584\beta+504\beta^2)\lambda_i^2\lambda_j + (384+360\beta+42\beta^3)\lambda_i^3, \\
\xi_{20}^{(2)}(\lambda_j;\lambda_i;\beta) &= (18720+1440\beta)\lambda_j^2 + (5292\beta+432\beta^2-12312)\lambda_i\lambda_j \\
&\quad +(864+36\beta^3-1584\beta+504\beta^2)\lambda_i^2, \\
\xi_{20}^{(3)}(\lambda_j;\lambda_i;\beta) &= (37440+2880\beta)\lambda_j + (5292\beta+432\beta^2-12312)\lambda_i, \\
\xi_{20}^{(4)}(\lambda_j;\lambda_i;\beta) &= 37440+2880\beta > 0 \text{ for all } \beta > -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{20}(\lambda_i;\lambda_i;\beta) &= 30\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -2, \\
\xi_{20}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(\beta+3)(13\beta^2+81\beta+74) > 0 \text{ for all } \beta > -3, \\
\xi_{20}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(202+143\beta+26\beta^2+\beta^3) > 0 \text{ for all } \beta > -2.2618, \\
\xi_{20}^{(3)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i(698+227\beta+12\beta^2) > 0 \text{ for all } \beta > -3.8643.
\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-1008+504\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^5 \\
&\quad +(96\beta+1248)(\lambda_j^3+\lambda_i^3)\lambda_l^5 \\
&\quad +(-1200+90\beta^2+420\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^4 \\
&\quad +(120\beta+1560)(\lambda_j^4+\lambda_i^4)\lambda_l^4 \\
&\quad +(1080+270\beta^2-1080\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&\quad +(-720\beta+108\beta^2+18\beta^3+864)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^3 \\
&\quad +(18\beta^2-1248+588\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^3 \\
&\quad +(-792+54\beta^2+288\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l^2 \\
&\quad +(-960+480\beta-36\beta^2+18\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 \\
&\quad +(1008+18\beta^2+180\beta)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\lambda_l \\
&\quad +(24\beta+312)\lambda_i^4\lambda_j^4,
\end{aligned}$$

$$
\begin{aligned}
\xi_3^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-5040+2520\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^4 \\
&+(6240+480\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^4 \\
&+(-4800+1680\beta+360\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^3 \\
&+(6240+480\beta)(\lambda_j^4+\lambda_i^4)\lambda_l^3 \\
&+(4320-4320\beta+1080\beta^2)\lambda_i^2\lambda_j^2\lambda_l^3 \\
&+(54\beta^3-2160\beta+2592+324\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^2 \\
&+(1764\beta+54\beta^2-3744)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^2 \\
&+(-1584+576\beta+108\beta^2)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l \\
&+(36\beta^3+960\beta-1920-72\beta^2)\lambda_i^3\lambda_j^3\lambda_l \\
&+(1008+18\beta^2+180\beta)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3), \\
\xi_3^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-20160+10080\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^3 \\
&+(24960+1920\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^3 \\
&+(5040\beta-14400+1080\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^2 \\
&+(18720+1440\beta)(\lambda_j^4+\lambda_i^4)\lambda_l^2 \\
&+(-12960\beta+3240\beta^2+12960)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&+(108\beta^3-4320\beta+648\beta^2+5184)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l \\
&+(108\beta^2-7488+3528\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l \\
&+(36\beta^3+960\beta-1920-72\beta^2)\lambda_i^3\lambda_j^3 \\
&+(-1584+576\beta+108\beta^2)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2), \\
\xi_3^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-60480+30240\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^2 \\
&+(5760\beta+74880)(\lambda_j^3+\lambda_i^3)\lambda_l^2 \\
&+(37440+2880\beta)(\lambda_j^4+\lambda_i^4)\lambda_l \\
&+(10080\beta-28800+2160\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l \\
&+(6480\beta^2-25920\beta+25920)\lambda_i^2\lambda_j^2\lambda_l \\
&+(108\beta^3-4320\beta+648\beta^2+5184)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2) \\
&+(108\beta^2-7488+3528\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j), \\
\xi_3^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 2(-60480+30240\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l \\
&+2(5760\beta+74880)(\lambda_j^3+\lambda_i^3)\lambda_l \\
&+(37440+2880\beta)(\lambda_j^4+\lambda_i^4) \\
&+(10080\beta-28800+2160\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j) \\
&+(6480\beta^2-25920\beta+25920)\lambda_i^2\lambda_j^2, \\
\xi_3^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 2(-60480+30240\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2) \\
&+2(5760\beta+74880)(\lambda_j^3+\lambda_i^3).
\end{aligned}
$$

Define $\xi_{35}(\lambda_j;\lambda_i;\beta)=\xi_3^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{35} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{35} with respect to λ_j gives

$$\begin{aligned}
\xi_{35}(\lambda_j;\lambda_i;\beta) &= (11520\beta+149760)\lambda_j^3+(-120960+60480\beta)\lambda_i\lambda_j^2\\
&\quad +(-120960+60480\beta)\lambda_i^2\lambda_j+(11520\beta+149760)\lambda_i^3,\\
\xi_{35}^{(1)}(\lambda_j;\lambda_i;\beta) &= (34560\beta+449280)\lambda_j^2+(-241920+120960\beta)\lambda_i\lambda_j\\
&\quad +(-120960+60480\beta)\lambda_i^2,\\
\xi_{35}^{(2)}(\lambda_j;\lambda_i;\beta) &= (69120\beta+898560)\lambda_j+(-241920+120960\beta)\lambda_i,\\
\xi_{35}^{(3)}(\lambda_j;\lambda_i;\beta) &= 69120\beta+898560 \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{35}(\lambda_i;\lambda_i;\beta) &= 28800\lambda_i^3(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5},\\
\xi_{35}^{(1)}(\lambda_i;\lambda_i;\beta) &= 43200\lambda_i^2(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5},\\
\xi_{35}^{(2)}(\lambda_i;\lambda_i;\beta) &= 17280\lambda_i(38+11\beta) \geq 0 \text{ for all } \beta \geq -3.4545.
\end{aligned}$$

Hence, $\xi_3^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{34}(\lambda_j;\lambda_i;\beta)=\xi_3^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{34} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{34} with respect to λ_j gives

$$\begin{aligned}
\xi_{34}(\lambda_j;\lambda_i;\beta) &= (14400\beta+187200)\lambda_j^4+(70560\beta-149760+2160\beta^2)\lambda_i\lambda_j^3\\
&\quad +(-95040+34560\beta+6480\beta^2)\lambda_i^2\lambda_j^2+(21600\beta+120960+2160\beta^2)\lambda_i^3\lambda_j\\
&\quad +(37440+2880\beta)\lambda_i^4,\\
\xi_{34}^{(1)}(\lambda_j;\lambda_i;\beta) &= (57600\beta+748800)\lambda_j^3+(211680\beta-449280+6480\beta^2)\lambda_i\lambda_j^2\\
&\quad +(-190080+69120\beta+12960\beta^2)\lambda_i^2\lambda_j+(21600\beta+120960+2160\beta^2)\lambda_i^3,\\
\xi_{34}^{(2)}(\lambda_j;\lambda_i;\beta) &= (172800\beta+2246400)\lambda_j^2+(423360\beta-898560+12960\beta^2)\lambda_i\lambda_j\\
&\quad +(-190080+69120\beta+12960\beta^2)\lambda_i^2,\\
\xi_{34}^{(3)}(\lambda_j;\lambda_i;\beta) &= (345600\beta+4492800)\lambda_j+(423360\beta-898560+12960\beta^2)\lambda_i,\\
\xi_{34}^{(4)}(\lambda_j;\lambda_i;\beta) &= 345600\beta+4492800 \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{34}(\lambda_i;\lambda_i;\beta) &= 3600\lambda_i^4(28+40\beta+3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412,\\
\xi_{34}^{(1)}(\lambda_i;\lambda_i;\beta) &= 7200\lambda_i^3(\beta+16)(3\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3},\\
\xi_{34}^{(2)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^2(77\beta+134+3\beta^2) \geq 0 \text{ for all } \beta \geq -1.8776,\\
\xi_{34}^{(3)}(\lambda_i;\lambda_i;\beta) &= 4320\lambda_i(178\beta+832+3\beta^2) \geq 0 \text{ for all } \beta \geq -5.1151.
\end{aligned}$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{33}(\lambda_j;\lambda_i;\beta) = \xi_3^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{33} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{33} with respect to λ_j gives

$$\begin{aligned}
\xi_{33}(\lambda_j;\lambda_i;\beta) &= (8640\beta + 112320)\lambda_j^4 + (-96768 + 43848\beta + 2268\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(-29376 + 7128\beta^2 + 108\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(11520\beta + 51264 + 2808\beta^2 + 108\beta^3)\lambda_i^3\lambda_j \\
&\quad +(29952 + 6408\beta + 108\beta^2)\lambda_i^4, \\
\xi_{33}^{(1)}(\lambda_j;\lambda_i;\beta) &= (34560\beta + 449280)\lambda_j^3 + (-290304 + 131544\beta + 6804\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(-58752 + 14256\beta^2 + 216\beta^3)\lambda_i^2\lambda_j \\
&\quad +(11520\beta + 51264 + 2808\beta^2 + 108\beta^3)\lambda_i^3, \\
\xi_{33}^{(2)}(\lambda_j;\lambda_i;\beta) &= (103680\beta + 1347840)\lambda_j^2 + (-580608 + 263088\beta + 13608\beta^2)\lambda_i\lambda_j \\
&\quad +(-58752 + 14256\beta^2 + 216\beta^3)\lambda_i^2, \\
\xi_{33}^{(3)}(\lambda_j;\lambda_i;\beta) &= (207360\beta + 2695680)\lambda_j + (-580608 + 263088\beta + 13608\beta^2)\lambda_i, \\
\xi_{33}^{(4)}(\lambda_j;\lambda_i;\beta) &= 207360\beta + 2695680 \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{33}(\lambda_i;\lambda_i;\beta) &= 216\lambda_i^4(326\beta + 312 + 57\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -1.2060, \\
\xi_{33}^{(1)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^3(4934\beta + 4208 + 663\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -0.9803, \\
\xi_{33}^{(2)}(\lambda_i;\lambda_i;\beta) &= 216\lambda_i^2(1698\beta + 3280 + 129\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -2.34, \\
\xi_{33}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1944\lambda_i(242\beta + 1088 + 7\beta^2) \geq 0 \text{ for all } \beta \geq -5.3121.
\end{aligned}$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{32}(\lambda_j;\lambda_i;\beta) = \xi_3^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{32} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}
\xi_{32}(\lambda_j;\lambda_i;\beta) &= (43680 + 3360\beta)\lambda_j^4 + (1188\beta^2 - 42048 + 18648\beta)\lambda_i\lambda_j^3 \\
&\quad +(-3600 - 6624\beta + 108\beta^3 + 3996\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(13824 + 1656\beta^2 + 144\beta^3 + 3600\beta)\lambda_i^3\lambda_j \\
&\quad +(5544\beta + 9648 + 216\beta^2)\lambda_i^4, \\
\xi_{32}^{(1)}(\lambda_j;\lambda_i;\beta) &= (174720 + 13440\beta)\lambda_j^3 + (3564\beta^2 - 126144 + 55944\beta)\lambda_i\lambda_j^2 \\
&\quad +(-7200 - 13248\beta + 216\beta^3 + 7992\beta^2)\lambda_i^2\lambda_j \\
&\quad +(13824 + 1656\beta^2 + 144\beta^3 + 3600\beta)\lambda_i^3, \\
\xi_{32}^{(2)}(\lambda_j;\lambda_i;\beta) &= (524160 + 40320\beta)\lambda_j^2 + (7128\beta^2 - 252288 + 111888\beta)\lambda_i\lambda_j \\
&\quad +(-7200 - 13248\beta + 216\beta^3 + 7992\beta^2)\lambda_i^2, \\
\xi_{32}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1048320 + 80640\beta)\lambda_j + (7128\beta^2 - 252288 + 111888\beta)\lambda_i, \\
\xi_{32}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1048320 + 80640\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i;\lambda_i;\beta) &= 84\lambda_i^4(256+292\beta+84\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq -1.4510,\\
\xi_{32}^{(1)}(\lambda_i;\lambda_i;\beta) &= 12\lambda_i^3(4600+4978\beta+1101\beta^2+30\beta^3)\geq 0 \text{ for all } \beta\geq -1.2667,\\
\xi_{32}^{(2)}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i^2(3676+1930\beta+210\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq -2.6278,\\
\xi_{32}^{(3)}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i(11056+2674\beta+99\beta^2)\geq 0 \text{ for all } \beta\geq -5.0961.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j;\lambda_i;\beta)=\xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{31} is non-negative for all $\lambda_j\geq\lambda_i\geq 0$ for all $\beta\geq$ Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j;\lambda_i;\beta) &= (12480+960\beta)\lambda_j^4+(5964\beta+414\beta^2-13584)\lambda_i\lambda_j^3\\
&\quad +(288+54\beta^3-3384\beta+1512\beta^2)\lambda_i^2\lambda_j^2\\
&\quad +(3120+630\beta^2+90\beta^3+1140\beta)\lambda_i^3\lambda_j\\
&\quad +(3000\beta+1920+180\beta^2)\lambda_i^4,\\
\xi_{31}^{(1)}(\lambda_j;\lambda_i;\beta) &= (49920+3840\beta)\lambda_j^3+(17892\beta+1242\beta^2-40752)\lambda_i\lambda_j^2\\
&\quad +(576+108\beta^3-6768\beta+3024\beta^2)\lambda_i^2\lambda_j\\
&\quad +(3120+630\beta^2+90\beta^3+1140\beta)\lambda_i^3,\\
\xi_{31}^{(2)}(\lambda_j;\lambda_i;\beta) &= (149760+11520\beta)\lambda_j^2+(35784\beta+2484\beta^2-81504)\lambda_i\lambda_j\\
&\quad +(576+108\beta^3-6768\beta+3024\beta^2)\lambda_i^2,\\
\xi_{31}^{(3)}(\lambda_j;\lambda_i;\beta) &= (299520+23040\beta)\lambda_j+(35784\beta+2484\beta^2-81504)\lambda_i,\\
\xi_{31}^{(4)}(\lambda_j;\lambda_i;\beta) &= 299520+23040\beta\geq 0 \text{ for all } \beta\geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{31}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i^4(88+160\beta+57\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq -0.7350,\\
\xi_{31}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(2144+2684\beta+816\beta^2+33\beta^3)\geq 0 \text{ for all } \beta\geq -1.2494,\\
\xi_{31}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(1912+1126\beta+153\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq -2.5154,\\
\xi_{31}^{(3)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i(6056+1634\beta+69\beta^2)\geq 0 \text{ for all } \beta\geq -4.5996.
\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j;\lambda_i;\beta)=\xi_3(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}
\xi_{30}(\lambda_j;\lambda_i;\beta) &= (12480+960\beta)\lambda_j^4+(5964\beta+414\beta^2-13584)\lambda_i\lambda_j^3 \\
&\quad +(288+54\beta^3-3384\beta+1512\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(3120+630\beta^2+90\beta^3+1140\beta)\lambda_i^3\lambda_j \\
&\quad +(3000\beta+1920+180\beta^2)\lambda_i^4, \\
\xi_{30}^{(1)}(\lambda_j;\lambda_i;\beta) &= (49920+3840\beta)\lambda_j^3+(17892\beta+1242\beta^2-40752)\lambda_i\lambda_j^2 \\
&\quad +(576+108\beta^3-6768\beta+3024\beta^2)\lambda_i^2\lambda_j \\
&\quad +(3120+630\beta^2+90\beta^3+1140\beta)\lambda_i^3, \\
\xi_{30}^{(2)}(\lambda_j;\lambda_i;\beta) &= (149760+11520\beta)\lambda_j^2+(35784\beta+2484\beta^2-81504)\lambda_i\lambda_j \\
&\quad +(576+108\beta^3-6768\beta+3024\beta^2)\lambda_i^2, \\
\xi_{30}^{(3)}(\lambda_j;\lambda_i;\beta) &= (299520+23040\beta)\lambda_j+(35784\beta+2484\beta^2-81504)\lambda_i, \\
\xi_{30}^{(4)}(\lambda_j;\lambda_i;\beta) &= 299520+23040\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{30}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i^4(88+160\beta+57\beta^2+3\beta^3) > 0 \text{ for all } \beta > -0.7350, \\
\xi_{30}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6\lambda_i^3(2144+2684\beta+816\beta^2+33\beta^3) > 0 \text{ for all } \beta > -1.2494, \\
\xi_{30}^{(2)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i^2(1912+1126\beta+153\beta^2+3\beta^3) > 0 \text{ for all } \beta > -2.5154, \\
\xi_{30}^{(3)}(\lambda_i;\lambda_i;\beta) &= 36\lambda_i(6056+1634\beta+69\beta^2) > 0 \text{ for all } \beta > -4.5996.
\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &= (-1008+504\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^4 \\
&\quad +(96\beta+1248)(\lambda_j^3+\lambda_i^3)\lambda_l^4 \\
&\quad +(72\beta^2-960+336\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^3 \\
&\quad +(96\beta+1248)(\lambda_i^4+\lambda_j^4)\lambda_l^3 \\
&\quad +(864-864\beta+216\beta^2)\lambda_i^2\lambda_j^2\lambda_l^3 \\
&\quad +(-1008+504\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^2 \\
&\quad +(864-864\beta+216\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^2 \\
&\quad +(72\beta^2-960+336\beta)\lambda_i^3\lambda_j^3\lambda_l \\
&\quad +(-1008+504\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l \\
&\quad +(96\beta+1248)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3),
\end{aligned}$$

$$
\begin{aligned}
\xi_4^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 4(-1008+504\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^3 \\
&\quad +4(96\beta+1248)(\lambda_j^3+\lambda_i^3)\lambda_l^3 \\
&\quad +3(72\beta^2-960+336\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^2 \\
&\quad +3(96\beta+1248)(\lambda_i^4+\lambda_j^4)\lambda_l^2 \\
&\quad +3(864-864\beta+216\beta^2)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&\quad +2(-1008+504\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l \\
&\quad +2(864-864\beta+216\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l \\
&\quad +(72\beta^2-960+336\beta)\lambda_i^3\lambda_j^3 \\
&\quad +(-1008+504\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2), \\
\xi_4^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 12(-1008+504\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l^2 \\
&\quad +12(96\beta+1248)(\lambda_j^3+\lambda_i^3)\lambda_l^2 \\
&\quad +6(72\beta^2-960+336\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l \\
&\quad +6(96\beta+1248)(\lambda_i^4+\lambda_j^4)\lambda_l \\
&\quad +6(864-864\beta+216\beta^2)\lambda_i^2\lambda_j^2\lambda_l \\
&\quad +2(-1008+504\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j) \\
&\quad +2(864-864\beta+216\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2), \\
\xi_4^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(-1008+504\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2)\lambda_l \\
&\quad +24(96\beta+1248)(\lambda_j^3+\lambda_i^3)\lambda_l \\
&\quad +6(72\beta^2-960+336\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j) \\
&\quad +6(96\beta+1248)(\lambda_i^4+\lambda_j^4) \\
&\quad +6(864-864\beta+216\beta^2)\lambda_i^2\lambda_j^2, \\
\xi_4^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(-1008+504\beta)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2) \\
&\quad +24(96\beta+1248)(\lambda_j^3+\lambda_i^3).
\end{aligned}
$$

Define $\xi_{44}(\lambda_j;\lambda_i;\beta)=\xi_4^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{44} is non-negative for all $\lambda_j\geq\lambda_i\geq 0$ for all $\beta\geq$ Differentiating ξ_{44} with respect to λ_j gives

$$
\begin{aligned}
\xi_{44}(\lambda_j;\lambda_i;\beta) &= (2304\beta+29952)\lambda_j^3+(-24192+12096\beta)\lambda_i\lambda_j^2 \\
&\quad +(-24192+12096\beta)\lambda_i^2\lambda_j+(2304\beta+29952)\lambda_i^3, \\
\xi_{44}^{(1)}(\lambda_j;\lambda_i;\beta) &= (6912\beta+89856)\lambda_j^2+(-48384+24192\beta)\lambda_i\lambda_j \\
&\quad +(-24192+12096\beta)\lambda_i^2, \\
\xi_{44}^{(2)}(\lambda_j;\lambda_i;\beta) &= (13824\beta+179712)\lambda_j+(-48384+24192\beta)\lambda_i, \\
\xi_{44}^{(3)}(\lambda_j;\lambda_i;\beta) &= 13824\beta+179712\geq 0 \text{ for all } \beta\geq 13.
\end{aligned}
$$

Now

$$\begin{aligned}
\xi_{44}(\lambda_i;\lambda_i;\beta) &= 5760\lambda_i^3(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5},\\
\xi_{44}^{(1)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^2(2+5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5},\\
\xi_{44}^{(2)}(\lambda_i;\lambda_i;\beta) &= 3456\lambda_i(38+11\beta) \geq 0 \text{ for all } \beta \geq -3.4545.
\end{aligned}$$

Hence, $\xi_4^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{43}(\lambda_j;\lambda_i;\beta) = \xi_4^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{43} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{43} with respect to λ_j gives

$$\begin{aligned}
\xi_{43}(\lambda_j;\lambda_i;\beta) &= (2880\beta+37440)\lambda_j^4 + (14112\beta-29952+432\beta^2)\lambda_i\lambda_j^3\\
&\quad +(-19008+6912\beta+1296\beta^2)\lambda_i^2\lambda_j^2 + (4320\beta+24192+432\beta^2)\lambda_i^3\lambda_j\\
&\quad +(576\beta+7488)\lambda_i^4,\\
\xi_{43}^{(1)}(\lambda_j;\lambda_i;\beta) &= (149760+11520\beta)\lambda_j^3 + (42336\beta-89856+1296\beta^2)\lambda_i\lambda_j^2\\
&\quad +(-38016+13824\beta+2592\beta^2)\lambda_i^2\lambda_j + (4320\beta+24192+432\beta^2)\lambda_i^3,\\
\xi_{43}^{(2)}(\lambda_j;\lambda_i;\beta) &= (449280+34560\beta)\lambda_j^2 + (84672\beta-179712+2592\beta^2)\lambda_i\lambda_j\\
&\quad +(-38016+13824\beta+2592\beta^2)\lambda_i^2,\\
\xi_{43}^{(3)}(\lambda_j;\lambda_i;\beta) &= (898560+69120\beta)\lambda_j + (84672\beta-179712+2592\beta^2)\lambda_i,\\
\xi_{43}^{(4)}(\lambda_j;\lambda_i;\beta) &= 898560+69120\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{43}(\lambda_i;\lambda_i;\beta) &= 720\lambda_i^4(28+40\beta+3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412,\\
\xi_{43}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1440\lambda_i^3(\beta+16)(3\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3},\\
\xi_{43}^{(2)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i^2(134+77\beta+3\beta^2) \geq 0 \text{ for all } \beta \geq -1.8776,\\
\xi_{43}^{(3)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i(832+178\beta+3\beta^2) \geq 0 \text{ for all } \beta \geq -5.1151.
\end{aligned}$$

Hence, $\xi_4^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{42}(\lambda_j;\lambda_i;\beta) = \xi_4^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{42} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned}
\xi_{42}(\lambda_j;\lambda_i;\beta) &= (1728\beta+22464)\lambda_j^4 + (-19872+9072\beta+432\beta^2)\lambda_i\lambda_j^3\\
&\quad +(-5184-864\beta+1728\beta^2)\lambda_i^2\lambda_j^2 + (1440\beta+10944+864\beta^2)\lambda_i^3\lambda_j\\
&\quad +(1584\beta+5472)\lambda_i^4,\\
\xi_{42}^{(1)}(\lambda_j;\lambda_i;\beta) &= (6912\beta+89856)\lambda_j^3 + (-59616+27216\beta+1296\beta^2)\lambda_i\lambda_j^2\\
&\quad +(-10368-1728\beta+3456\beta^2)\lambda_i^2\lambda_j + (1440\beta+10944+864\beta^2)\lambda_i^3,\\
\xi_{42}^{(2)}(\lambda_j;\lambda_i;\beta) &= (20736\beta+269568)\lambda_j^2 + (-119232+54432\beta+2592\beta^2)\lambda_i\lambda_j\\
&\quad +(-10368-1728\beta+3456\beta^2)\lambda_i^2,\\
\xi_{42}^{(3)}(\lambda_j;\lambda_i;\beta) &= (41472\beta+539136)\lambda_j + (-119232+54432\beta+2592\beta^2)\lambda_i,\\
\xi_{42}^{(4)}(\lambda_j;\lambda_i;\beta) &= 41472\beta+539136 \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{42}(\lambda_i;\lambda_i;\beta) &= 432\lambda_i^4(7\beta+16)(\beta+2) \geq 0 \text{ for all } \beta \geq -2,\\
\xi_{42}^{(1)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^3(235\beta+214+39\beta^2) \geq 0 \text{ for all } \beta \geq -1.1181,\\
\xi_{42}^{(2)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^2(85\beta+162+7\beta^2) \geq 0 \text{ for all } \beta \geq -2.3675,\\
\xi_{42}^{(3)}(\lambda_i;\lambda_i;\beta) &= 2592\lambda_i(37\beta+162+\beta^2) \geq 0 \text{ for all } \beta \geq -5.0743.
\end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{41}(\lambda_j;\lambda_i;\beta) = \xi_4^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}
\xi_{41}(\lambda_j;\lambda_i;\beta) &= (8736+672\beta)\lambda_j^4 + (-8928+4032\beta+216\beta^2)\lambda_i\lambda_j^3\\
&\quad +(-720-1800\beta+1080\beta^2)\lambda_i^2\lambda_j^2 + (2880+720\beta^2)\lambda_i^3\lambda_j\\
&\quad +(720+1800\beta)\lambda_i^4,\\
\xi_{41}^{(1)}(\lambda_j;\lambda_i;\beta) &= (34944+2688\beta)\lambda_j^3 + (-26784+12096\beta+648\beta^2)\lambda_i\lambda_j^2\\
&\quad +(-1440-3600\beta+2160\beta^2)\lambda_i^2\lambda_j + (2880+720\beta^2)\lambda_i^3,\\
\xi_{41}^{(2)}(\lambda_j;\lambda_i;\beta) &= (104832+8064\beta)\lambda_j^2 + (-53568+24192\beta+1296\beta^2)\lambda_i\lambda_j\\
&\quad +(-1440-3600\beta+2160\beta^2)\lambda_i^2,\\
\xi_{41}^{(3)}(\lambda_j;\lambda_i;\beta) &= (209664+16128\beta)\lambda_j + (-53568+24192\beta+1296\beta^2)\lambda_i,\\
\xi_{41}^{(4)}(\lambda_j;\lambda_i;\beta) &= 209664+16128\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{41}(\lambda_i;\lambda_i;\beta) &= 672\lambda_i^4(3\beta+4)(\beta+1) \geq 0 \text{ for all } \beta \geq -1,\\
\xi_{41}^{(1)}(\lambda_i;\lambda_i;\beta) &= 24\lambda_i^3(400+466\beta+147\beta^2) \geq 0 \text{ for all } \beta,\\
\xi_{41}^{(2)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^2(346+199\beta+24\beta^2) \geq 0 \text{ for all } \beta \geq -2.4811,\\
\xi_{41}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(1084+280\beta+9\beta^2) \geq 0 \text{ for all } \beta \geq -4.5315.
\end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j;\lambda_i;\beta) = \xi_4(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}
\xi_{40}(\lambda_j;\lambda_i;\beta) &= (192\beta+2496)\lambda_j^4 + (-2976+1344\beta+72\beta^2)\lambda_i\lambda_j^3\\
&\quad +(-288-720\beta+432\beta^2)\lambda_i^2\lambda_j^2 + (1440+360\beta^2)\lambda_i^3\lambda_j\\
&\quad +(1200\beta+480)\lambda_i^4,\\
\xi_{40}^{(1)}(\lambda_j;\lambda_i;\beta) &= (768\beta+9984)\lambda_j^3 + (-8928+4032\beta+216\beta^2)\lambda_i\lambda_j^2\\
&\quad +(-576-1440\beta+864\beta^2)\lambda_i^2\lambda_j + (1440+360\beta^2)\lambda_i^3,\\
\xi_{40}^{(2)}(\lambda_j;\lambda_i;\beta) &= (2304\beta+29952)\lambda_j^2 + (-17856+8064\beta+432\beta^2)\lambda_i\lambda_j\\
&\quad +(-576-1440\beta+864\beta^2)\lambda_i^2,\\
\xi_{40}^{(3)}(\lambda_j;\lambda_i;\beta) &= (4608\beta+59904)\lambda_j + (-17856+8064\beta+432\beta^2)\lambda_i,\\
\xi_{40}^{(4)}(\lambda_j;\lambda_i;\beta) &= 4608\beta+59904 \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{40}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i^4(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1,\\
\xi_{40}^{(1)}(\lambda_i;\lambda_i;\beta) &= 480\lambda_i^3(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1,\\
\xi_{40}^{(2)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^2(62\beta+80+9\beta^2) > 0 \text{ for all } \beta > -1.7195,\\
\xi_{40}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(88\beta+292+3\beta^2) > 0 \text{ for all } \beta > -3.8141.
\end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.13 Let

$$\begin{aligned}
\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (-1296\beta+324\beta^2+1296)t(2,2,2,4)\\
&\quad +(336\beta-672)t(1,1,4,4)\\
&\quad +(-864+108\beta^2+216\beta)t(1,2,3,4)\\
&\quad +(12\beta^2+1504+184\beta)t(0,3,3,4)\\
&\quad +(132\beta+456)t(0,2,4,4)\\
&\quad +(24\beta^3+864\beta-1920)t(1,3,3,3)\\
&\quad +(1152+72\beta^3-144\beta^2-576\beta)t(2,2,3,3),
\end{aligned}$$

where

$$t(a,b,c,d) = \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\mathrm{perm}(a,b,c,d)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_k^{n_4}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(a,b,c)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 4(-1296\beta+324\beta^2+1296)\lambda_k^3\hat{s}(2,2,2)\\
&+4(336\beta-672)\lambda_k^3\hat{s}(1,1,4)\\
&+4(-864+108\beta^2+216\beta)\lambda_k^3\hat{s}(1,2,3)\\
&+4(12\beta^2+1504+184\beta)\lambda_k^3\hat{s}(0,3,3)\\
&+4(132\beta+456)\lambda_k^3\hat{s}(0,2,4)\\
&+3(-576\beta+1152+72\beta^3-144\beta^2)\lambda_k^2\hat{s}(2,2,3)\\
&+3(-864+108\beta^2+216\beta\lambda_k^2\hat{s}(1,2,4)\\
&+3(24\beta^3+864\beta-1920)\lambda_k^2\hat{s}(1,3,3)\\
&+3(12\beta^2+1504+184\beta)\lambda_k^2\hat{s}(0,3,4)\\
&+2(132\beta+456)\lambda_k\hat{s}(0,4,4)\\
&+2(-576\beta+1152+72\beta^3-144\beta^2)\lambda_k\hat{s}(2,3,3)\\
&+2(-1296\beta+324\beta^2+1296)\lambda_k\hat{s}(2,2,4)\\
&+2(-864+108\beta^2+216\beta)\lambda_k\hat{s}(1,3,4)\\
&+(336\beta-672)\hat{s}(1,4,4)\\
&+(24\beta^3+864\beta-1920)\hat{s}(3,3,3)\\
&+(-864+108\beta^2+216\beta)\hat{s}(2,3,4),\\
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (3888\beta^2-15552\beta+15552)\lambda_k^2\hat{s}(2,2,2)\\
&+(2592\beta+1296\beta^2-10368)\lambda_k^2\hat{s}(1,2,3)\\
&+(-8064+4032\beta)\lambda_k^2\hat{s}(1,1,4)\\
&+(18048+144\beta^2+2208\beta)\lambda_k^2\hat{s}(0,3,3)\\
&+(5472+1584\beta)\lambda_k^2\hat{s}(0,2,4)\\
&+(432\beta^3-3456\beta-864\beta^2+6912)\lambda_k\hat{s}(2,2,3)\\
&+(9024+1104\beta+72\beta^2)\lambda_k\hat{s}(0,3,4)\\
&+(-5184+1296\beta+648\beta^2)\lambda_k\hat{s}(1,2,4)\\
&+(5184\beta-11520+144\beta^3)\lambda_k\hat{s}(1,3,3)\\
&+(648\beta^2+2592-2592\beta)\hat{s}(2,4,4)\\
&+(-1728+216\beta^2+432\beta)\hat{s}(1,3,4)\\
&+(2304+144\beta^3-1152\beta-288\beta^2)\hat{s}(2,3,3)\\
&+(264\beta+912)\hat{s}(0,4,4),
\end{aligned}$$

$$
\begin{aligned}
\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) \;=\;& (7776\beta^2+31104-31104\beta)\lambda_k\hat{s}(2,2,2)\\
&+(10944+3168\beta)\lambda_k\hat{s}(0,2,4)\\
&+(8064\beta-16128)\lambda_k\hat{s}(1,1,4)\\
&+(5184\beta+2592\beta^2-20736)\lambda_k\hat{s}(1,2,3)\\
&+(4416\beta+288\beta^2+36096)\lambda_k\hat{s}(0,3,3)\\
&+(432\beta^3-3456\beta-864\beta^2+6912)\hat{s}(2,2,3)\\
&+(9024+1104\beta+72\beta^2)\hat{s}(0,3,4)\\
&+(-5184+1296\beta+648\beta^2)\hat{s}(1,2,4)\\
&+(5184\beta-11520+144\beta^3)\hat{s}(1,3,3),\\
\xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) \;=\;& (7776\beta^2+31104-31104\beta)\hat{s}(2,2,2)\\
&+(10944+3168\beta)\hat{s}(0,2,4)\\
&+(8064\beta-16128)\hat{s}(1,1,4)\\
&+(5184\beta+2592\beta^2-20736)\hat{s}(1,2,3)\\
&+(4416\beta+288\beta^2+36096)\hat{s}(0,3,3).
\end{aligned}
$$

Define

$$
\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l^2,\\
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l,\\
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta).
\end{aligned}
$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$
\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (132\beta+456)(\lambda_j^2+\lambda_i^2)\lambda_l^6+(336\beta-672)\lambda_i\lambda_j\lambda_l^6\\
&\quad +(-1728+216\beta^2+432\beta)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^5\\
&\quad +(3008+368\beta+24\beta^2)(\lambda_j^3+\lambda_i^3)\lambda_l^5\\
&\quad +(24\beta^3-3648+216\beta^2+1296\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^4\\
&\quad +(2416+448\beta+12\beta^2)(\lambda_j^4+\lambda_i^4)\lambda_l^4\\
&\quad +(-3168\beta+72\beta^3+3744+504\beta^2)\lambda_i^2\lambda_j^2\lambda_l^4\\
&\quad +(-720\beta+144\beta^3-72\beta^2+576)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^3\\
&\quad +(1104\beta+216\beta^2-3072)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^3\\
&\quad +(540\beta^2+480-600\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l^2\\
&\quad +(120\beta^3-120\beta^2+320+1520\beta)\lambda_i^3\lambda_j^3\lambda_l^2\\
&\quad +(1280+240\beta^2+800\beta)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\lambda_l\\
&\quad +(600\beta+240)\lambda_i^4\lambda_j^4,
\end{aligned}
$$

Define $\xi_{06}(\lambda_j;\lambda_i;\beta)=\xi_0^{(6)}(\lambda_l;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{06} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{06} with respect to λ_j gives

$$
\begin{aligned}
\xi_{06}(\lambda_j;\lambda_i;\beta) &= (95040\beta+328320)\lambda_j^2+(-483840+241920\beta)\lambda_i\lambda_j\\
&\quad +(95040\beta+328320)\lambda_i^2,\\
\xi_{06}^{(1)}(\lambda_j;\lambda_i;\beta) &= (190080\beta+656640)\lambda_j+(-483840+241920\beta)\lambda_i,\\
\xi_{06}^{(2)}(\lambda_j;\lambda_i;\beta) &= 190080\beta+656640\geq 0 \text{ for all } \beta\geq -3.4545.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{06}(\lambda_i;\lambda_i;\beta) &= 86400\lambda_i^2(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5},\\
\xi_{06}^{(1)}(\lambda_i;\lambda_i;\beta) &= 86400\lambda_i(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}
$$

Hence, $\xi_0^{(6)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{05}(\lambda_j;\lambda_i;\beta)=\xi_0^{(5)}(\lambda_j;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{05} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{05} with

respect to λ_j gives

$$\begin{aligned}
\xi_{05}(\lambda_j;\lambda_i;\beta) &= (139200\beta+689280+2880\beta^2)\lambda_j^3\\
&\quad +(-691200+293760\beta+25920\beta^2)\lambda_i\lambda_j^2\\
&\quad +(146880\beta+120960+25920\beta^2)\lambda_i^2\lambda_j\\
&\quad +(2880\beta^2+44160\beta+360960)\lambda_i^3,\\
\xi_{05}^{(1)}(\lambda_j;\lambda_i;\beta) &= (417600\beta+2067840+8640\beta^2)\lambda_j^2\\
&\quad +(-1382400+587520\beta+51840\beta^2)\lambda_i\lambda_j\\
&\quad +(146880\beta+120960+25920\beta^2)\lambda_i^2,\\
\xi_{05}^{(2)}(\lambda_j;\lambda_i;\beta) &= (835200\beta+4135680+17280\beta^2)\lambda_j\\
&\quad +(-1382400+587520\beta+51840\beta^2)\lambda_i,\\
\xi_{05}^{(3)}(\lambda_j;\lambda_i;\beta) &= 835200\beta+4135680+17280\beta^2 \geq 0 \text{ for all } \beta \geq -5.6007.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{05}(\lambda_i;\lambda_i;\beta) &= 9600\lambda_i^3(\beta+10)(6\beta+5) \geq 0 \text{ for all } \beta \geq -\frac{5}{6},\\
\xi_{05}^{(1)}(\lambda_i;\lambda_i;\beta) &= 28800\lambda_i^2(28+40\beta+3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412,\\
\xi_{05}^{(2)}(\lambda_i;\lambda_i;\beta) &= 5760\lambda_i(247\beta+478+12\beta^2) \geq 0 \text{ for all } \beta \geq -2.1624.
\end{aligned}$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{04}(\lambda_j;\lambda_i;\beta) = \xi_0^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{04} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned}
\xi_{04}(\lambda_j;\lambda_i;\beta) &= (102432\beta+583104+3168\beta^2)\lambda_j^4\\
&\quad +(203904\beta+31104\beta^2-536832+576\beta^3)\lambda_i\lambda_j^3\\
&\quad +(23328\beta+46656+38016\beta^2+1728\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(8064\beta^2+75264\beta+273408+576\beta^3)\lambda_i^3\lambda_j\\
&\quad +(57984+288\beta^2+10752\beta)\lambda_i^4,\\
\xi_{04}^{(1)}(\lambda_j;\lambda_i;\beta) &= (203904\beta+31104\beta^2-536832+576\beta^3)\lambda_j^3\\
&\quad +2(23328\beta+46656+38016\beta^2+1728\beta^3)\lambda_i\lambda_j^2\\
&\quad +3(8064\beta^2+75264\beta+273408+576\beta^3)\lambda_i^2\lambda_j\\
&\quad +4(57984+10752\beta+288\beta^2)\lambda_i^3,\\
\xi_{04}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1229184\beta+6997248+38016\beta^2)\lambda_j^2\\
&\quad +(1223424\beta+186624\beta^2-3220992+3456\beta^3)\lambda_i\lambda_j\\
&\quad +(46656\beta+93312+76032\beta^2+3456\beta^3)\lambda_i^2,\\
\xi_{04}^{(3)}(\lambda_j;\lambda_i;\beta) &= (2458368\beta+13994496+76032\beta^2)\lambda_j\\
&\quad +(1223424\beta+186624\beta^2-3220992+3456\beta^3)\lambda_i,\\
\xi_{04}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2458368\beta+13994496+76032\beta^2 \geq 0 \text{ for all } \beta \geq -7.3746.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{04}(\lambda_i;\lambda_i;\beta) &= 960\lambda_i^4(433\beta+442+84\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -1.3643,\\
\xi_{04}^{(1)}(\lambda_i;\lambda_i;\beta) &= 2880\lambda_i^3(397\beta+378+66\beta^2+2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1726,\\
\xi_{04}^{(2)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i^2(4339\beta+6718+522\beta^2+12\beta^3) \geq 0 \text{ for all } \beta \geq -2.0134,\\
\xi_{04}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1152\lambda_i(3196\beta+9352+228\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -4.0159.
\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j;\lambda_i;\beta) = \xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$, where all derivatives are with respect to λ_l. We wish to show that ξ_{03} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}
\xi_{03}(\lambda_j;\lambda_i;\beta) &= (48672\beta+293184+1728\beta^2)\lambda_j^4\\
&\quad +(103968\beta-290304+19440\beta^2+576\beta^3)\lambda_i\lambda_j^3\\
&\quad +(-38592\beta+44352+24624\beta^2+2592\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(96384+6192\beta^2+48864\beta+1440\beta^3)\lambda_i^3\lambda_j\\
&\quad +(39552+1584\beta^2+17376\beta)\lambda_i^4,\\
\xi_{03}^{(1)}(\lambda_j;\lambda_i;\beta) &= (194688\beta+1172736+6912\beta^2)\lambda_j^3\\
&\quad +(311904\beta-870912+58320\beta^2+1728\beta^3)\lambda_i\lambda_j^2\\
&\quad +(-77184\beta+88704+49248\beta^2+5184\beta^3)\lambda_i^2\lambda_j\\
&\quad +(96384+6192\beta^2+48864\beta+1440\beta^3)\lambda_i^3,\\
\xi_{03}^{(2)}(\lambda_j;\lambda_i;\beta) &= (584064\beta+3518208+20736\beta^2)\lambda_j^2\\
&\quad +(623808\beta-1741824+116640\beta^2+3456\beta^3)\lambda_i\lambda_j\\
&\quad +(-77184\beta+88704+49248\beta^2+5184\beta^3)\lambda_i^2,\\
\xi_{03}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1168128\beta+7036416+41472\beta^2)\lambda_j\\
&\quad +(623808\beta-1741824+116640\beta^2+3456\beta^3)\lambda_i,\\
\xi_{03}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1168128\beta+7036416+41472\beta^2 \geq 0 \text{ for all } \beta \geq -8.7286.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{03}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i^4(\beta+3)(\beta+2)(8\beta+53) \geq 0 \text{ for all } \beta \geq -2,\\
\xi_{03}^{(1)}(\lambda_i;\lambda_i;\beta) &= 96\lambda_i^3(4982\beta+5072+1257\beta^2+87\beta^3) \geq 0 \text{ for all } \beta \geq -1.5772,\\
\xi_{03}^{(2)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i^2(1963\beta+3238+324\beta^2+15\beta^3) \geq 0 \text{ for all } \beta \geq -2.7086,\\
\xi_{03}^{(3)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i(2074\beta+6128+183\beta^2+4\beta^3) \geq 0 \text{ for all } \beta \geq -4.7120.
\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j;\lambda_i;\beta) = \xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$, where all derivatives are with respect to λ_l. We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with

respect to λ_j gives

$$\begin{aligned}
\xi_{02}(\lambda_j;\lambda_i;\beta) &= (102832+16696\beta+624\beta^2)\lambda_j^4\\
&\quad+(40896\beta-116928+8208\beta^2+288\beta^3)\lambda_i\lambda_j^3\\
&\quad+(28464-30936\beta+11016\beta^2+1728\beta^3)\lambda_i^2\lambda_j^2\\
&\quad+(21632\beta+2400\beta^2+20480+1392\beta^3)\lambda_i^3\lambda_j\\
&\quad+(11520+10800\beta+2520\beta^2)\lambda_i^4,\\
\xi_{02}^{(1)}(\lambda_j;\lambda_i;\beta) &= (411328+66784\beta+2496\beta^2)\lambda_j^3\\
&\quad+(122688\beta-350784+24624\beta^2+864\beta^3)\lambda_i\lambda_j^2\\
&\quad+(56928-61872\beta+22032\beta^2+3456\beta^3)\lambda_i^2\lambda_j\\
&\quad+(21632\beta+2400\beta^2+20480+1392\beta^3)\lambda_i^3,\\
\xi_{02}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1233984+200352\beta+7488\beta^2)\lambda_j^2\\
&\quad+(245376\beta-701568+49248\beta^2+1728\beta^3)\lambda_i\lambda_j\\
&\quad+(56928-61872\beta+22032\beta^2+3456\beta^3)\lambda_i^2,\\
\xi_{02}^{(3)}(\lambda_j;\lambda_i;\beta) &= (2467968+400704\beta+14976\beta^2)\lambda_j\\
&\quad+(245376\beta-701568+49248\beta^2+1728\beta^3)\lambda_i,\\
\xi_{02}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2467968+400704\beta+14976\beta^2\geq 0 \text{ for all } \beta\geq -9.6124.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{02}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i^4(\beta+3)(\beta+2)(71\beta+161)\geq 0 \text{ for all } \beta\geq -2,\\
\xi_{02}^{(1)}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i^3(\beta+3)(\beta+2)(119\beta+479)\geq 0 \text{ for all } \beta\geq -2,\\
\xi_{02}^{(2)}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i^2(12278+7997\beta+1641\beta^2+108\beta^3)\geq 0 \text{ for all } \beta\geq -3.1439,\\
\xi_{02}^{(3)}(\lambda_i;\lambda_i;\beta) &= 96\lambda_i(\beta+10)(18\beta^2+489\beta+1840)\geq 0 \text{ for all } \beta\geq -4.5122.
\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$, where all derivatives are with respect to λ_l. We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with

respect to λ_j gives

$$
\begin{aligned}
\xi_{01}(\lambda_j;\lambda_i;\beta) &= (27440+4424\beta+168\beta^2)\lambda_j^4 \\
&\quad +(12672\beta-36480+2592\beta^2+96\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(11760-13080\beta+3960\beta^2+720\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(768\beta^2+4096+8704\beta+768\beta^3)\lambda_i^3\lambda_j \\
&\quad +(2016\beta^2+2688+4704\beta)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= (109760+17696\beta+672\beta^2)\lambda_j^3 \\
&\quad +(38016\beta-109440+7776\beta^2+288\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(23520-26160\beta+7920\beta^2+1440\beta^3)\lambda_i^2\lambda_j \\
&\quad +(768\beta^2+4096+8704\beta+768\beta^3)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= (329280+53088\beta+2016\beta^2)\lambda_j^2 \\
&\quad +(76032\beta-218880+15552\beta^2+576\beta^3)\lambda_i\lambda_j \\
&\quad +(23520-26160\beta+7920\beta^2+1440\beta^3)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j;\lambda_i;\beta) &= (658560+106176\beta+4032\beta^2)\lambda_j \\
&\quad +(76032\beta-218880+15552\beta^2+576\beta^3)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1344(3\beta+49)(\beta+10) \geq 0 \text{ for all } \beta \geq -10.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 1584\lambda_i^4(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i^3(\beta+3)(\beta+2)(52\beta+97) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{01}^{(2)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^2(\beta+3)(14\beta^2+135\beta+310) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i;\lambda_i;\beta) &= 192\lambda_i(\beta+10)(3\beta^2+72\beta+229) \geq 0 \text{ for all } \beta \geq -3.7740.
\end{aligned}
$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$, where all derivatives are with respect to λ_l. We wish to show that ξ_{00} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with

respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= (948\beta+5880+36\beta^2)\lambda_j^4+(3168\beta-9120+648\beta^2+24\beta^3)\lambda_i\lambda_j^3\\
&\quad +(-3924\beta+3528+1188\beta^2+216\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(1536+3264\beta+288\beta^2+288\beta^3)\lambda_i^3\lambda_j\\
&\quad +(1344+2352\beta+1008\beta^2)\lambda_i^4,\\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3792\beta+23520+144\beta^2)\lambda_j^3+(9504\beta-27360+1944\beta^2+72\beta^3)\lambda_i\lambda_j^2\\
&\quad +(-7848\beta+7056+2376\beta^2+432\beta^3)\lambda_i^2\lambda_j\\
&\quad +(1536+3264\beta+288\beta^2+288\beta^3)\lambda_i^3,\\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= (11376\beta+70560+432\beta^2)\lambda_j^2+(19008\beta-54720+3888\beta^2+144\beta^3)\lambda_i\lambda_j\\
&\quad +(-7848\beta+7056+2376\beta^2+432\beta^3)\lambda_i^2,\\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= (22752\beta+141120+864\beta^2)\lambda_j+(19008\beta-54720+3888\beta^2+144\beta^3)\lambda_i,\\
\xi_{00}^{(4)}(\lambda_j;\lambda_i;\beta) &= 288(3\beta+49)(\beta+10)\geq 0 \text{ for all } \beta\geq -10.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 528\lambda_i^4(\beta+3)(\beta+2)(\beta+1)>0 \text{ for all } \beta>-1,\\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 792\lambda_i^3(\beta+3)(\beta+2)(\beta+1)>0 \text{ for all } \beta>-1,\\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 72\lambda_i^2(\beta+3)(\beta+2)(8\beta+53)>0 \text{ for all } \beta>-2,\\
\xi_{00}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(\beta+10)(\beta+20)(\beta+3)>0 \text{ for all } \beta>-3.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l\geq\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (528\beta+1824)(\lambda_j^2+\lambda_i^2)\lambda_l^6+(-2688+1344\beta)\lambda_i\lambda_j\lambda_l^6\\
&\quad +(1512\beta-6048+756\beta^2)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^5\\
&\quad +(10528+84\beta^2+1288\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^5\\
&\quad +(-10944+648\beta^2+3888\beta+72\beta^3)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^4\\
&\quad +(7248+36\beta^2+1344\beta)(\lambda_j^4+\lambda_i^4)\lambda_l^4\\
&\quad +(11232+216\beta^3-9504\beta+1512\beta^2)\lambda_i^2\lambda_j^2\lambda_l^4\\
&\quad +(-1800\beta+1440+360\beta^3-180\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^3\\
&\quad +(2760\beta-7680+540\beta^2)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^3\\
&\quad +(1080\beta^2+960-1200\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l^2\\
&\quad +(3040\beta+240\beta^3+640-240\beta^2)\lambda_i^3\lambda_j^3\lambda_l^2\\
&\quad +(1200\beta+360\beta^2+1920)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\lambda_l\\
&\quad +(240+600\beta)\lambda_i^4\lambda_j^4,
\end{aligned}$$

Define $\xi_{16}(\lambda_j;\lambda_i;\beta)=\xi_1^{(6)}(\lambda_l;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{16} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{16} with respect to λ_j gives

$$\begin{aligned}
\xi_{16}(\lambda_j;\lambda_i;\beta) &= (380160\beta+1313280)\lambda_j^2+(967680\beta-1935360)\lambda_i\lambda_j \\
&\quad +(380160\beta+1313280)\lambda_i^2, \\
\xi_{16}^{(1)}(\lambda_j;\lambda_i;\beta) &= (760320\beta+2626560)\lambda_j+(967680\beta-1935360)\lambda_i, \\
\xi_{16}^{(2)}(\lambda_j;\lambda_i;\beta) &= 760320\beta+2626560\geq 0 \text{ for all } \beta\geq -3.4545.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{16}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i^2(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}, \\
\xi_{16}^{(1)}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}$$

Hence, $\xi_1^{(6)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{15}(\lambda_j;\lambda_i;\beta)=\xi_1^{(5)}(\lambda_j;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{15} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{15} with respect to λ_j gives

$$\begin{aligned}
\xi_{15}(\lambda_j;\lambda_i;\beta) &= (534720\beta+2576640+10080\beta^2)\lambda_j^3 \\
&\quad +(1149120\beta-2661120+90720\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(561600\beta+587520+90720\beta^2)\lambda_i^2\lambda_j \\
&\quad +(1263360+10080\beta^2+154560\beta)\lambda_i^3, \\
\xi_{15}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1604160\beta+7729920+30240\beta^2)\lambda_j^2 \\
&\quad +(2298240\beta-5322240+181440\beta^2)\lambda_i\lambda_j \\
&\quad +(561600\beta+587520+90720\beta^2)\lambda_i^2, \\
\xi_{15}^{(2)}(\lambda_j;\lambda_i;\beta) &= (3208320\beta+15459840+60480\beta^2)\lambda_j \\
&\quad +(2298240\beta-5322240+181440\beta^2)\lambda_i, \\
\xi_{15}^{(3)}(\lambda_j;\lambda_i;\beta) &= 3208320\beta+15459840+60480\beta^2\geq 0 \text{ for all } \beta\geq -5.3603.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{15}(\lambda_i;\lambda_i;\beta) &= 9600\lambda_i^3(250\beta+184+21\beta^2)\geq 0 \text{ for all } \beta\geq -0.7882, \\
\xi_{15}^{(1)}(\lambda_i;\lambda_i;\beta) &= 14400\lambda_i^2(310\beta+208+21\beta^2)\geq 0 \text{ for all } \beta\geq -0.7046, \\
\xi_{15}^{(2)}(\lambda_i;\lambda_i;\beta) &= 11520\lambda_i(478\beta+880+21\beta^2)\geq 0 \text{ for all } \beta\geq -2.0203.
\end{aligned}$$

Hence, $\xi_1^{(5)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{14}(\lambda_j;\lambda_i;\beta)=\xi_1^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{14} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{14} with

respect to λ_j gives

$$\begin{aligned}
\xi_{14}(\lambda_j;\lambda_i;\beta) &= (376896\beta+2093952+10944\beta^2)\lambda_j^4\\
&\quad +(106272\beta^2+758592\beta-1956096+1728\beta^3)\lambda_i\lambda_j^3\\
&\quad +(143424\beta+200448+127008\beta^2+5184\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(1000704+25632\beta^2+247872\beta+1728\beta^3)\lambda_i^3\lambda_j\\
&\quad +(173952+32256\beta+864\beta^2)\lambda_i^4,\\
\xi_{14}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1507584\beta+8375808+43776\beta^2)\lambda_j^3\\
&\quad +(318816\beta^2+2275776\beta-5868288+5184\beta^3)\lambda_i\lambda_j^2\\
&\quad +(286848\beta+400896+254016\beta^2+10368\beta^3)\lambda_i^2\lambda_j\\
&\quad +(1000704+25632\beta^2+247872\beta+1728\beta^3)\lambda_i^3,\\
\xi_{14}^{(2)}(\lambda_j;\lambda_i;\beta) &= (4522752\beta+25127424+131328\beta^2)\lambda_j^2\\
&\quad +(637632\beta^2+4551552\beta-11736576+10368\beta^3)\lambda_i\lambda_j\\
&\quad +(286848\beta+400896+254016\beta^2+10368\beta^3)\lambda_i^2,\\
\xi_{14}^{(3)}(\lambda_j;\lambda_i;\beta) &= (9045504\beta+50254848+262656\beta^2)\lambda_j\\
&\quad +(637632\beta^2+4551552\beta-11736576+10368\beta^3)\lambda_i,\\
\xi_{14}^{(4)}(\lambda_j;\lambda_i;\beta) &= 9045504\beta+50254848+262656\beta^2 \geq 0 \text{ for all } \beta \geq -6.9640.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{14}(\lambda_i;\lambda_i;\beta) &= 960\lambda_i^4(1624\beta+1576+282\beta^2+9\beta^3) \geq 0 \text{ for all } \beta \geq -1.2181,\\
\xi_{14}^{(1)}(\lambda_i;\lambda_i;\beta) &= 960\lambda_i^3(4498\beta+4072+669\beta^2+18\beta^3) \geq 0 \text{ for all } \beta \geq -1.0710,\\
\xi_{14}^{(2)}(\lambda_i;\lambda_i;\beta) &= 2304\lambda_i^2(4063\beta+5986+444\beta^2+9\beta^3) \geq 0 \text{ for all } \beta \geq -1.8231,\\
\xi_{14}^{(3)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i(23606\beta+66872+1563\beta^2+18\beta^3) \geq 0 \text{ for all } \beta \geq -3.7012.
\end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j;\lambda_i;\beta) = \xi_1^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$, where all derivatives are with respect to λ_l. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with

respect to λ_j gives

$$
\begin{aligned}
\xi_{13}(\lambda_j;\lambda_i;\beta) &= (1024512+172896\beta+5904\beta^2)\lambda_j^4 \\
&\quad +(-994176+361872\beta+64152\beta^2+1728\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(134208-84816\beta+80568\beta^2+7344\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(19512\beta^2+159792\beta+377664+3888\beta^3)\lambda_i^3\lambda_j \\
&\quad +(127872+48816\beta+4104\beta^2)\lambda_i^4, \\
\xi_{13}^{(1)}(\lambda_j;\lambda_i;\beta) &= (4098048+691584\beta+23616\beta^2)\lambda_j^3 \\
&\quad +(-2982528+1085616\beta+192456\beta^2+5184\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(268416-169632\beta+161136\beta^2+14688\beta^3)\lambda_i^2\lambda_j \\
&\quad +(19512\beta^2+159792\beta+377664+3888\beta^3)\lambda_i^3, \\
\xi_{13}^{(2)}(\lambda_j;\lambda_i;\beta) &= (12294144+2074752\beta+70848\beta^2)\lambda_j^2 \\
&\quad +(-5965056+2171232\beta+384912\beta^2+10368\beta^3)\lambda_i\lambda_j \\
&\quad +(268416-169632\beta+161136\beta^2+14688\beta^3)\lambda_i^2, \\
\xi_{13}^{(3)}(\lambda_j;\lambda_i;\beta) &= (24588288+4149504\beta+141696\beta^2)\lambda_j \\
&\quad +(-5965056+2171232\beta+384912\beta^2+10368\beta^3)\lambda_i, \\
\xi_{13}^{(4)}(\lambda_j;\lambda_i;\beta) &= 24588288+4149504\beta+141696\beta^2 \geq 0 \text{ for all } \beta \geq -8.2495.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{13}(\lambda_i;\lambda_i;\beta) &= 480\lambda_i^4(1396+1372\beta+363\beta^2+27\beta^3) \geq 0 \text{ for all } \beta \geq -1.6479, \\
\xi_{13}^{(1)}(\lambda_i;\lambda_i;\beta) &= 240\lambda_i^3(\beta+10)(99\beta^2+663\beta+734) \geq 0 \text{ for all } \beta \geq -1.3996, \\
\xi_{13}^{(2)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^2(7636+4718\beta+714\beta^2+29\beta^3) \geq 0 \text{ for all } \beta \geq -2.4138, \\
\xi_{13}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(129328+43894\beta+3657\beta^2+72\beta^3) \geq 0 \text{ for all } \beta \geq -4.4547.
\end{aligned}
$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j;\lambda_i;\beta) = \xi_1^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$, where all derivatives are with respect to λ_l. We wish to show that ξ_{12} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with

respect to λ_j gives

$$\begin{aligned}
\xi_{12}(\lambda_j;\lambda_i;\beta) &= (352256+2112\beta^2+57728\beta)\lambda_j^4 \\
&\quad +(864\beta^3-379008+133776\beta+26136\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(34344\beta^2+4752\beta^3+79104-81168\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(89152+67696\beta+3504\beta^3+7896\beta^2)\lambda_i^3\lambda_j \\
&\quad +(42816+30288\beta+5832\beta^2)\lambda_i^4, \\
\xi_{12}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1409024+8448\beta^2+230912\beta)\lambda_j^3 \\
&\quad +(2592\beta^3-1137024+401328\beta+78408\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(68688\beta^2+9504\beta^3+158208-162336\beta)\lambda_i^2\lambda_j \\
&\quad +(89152+67696\beta+3504\beta^3+7896\beta^2)\lambda_i^3, \\
\xi_{12}^{(2)}(\lambda_j;\lambda_i;\beta) &= (4227072+25344\beta^2+692736\beta)\lambda_j^2 \\
&\quad +(5184\beta^3-2274048+802656\beta+156816\beta^2)\lambda_i\lambda_j \\
&\quad +(68688\beta^2+9504\beta^3+158208-162336\beta)\lambda_i^2, \\
\xi_{12}^{(3)}(\lambda_j;\lambda_i;\beta) &= (8454144+50688\beta^2+1385472\beta)\lambda_j \\
&\quad +(5184\beta^3-2274048+802656\beta+156816\beta^2)\lambda_i, \\
\xi_{12}^{(4)}(\lambda_j;\lambda_i;\beta) &= 8454144+50688\beta^2+1385472\beta \geq 0 \text{ for all } \beta \geq -9.1957.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{12}(\lambda_i;\lambda_i;\beta) &= 480\lambda_i^4(\beta+3)(\beta+2)(19\beta+64) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{12}^{(1)}(\lambda_i;\lambda_i;\beta) &= 240\lambda_i^3(2164+681\beta^2+2240\beta+65\beta^3) \geq 0 \text{ for all } \beta \geq -1.7098, \\
\xi_{12}^{(2)}(\lambda_i;\lambda_i;\beta) &= 96\lambda_i^2(21992+2613\beta^2+13886\beta+153\beta^3) \geq 0 \text{ for all } \beta \geq -2.8871, \\
\xi_{12}^{(3)}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i(128752+4323\beta^2+45586\beta+108\beta^3) \geq 0 \text{ for all } \beta \geq -4.6015.
\end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j;\lambda_i;\beta) = \xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$, where all derivatives are with respect to λ_l. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with

respect to λ_j gives

$$
\begin{aligned}
\xi_{11}(\lambda_j;\lambda_i;\beta) &= (92576+14984\beta+564\beta^2)\lambda_j^4 \\
&\quad +(288\beta^3+7992\beta^2+39456\beta-113184)\lambda_i\lambda_j^3 \\
&\quad +(-35088\beta+11448\beta^2+31872+1944\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(16384+23872\beta+1848\beta^3+2352\beta^2)\lambda_i^3\lambda_j \\
&\quad +(9792+12456\beta+4284\beta^2)\lambda_i^4, \\
\xi_{11}^{(1)}(\lambda_j;\lambda_i;\beta) &= (370304+59936\beta+2256\beta^2)\lambda_j^3 \\
&\quad +(864\beta^3+23976\beta^2+118368\beta-339552)\lambda_i\lambda_j^2 \\
&\quad +(-70176\beta+22896\beta^2+63744+3888\beta^3)\lambda_i^2\lambda_j \\
&\quad +(16384+23872\beta+1848\beta^3+2352\beta^2)\lambda_i^3, \\
\xi_{11}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1110912+179808\beta+6768\beta^2)\lambda_j^2 \\
&\quad +(1728\beta^3+47952\beta^2+236736\beta-679104)\lambda_i\lambda_j \\
&\quad +(-70176\beta+22896\beta^2+63744+3888\beta^3)\lambda_i^2, \\
\xi_{11}^{(3)}(\lambda_j;\lambda_i;\beta) &= (2221824+359616\beta+13536\beta^2)\lambda_j \\
&\quad +(1728\beta^3+47952\beta^2+236736\beta-679104)\lambda_i, \\
\xi_{11}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2221824+359616\beta+13536\beta^2 \geq 0 \text{ for all } \beta \geq -9.7745.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{11}(\lambda_i;\lambda_i;\beta) &= 240\lambda_i^4(17\beta+26)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{26}{17}, \\
\xi_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1320\lambda_i^3(5\beta+14)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(2)}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i^2(10324+7216\beta+1617\beta^2+117\beta^3) \geq 0 \text{ for all } \beta \geq -3.1148, \\
\xi_{11}^{(3)}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i(\beta+10)(36\beta^2+921\beta+3214) \geq 0 \text{ for all } \beta \geq -4.1691.
\end{aligned}
$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j;\lambda_i;\beta) = \xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$, where all derivatives are with respect to λ_l. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with

respect to λ_j gives

$$\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= (19600+3160\beta+120\beta^2)\lambda_j^4\\
&\quad +(72\beta^3+1944\beta^2+9504\beta-27360)\lambda_i\lambda_j^3\\
&\quad +(-10464\beta+3168\beta^2+9408+576\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(3584+7616\beta+672\beta^3+672\beta^2)\lambda_i^3\lambda_j\\
&\quad +(2688+4704\beta+2016\beta^2)\lambda_i^4,\\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= (78400+12640\beta+480\beta^2)\lambda_j^3\\
&\quad +(216\beta^3+5832\beta^2+28512\beta-82080)\lambda_i\lambda_j^2\\
&\quad +(-20928\beta+6336\beta^2+18816+1152\beta^3)\lambda_i^2\lambda_j\\
&\quad +(3584+7616\beta+672\beta^3+672\beta^2)\lambda_i^3,\\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= (235200+37920\beta+1440\beta^2)\lambda_j^2\\
&\quad +(432\beta^3+11664\beta^2+57024\beta-164160)\lambda_i\lambda_j\\
&\quad +(-20928\beta+6336\beta^2+18816+1152\beta^3)\lambda_i^2,\\
\xi_{10}^{(3)}(\lambda_j;\lambda_i;\beta) &= (470400+75840\beta+2880\beta^2)\lambda_j\\
&\quad +(432\beta^3+11664\beta^2+57024\beta-164160)\lambda_i,\\
\xi_{10}^{(4)}(\lambda_j;\lambda_i;\beta) &= 960(3\beta+49)(\beta+10)\geq 0 \text{ for all } \beta\geq -10.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 1320\lambda_i^4(\beta+3)(\beta+2)(\beta+1)>0 \text{ for all } \beta>-1,\\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 120\lambda_i^3(17\beta+26)(\beta+3)(\beta+2)>0 \text{ for all } \beta>-\frac{26}{17},\\
\xi_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^2(\beta+3)(11\beta^2+102\beta+208)>0 \text{ for all } \beta>-3,\\
\xi_{10}^{(3)}(\lambda_i;\lambda_i;\beta) &= 48\lambda_i(\beta+10)(9\beta^2+213\beta+638)>0 \text{ for all } \beta>-3.5184.
\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l\geq\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= (5472+1584\beta)(\lambda_j^2+\lambda_i^2)\lambda_l^6+(-8064+4032\beta)\lambda_i\lambda_j\lambda_l^6\\
&+(3888\beta+1944\beta^2-15552)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^5\\
&+(3312\beta+216\beta^2+27072)(\lambda_j^3+\lambda_i^3)\lambda_l^5\\
&+(-23616+144\beta^3+8208\beta+1512\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^4\\
&+(2952\beta+15408+72\beta^2)(\lambda_j^4+\lambda_i^4)\lambda_l^4\\
&+(25056+3672\beta^2+432\beta^3-21600\beta)\lambda_i^2\lambda_j^2\lambda_l^4\\
&+(576\beta^3+144\beta^2-1152-2016\beta)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^3\\
&+(-14976+5760\beta+864\beta^2)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^3\\
&+(288\beta+2880+1296\beta^2)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l^2\\
&+(-144\beta^2+8832+6240\beta+288\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2\\
&+(7296+1536\beta+288\beta^2)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\lambda_l\\
&+(264\beta+912)\lambda_i^4\lambda_j^4,
\end{aligned}$$

Define $\xi_{26}(\lambda_j;\lambda_i;\beta)=\xi_2^{(6)}(\lambda_l;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{26} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{26} with respect to λ_j gives

$$\begin{aligned}
\xi_{26}(\lambda_j;\lambda_i;\beta) &= (1140480\beta+3939840)\lambda_j^2+(-5806080+2903040\beta)\lambda_i\lambda_j\\
&\quad+(1140480\beta+3939840)\lambda_i^2,\\
\xi_{26}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2280960\beta+7879680)\lambda_j+(-5806080+2903040\beta)\lambda_i,\\
\xi_{26}^{(2)}(\lambda_j;\lambda_i;\beta) &= 2280960\beta+7879680\geq 0 \text{ for all } \beta\geq 3.4545.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{26}(\lambda_i;\lambda_i;\beta) &= 1036800\lambda_i^2(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5},\\
\xi_{26}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1036800\lambda_i(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}$$

Hence, $\xi_2^{(6)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{25}(\lambda_j;\lambda_i;\beta)=\xi_2^{(5)}(\lambda_j;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{25} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{25} with

respect to λ_j gives

$$\begin{aligned}
\xi_{25}(\lambda_j;\lambda_i;\beta) &= (1537920\beta + 7188480 + 25920\beta^2)\lambda_j^3 \\
&\quad +(-7672320 + 3369600\beta + 233280\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(1607040\beta + 2073600 + 233280\beta^2)\lambda_i^2\lambda_j \\
&\quad +(397440\beta + 25920\beta^2 + 3248640)\lambda_i^3, \\
\xi_{25}^{(1)}(\lambda_j;\lambda_i;\beta) &= (4613760\beta + 21565440 + 77760\beta^2)\lambda_j^2 \\
&\quad +(-15344640 + 6739200\beta + 466560\beta^2)\lambda_i\lambda_j \\
&\quad +(1607040\beta + 2073600 + 233280\beta^2)\lambda_i^2, \\
\xi_{25}^{(2)}(\lambda_j;\lambda_i;\beta) &= (9227520\beta + 43130880 + 155520\beta^2)\lambda_j \\
&\quad +(-15344640 + 6739200\beta + 466560\beta^2)\lambda_i, \\
\xi_{25}^{(3)}(\lambda_j;\lambda_i;\beta) &= 9227520\beta + 43130880 + 155520\beta^2 \geq 0 \text{ for all } \beta \geq -5.1151.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{25}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i^3(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\
\xi_{25}^{(1)}(\lambda_i;\lambda_i;\beta) &= 259200\lambda_i^2(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\
\xi_{25}^{(2)}(\lambda_i;\lambda_i;\beta) &= 207360\lambda_i(77\beta + 134 + 3\beta^2) \geq 0 \text{ for all } \beta \geq -1.8776.
\end{aligned}$$

Hence, $\xi_2^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{24}(\lambda_j;\lambda_i;\beta) = \xi_2^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{24} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{24} with respect to λ_j gives

$$\begin{aligned}
\xi_{24}(\lambda_j;\lambda_i;\beta) &= (1038528\beta + 5588352 + 27648\beta^2)\lambda_j^4 \\
&\quad +(2115072\beta - 5336064 + 269568\beta^2 + 3456\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(518400\beta + 705024 + 10368\beta^3 + 321408\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(594432\beta + 62208\beta^2 + 2681856 + 3456\beta^3)\lambda_i^3\lambda_j \\
&\quad +(369792 + 70848\beta + 1728\beta^2)\lambda_i^4, \\
\xi_{24}^{(1)}(\lambda_j;\lambda_i;\beta) &= (4154112\beta + 22353408 + 110592\beta^2)\lambda_j^3 \\
&\quad +(6345216\beta - 16008192 + 808704\beta^2 + 10368\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(1036800\beta + 1410048 + 20736\beta^3 + 642816\beta^2)\lambda_i^2\lambda_j \\
&\quad +(594432\beta + 62208\beta^2 + 2681856 + 3456\beta^3)\lambda_i^3, \\
\xi_{24}^{(2)}(\lambda_j;\lambda_i;\beta) &= (12462336\beta + 67060224 + 331776\beta^2)\lambda_j^2 \\
&\quad +(12690432\beta - 32016384 + 1617408\beta^2 + 20736\beta^3)\lambda_i\lambda_j \\
&\quad +(1036800\beta + 1410048 + 20736\beta^3 + 642816\beta^2)\lambda_i^2, \\
\xi_{24}^{(3)}(\lambda_j;\lambda_i;\beta) &= (24924672\beta + 134120448 + 663552\beta^2)\lambda_j \\
&\quad +(12690432\beta - 32016384 + 1617408\beta^2 + 20736\beta^3)\lambda_i, \\
\xi_{24}^{(4)}(\lambda_j;\lambda_i;\beta) &= 24924672\beta + 134120448 + 663552\beta^2 \geq 0 \text{ for all } \beta \geq -6.5089.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{24}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^4(502\beta+464+79\beta^2+2\beta^3)\geq 0 \text{ for all } \beta\geq -1.1141,\\
\xi_{24}^{(1)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i^3(351\beta+302+47\beta^2+\beta^3)\geq 0 \text{ for all } \beta\geq -0.9885,\\
\xi_{24}^{(2)}(\lambda_i;\lambda_i;\beta) &= 20736\lambda_i^2(1263\beta+1758+125\beta^2+2\beta^3)\geq 0 \text{ for all } \beta\geq -1.6562,\\
\xi_{24}^{(3)}(\lambda_i;\lambda_i;\beta) &= 20736\lambda_i(1814\beta+4924+110\beta^2+\beta^3)\geq 0 \text{ for all } \beta\geq -3.3897.
\end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{23}(\lambda_j;\lambda_i;\beta)=\xi_2^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$, where all derivatives are with respect to λ_l. We wish to show that ξ_{23} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned}
\xi_{23}(\lambda_j;\lambda_i;\beta) &= (2650752+459648\beta+14688\beta^2)\lambda_j^4\\
&\quad +(948672\beta+3456\beta^3+158112\beta^2-2557440)\lambda_i\lambda_j^3\\
&\quad +(205632\beta^2+13824\beta^3+317952-107136\beta)\lambda_i^2\lambda_j^2\\
&\quad +(1050624+383616\beta+6912\beta^3+50112\beta^2)\lambda_i^3\lambda_j\\
&\quad +(279936+105408\beta+6912\beta^2)\lambda_i^4,\\
\xi_{23}^{(1)}(\lambda_j;\lambda_i;\beta) &= (10603008+1838592\beta+58752\beta^2)\lambda_j^3\\
&\quad +(2846016\beta+10368\beta^3+474336\beta^2-7672320)\lambda_i\lambda_j^2\\
&\quad +(411264\beta^2+27648\beta^3+635904-214272\beta)\lambda_i^2\lambda_j\\
&\quad +(1050624+383616\beta+6912\beta^3+50112\beta^2)\lambda_i^3,\\
\xi_{23}^{(2)}(\lambda_j;\lambda_i;\beta) &= (31809024+5515776\beta+176256\beta^2)\lambda_j^2\\
&\quad +(5692032\beta+20736\beta^3+948672\beta^2-15344640)\lambda_i\lambda_j\\
&\quad +(411264\beta^2+27648\beta^3+635904-214272\beta)\lambda_i^2,\\
\xi_{23}^{(3)}(\lambda_j;\lambda_i;\beta) &= (63618048+11031552\beta+352512\beta^2)\lambda_j\\
&\quad +(5692032\beta+20736\beta^3+948672\beta^2-15344640)\lambda_i,\\
\xi_{23}^{(4)}(\lambda_j;\lambda_i;\beta) &= 63618048+11031552\beta+352512\beta^2\geq 0 \text{ for all } \beta\geq -7.6246.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{23}(\lambda_i;\lambda_i;\beta) &= 24192\lambda_i^4(\beta+4)(\beta^2+14\beta+18)\geq 0 \text{ for all } \beta\geq -1.4322,\\
\xi_{23}^{(1)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^3(5344+5618\beta+1151\beta^2+52\beta^3)\geq 0 \text{ for all } \beta\geq -1.2562,\\
\xi_{23}^{(2)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i^2(9896+6362\beta+889\beta^2+28\beta^3)\geq 0 \text{ for all } \beta\geq -2.1667,\\
\xi_{23}^{(3)}(\lambda_i;\lambda_i;\beta) &= 5184\lambda_i(9312+3226\beta+251\beta^2+4\beta^3)\geq 0 \text{ for all } \beta\geq -4.1213.
\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{22}(\lambda_j;\lambda_i;\beta)=\xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$, where all derivatives are with respect to λ_l. We wish to show that ξ_{22} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{22} with

respect to λ_j gives

$$
\begin{aligned}
\xi_{22}(\lambda_j;\lambda_i;\beta) &= (890496+149184\beta+5184\beta^2)\lambda_j^4\\
&\quad +(1728\beta^3+62208\beta^2-926208+331776\beta)\lambda_i\lambda_j^3\\
&\quad +(-145440\beta+86400\beta^2+152640+8640\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(268800+165120\beta+5760\beta^3+23040\beta^2)\lambda_i^3\lambda_j\\
&\quad +(100800+70560\beta+8640\beta^2)\lambda_i^4,\\
\xi_{22}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3561984+596736\beta+20736\beta^2)\lambda_j^3\\
&\quad +(5184\beta^3+186624\beta^2-2778624+995328\beta)\lambda_i\lambda_j^2\\
&\quad +(-290880\beta+172800\beta^2+305280+17280\beta^3)\lambda_i^2\lambda_j\\
&\quad +(268800+165120\beta+5760\beta^3+23040\beta^2)\lambda_i^3,\\
\xi_{22}^{(2)}(\lambda_j;\lambda_i;\beta) &= (10685952+1790208\beta+62208\beta^2)\lambda_j^2\\
&\quad +(10368\beta^3+373248\beta^2-5557248+1990656\beta)\lambda_i\lambda_j\\
&\quad +(-290880\beta+172800\beta^2+305280+17280\beta^3)\lambda_i^2,\\
\xi_{22}^{(3)}(\lambda_j;\lambda_i;\beta) &= (21371904+3580416\beta+124416\beta^2)\lambda_j\\
&\quad +(10368\beta^3+373248\beta^2-5557248+1990656\beta)\lambda_i,\\
\xi_{22}^{(4)}(\lambda_j;\lambda_i;\beta) &= 21371904+3580416\beta+124416\beta^2\geq 0 \text{ for all } \beta\geq -8.4507.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{22}(\lambda_i;\lambda_i;\beta) &= 1344\lambda_i^4(362+425\beta+138\beta^2+12\beta^3)\geq 0 \text{ for all } \beta\geq -1.4432,\\
\xi_{22}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1344\lambda_i^3(1010+1091\beta+300\beta^2+21\beta^3)\geq 0 \text{ for all } \beta\geq -1.4354,\\
\xi_{22}^{(2)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i^2(9434+6059\beta+1056\beta^2+48\beta^3)\geq 0 \text{ for all } \beta\geq -2.5842,\\
\xi_{22}^{(3)}(\lambda_i;\lambda_i;\beta) &= 3456\lambda_i(4576+1612\beta+144\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq -4.4300.
\end{aligned}
$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{21}(\lambda_j;\lambda_i;\beta)=\xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$, where all derivatives are with respect to λ_l. We wish to show that ξ_{21} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{21} with

respect to λ_j gives

$$
\begin{aligned}
\xi_{21}(\lambda_j;\lambda_i;\beta) &= (229824+1368\beta^2+37872\beta)\lambda_j^4\\
&\quad+(-265536+18360\beta^2+576\beta^3+93744\beta)\lambda_i\lambda_j^3\\
&\quad+(-62928\beta+27432\beta^2+3456\beta^3+57600)\lambda_i^2\lambda_j^2\\
&\quad+(62400+7560\beta^2+2880\beta^3+57360\beta)\lambda_i^3\lambda_j\\
&\quad+(29760+31200\beta+5760\beta^2)\lambda_i^4,\\
\xi_{21}^{(1)}(\lambda_j;\lambda_i;\beta) &= (919296+5472\beta^2+151488\beta)\lambda_j^3\\
&\quad+(-796608+55080\beta^2+1728\beta^3+281232\beta)\lambda_i\lambda_j^2\\
&\quad+(-125856\beta+54864\beta^2+6912\beta^3+115200)\lambda_i^2\lambda_j\\
&\quad+(62400+7560\beta^2+2880\beta^3+57360\beta)\lambda_i^3,\\
\xi_{21}^{(2)}(\lambda_j;\lambda_i;\beta) &= (2757888+16416\beta^2+454464\beta)\lambda_j^2\\
&\quad+(-1593216+110160\beta^2+3456\beta^3+562464\beta)\lambda_i\lambda_j\\
&\quad+(-125856\beta+54864\beta^2+6912\beta^3+115200)\lambda_i^2,\\
\xi_{21}^{(3)}(\lambda_j;\lambda_i;\beta) &= (5515776+32832\beta^2+908928\beta)\lambda_j\\
&\quad+(-1593216+110160\beta^2+3456\beta^3+562464\beta)\lambda_i,\\
\xi_{21}^{(4)}(\lambda_j;\lambda_i;\beta) &= 5515776+32832\beta^2+908928\beta\geq 0 \text{ for all } \beta\geq -13.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{21}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i^4(\beta+3)(4\beta^2+23\beta+22)\geq 0 \text{ for all } \beta\geq -1.2120,\\
\xi_{21}^{(1)}(\lambda_i;\lambda_i;\beta) &= 96\lambda_i^3(3128+1281\beta^2+3794\beta+120\beta^3)\geq 0 \text{ for all } \beta\geq -1.3977,\\
\xi_{21}^{(2)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i^2(2222+315\beta^2+1547\beta+18\beta^3)\geq 0 \text{ for all } \beta\geq -2.6483,\\
\xi_{21}^{(3)}(\lambda_i;\lambda_i;\beta) &= 432\lambda_i(9080+331\beta^2+3406\beta+8\beta^3)\geq 0 \text{ for all } \beta\geq -4.2200.
\end{aligned}
$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j;\lambda_i;\beta)=\xi_2(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$, where all derivatives are with respect to λ_l. We wish to show that ξ_{20} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{20} with

respect to λ_j gives

$$
\begin{aligned}
\xi_{20}(\lambda_j;\lambda_i;\beta) &= (47952+288\beta^2+7848\beta)\lambda_j^4 \\
&\quad +(4320\beta^2+21888\beta+144\beta^3-62208)\lambda_i\lambda_j^3 \\
&\quad +(-17856\beta+7056\beta^2+1008\beta^3+16704)\lambda_i^2\lambda_j^2 \\
&\quad +(18432+17280\beta+2016\beta^2+1008\beta^3)\lambda_i^3\lambda_j \\
&\quad +(11520+10800\beta+2520\beta^2)\lambda_i^4, \\
\xi_{20}^{(1)}(\lambda_j;\lambda_i;\beta) &= (191808+1152\beta^2+31392\beta)\lambda_j^3 \\
&\quad +(12960\beta^2+65664\beta+432\beta^3-186624)\lambda_i\lambda_j^2 \\
&\quad +(-35712\beta+14112\beta^2+2016\beta^3+33408)\lambda_i^2\lambda_j \\
&\quad +(18432+17280\beta+2016\beta^2+1008\beta^3)\lambda_i^3, \\
\xi_{20}^{(2)}(\lambda_j;\lambda_i;\beta) &= (575424+3456\beta^2+94176\beta)\lambda_j^2 \\
&\quad +(25920\beta^2+131328\beta+864\beta^3-373248)\lambda_i\lambda_j \\
&\quad +(-35712\beta+14112\beta^2+2016\beta^3+33408)\lambda_i^2, \\
\xi_{20}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1150848+6912\beta^2+188352\beta)\lambda_j \\
&\quad +(25920\beta^2+131328\beta+864\beta^3-373248)\lambda_i, \\
\xi_{20}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1728(\beta+18)(4\beta+37)\geq 0 \text{ for all } \beta\geq -\frac{37}{4}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{20}(\lambda_i;\lambda_i;\beta) &= 1080\lambda_i^4(2\beta+5)(\beta+3)(\beta+2)>0 \text{ for all } \beta>-2, \\
\xi_{20}^{(1)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^3(\beta+3)(4\beta^2+23\beta+22)>0 \text{ for all } \beta>-1.2120, \\
\xi_{20}^{(2)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i^2(818+151\beta^2+659\beta+10\beta^3)>0 \text{ for all } \beta>-2.1484, \\
\xi_{20}^{(3)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i(\beta+10)(\beta^2+28\beta+90)>0 \text{ for all } \beta>-3.7044.
\end{aligned}
$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$
\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) \;=\;& (10944+3168\beta)(\lambda_i^2+\lambda_j^2)\lambda_l^5+(8064\beta-16128)\lambda_i\lambda_j\lambda_l^5\\
&+(6480\beta+3240\beta^2-25920)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^4\\
&+(360\beta^2+45120+5520\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^4\\
&+(2592\beta^2-32256+10368\beta+144\beta^3)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\\
&+(72\beta^2+19968+4272\beta)(\lambda_j^4+\lambda_i^4)\\
&+(432\beta^3+38016-34560\beta+6912\beta^2)\lambda_i^2\lambda_j^2\lambda_l^3\\
&+(432\beta^3-13824+1728\beta+1728\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^2\\
&+(-21312+9360\beta+648\beta^2)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^2\\
&+(5760+4464\beta+648\beta^2)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l\\
&+(144\beta^3+288\beta^2+9600\beta+24576)\lambda_i^3\lambda_j^3\lambda_l\\
&+(9024+1104\beta+72\beta^2)\lambda_i^3\lambda_j^4+(9024+1104\beta+72\beta^2)\lambda_i^4\lambda_j^3.
\end{aligned}
$$

Define $\xi_{35}(\lambda_j;\lambda_i;\beta)=\xi_3^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{35} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{35} with respect to λ_j gives

$$
\begin{aligned}
\xi_{35}(\lambda_j;\lambda_i;\beta) &= (380160\beta+1313280)\lambda_j^2+(967680\beta-1935360)\lambda_i\lambda_j\\
&\quad+(380160\beta+1313280)\lambda_i^2,\\
\xi_{35}^{(1)}(\lambda_j;\lambda_i;\beta) &= (760320\beta+2626560)\lambda_j+(967680\beta-1935360)\lambda_i,\\
\xi_{35}^{(2)}(\lambda_j;\lambda_i;\beta) &= 760320\beta+2626560\geq 0 \text{ for all } \beta\geq -3.4545.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{35}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i^2(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5},\\
\xi_{35}^{(1)}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}
$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{34}(\lambda_j;\lambda_i;\beta)=\xi_3^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{34} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{34} with respect to λ_j gives

$$\begin{aligned}
\xi_{34}(\lambda_j;\lambda_i;\beta) &= (512640\beta+2396160+8640\beta^2)\lambda_j^3\\
&\quad +(1123200\beta-2557440+77760\beta^2)\lambda_i\lambda_j^2\\
&\quad +(535680\beta+691200+77760\beta^2)\lambda_i^2\lambda_j\\
&\quad +(1082880+132480\beta+8640\beta^2)\lambda_i^3,\\
\xi_{34}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1537920\beta+7188480+25920\beta^2)\lambda_j^2\\
&\quad +(2246400\beta-5114880+155520\beta^2)\lambda_i\lambda_j\\
&\quad +(535680\beta+691200+77760\beta^2)\lambda_i^2,\\
\xi_{34}^{(2)}(\lambda_j;\lambda_i;\beta) &= (3075840\beta+14376960+51840\beta^2)\lambda_j\\
&\quad +(2246400\beta-5114880+155520\beta^2)\lambda_i,\\
\xi_{34}^{(3)}(\lambda_j;\lambda_i;\beta) &= 3075840\beta+14376960+51840\beta^2\geq 0 \text{ for all } \beta\geq -5.1151.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{34}(\lambda_i;\lambda_i;\beta) &= 57600\lambda_i^3(28+40\beta+3\beta^2)\geq 0 \text{ for all } \beta\geq -0.7412,\\
\xi_{34}^{(1)}(\lambda_i;\lambda_i;\beta) &= 86400\lambda_i^2(\beta+16)(3\beta+2)\geq 0 \text{ for all } \beta\geq -\frac{2}{3},\\
\xi_{34}^{(2)}(\lambda_i;\lambda_i;\beta) &= 69120\lambda_i(77\beta+134+3\beta^2)\geq 0 \text{ for all } \beta\geq -1.8776.
\end{aligned}$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{33}(\lambda_j;\lambda_i;\beta) = \xi_3^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$, where all derivatives are with respect to λ_l. We wish to show that ξ_{33} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{33} with respect to λ_j gives

$$\begin{aligned}
\xi_{33}(\lambda_j;\lambda_i;\beta) &= (1859328+9072\beta^2+348192\beta)\lambda_j^4\\
&\quad +(701568\beta+93312\beta^2+864\beta^3-1783296)\lambda_i\lambda_j^3\\
&\quad +(138240\beta+119232\beta^2+2592\beta^3+262656)\lambda_i^2\lambda_j^2\\
&\quad +(889344+864\beta^3+194688\beta+24192\beta^2)\lambda_i^3\lambda_j\\
&\quad +(432\beta^2+25632\beta+119808)\lambda_i^4,\\
\xi_{33}^{(1)}(\lambda_j;\lambda_i;\beta) &= (7437312+36288\beta^2+1392768\beta)\lambda_j^3\\
&\quad +(2104704\beta+279936\beta^2+2592\beta^3-5349888)\lambda_i\lambda_j^2\\
&\quad +(276480\beta+238464\beta^2+5184\beta^3+525312)\lambda_i^2\lambda_j\\
&\quad +(889344+864\beta^3+194688\beta+24192\beta^2)\lambda_i^3,\\
\xi_{33}^{(2)}(\lambda_j;\lambda_i;\beta) &= (22311936+108864\beta^2+4178304\beta)\lambda_j^2\\
&\quad +(4209408\beta+559872\beta^2+5184\beta^3-10699776)\lambda_i\lambda_j\\
&\quad +(276480\beta+238464\beta^2+5184\beta^3+525312)\lambda_i^2,\\
\xi_{33}^{(3)}(\lambda_j;\lambda_i;\beta) &= (44623872+217728\beta^2+8356608\beta)\lambda_j\\
&\quad +(4209408\beta+559872\beta^2+5184\beta^3-10699776)\lambda_i,\\
\xi_{33}^{(4)}(\lambda_j;\lambda_i;\beta) &= 44623872+217728\beta^2+8356608\beta\geq 0 \text{ for all } \beta\geq -6.4107.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{33}(\lambda_i;\lambda_i;\beta) &= 4320\lambda_i^4(312+57\beta^2+326\beta+\beta^3) \geq 0 \text{ for all } \beta \geq -1.2060,\\
\xi_{33}^{(1)}(\lambda_i;\lambda_i;\beta) &= 2880\lambda_i^3(1216+201\beta^2+1378\beta+3\beta^3) \geq 0 \text{ for all } \beta \geq -1.0368,\\
\xi_{33}^{(2)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i^2(7024+525\beta^2+5014\beta+6\beta^3) \geq 0 \text{ for all } \beta \geq -1.6963,\\
\xi_{33}^{(3)}(\lambda_i;\lambda_i;\beta) &= 5184\lambda_i(6544+150\beta^2+2424\beta+\beta^3) \geq 0 \text{ for all } \beta \geq -3.3980.
\end{aligned}$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{32}(\lambda_j;\lambda_i;\beta) = \xi_3^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{32} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}
\xi_{32}(\lambda_j;\lambda_i;\beta) &= (155232\beta+880128+4752\beta^2)\lambda_j^4\\
&\quad +(-869760+319968\beta+864\beta^3+55728\beta^2)\lambda_i\lambda_j^3\\
&\quad +(83808\beta^2+3456\beta^3+108288-62784\beta)\lambda_i^2\lambda_j^2\\
&\quad +(23328\beta^2+1728\beta^3+320256+131904\beta)\lambda_i^3\lambda_j\\
&\quad +(77184+1728\beta^2+44352\beta)\lambda_i^4,\\
\xi_{32}^{(1)}(\lambda_j;\lambda_i;\beta) &= (620928\beta+3520512+19008\beta^2)\lambda_j^3\\
&\quad +(-2609280+959904\beta+2592\beta^3+167184\beta^2)\lambda_i\lambda_j^2\\
&\quad +(167616\beta^2+6912\beta^3+216576-125568\beta)\lambda_i^2\lambda_j\\
&\quad +(23328\beta^2+1728\beta^3+320256+131904\beta)\lambda_i^3,\\
\xi_{32}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1862784\beta+10561536+57024\beta^2)\lambda_j^2\\
&\quad +(-5218560+1919808\beta+5184\beta^3+334368\beta^2)\lambda_i\lambda_j\\
&\quad +(167616\beta^2+6912\beta^3+216576-125568\beta)\lambda_i^2,\\
\xi_{32}^{(3)}(\lambda_j;\lambda_i;\beta) &= (3725568\beta+21123072+114048\beta^2)\lambda_j\\
&\quad +(-5218560+1919808\beta+5184\beta^3+334368\beta^2)\lambda_i,\\
\xi_{32}^{(4)}(\lambda_j;\lambda_i;\beta) &= 3725568\beta+21123072+114048\beta^2 \geq 0 \text{ for all } \beta \geq -7.3020.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i;\lambda_i;\beta) &= 2016\lambda_i^4(292\beta+256+84\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4510,\\
\xi_{32}^{(1)}(\lambda_i;\lambda_i;\beta) &= 432\lambda_i^3(3674\beta+3352+873\beta^2+26\beta^3) \geq 0 \text{ for all } \beta \geq -1.2961,\\
\xi_{32}^{(2)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i^2(12698\beta+19304+1941\beta^2+42\beta^3) \geq 0 \text{ for all } \beta \geq -2.2678,\\
\xi_{32}^{(3)}(\lambda_i;\lambda_i;\beta) &= 2592\lambda_i(2178\beta+6136+173\beta^2+2\beta^3) \geq 0 \text{ for all } \beta \geq -4.0728.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j;\lambda_i;\beta) = \xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{31} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j;\lambda_i;\beta) &= (295104 + 1656\beta^2 + 50736\beta)\lambda_j^4 \\
&\quad +(432\beta^3 + 116064\beta - 323712 + 22032\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(43200 + 37800\beta^2 + 2160\beta^3 - 54000\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(80640 + 66240\beta + 1440\beta^3 + 12960\beta^2)\lambda_i^3\lambda_j \\
&\quad +(23040 + 36000\beta + 2160\beta^2)\lambda_i^4, \\
\xi_{31}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1180416 + 6624\beta^2 + 202944\beta)\lambda_j^3 \\
&\quad +(1296\beta^3 + 348192\beta - 971136 + 66096\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(86400 + 75600\beta^2 + 4320\beta^3 - 108000\beta)\lambda_i^2\lambda_j \\
&\quad +(80640 + 66240\beta + 1440\beta^3 + 12960\beta^2)\lambda_i^3, \\
\xi_{31}^{(2)}(\lambda_j;\lambda_i;\beta) &= (3541248 + 19872\beta^2 + 608832\beta)\lambda_j^2 \\
&\quad +(2592\beta^3 + 696384\beta - 1942272 + 132192\beta^2)\lambda_i\lambda_j \\
&\quad +(86400 + 75600\beta^2 + 4320\beta^3 - 108000\beta)\lambda_i^2, \\
\xi_{31}^{(3)}(\lambda_j;\lambda_i;\beta) &= (7082496 + 39744\beta^2 + 1217664\beta)\lambda_j \\
&\quad +(2592\beta^3 + 696384\beta - 1942272 + 132192\beta^2)\lambda_i, \\
\xi_{31}^{(4)}(\lambda_j;\lambda_i;\beta) &= 7082496 + 39744\beta^2 + 1217664\beta \geq 0 \text{ for all } \beta \geq -7.8046.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{31}(\lambda_i;\lambda_i;\beta) &= 1344\lambda_i^4(88 + 57\beta^2 + 160\beta + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350, \\
\xi_{31}^{(1)}(\lambda_i;\lambda_i;\beta) &= 336\lambda_i^3(1120 + 480\beta^2 + 1516\beta + 21\beta^3) \geq 0 \text{ for all } \beta \geq -1.1099, \\
\xi_{31}^{(2)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^2(11704 + 1581\beta^2 + 8314\beta + 48\beta^3) \geq 0 \text{ for all } \beta \geq -2.5399, \\
\xi_{31}^{(3)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i(17848 + 597\beta^2 + 6646\beta + 9\beta^3) \geq 0 \text{ for all } \beta \geq -4.1067.
\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j;\lambda_i;\beta) = \xi_3(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{30} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}
\xi_{30}(\lambda_j;\lambda_i;\beta) &= (76032 + 432\beta^2 + 12960\beta)\lambda_j^4 \\
&\quad +(144\beta^3 + 6480\beta^2 - 95616 + 34272\beta)\lambda_i\lambda_j^3 \\
&\quad +(-18720\beta + 12528\beta^2 + 14976 + 864\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(32640 + 28320\beta + 720\beta^3 + 5040\beta^2)\lambda_i^3\lambda_j \\
&\quad +(13440 + 19200\beta + 1440\beta^2)\lambda_i^4, \\
\xi_{30}^{(1)}(\lambda_j;\lambda_i;\beta) &= (304128 + 1728\beta^2 + 51840\beta)\lambda_j^3 \\
&\quad +(432\beta^3 + 19440\beta^2 - 286848 + 102816\beta)\lambda_i\lambda_j^2 \\
&\quad +(-37440\beta + 25056\beta^2 + 29952 + 1728\beta^3)\lambda_i^2\lambda_j \\
&\quad +(32640 + 28320\beta + 720\beta^3 + 5040\beta^2)\lambda_i^3, \\
\xi_{30}^{(2)}(\lambda_j;\lambda_i;\beta) &= (912384 + 5184\beta^2 + 155520\beta)\lambda_j^2 \\
&\quad +(864\beta^3 + 38880\beta^2 - 573696 + 205632\beta)\lambda_i\lambda_j \\
&\quad +(-37440\beta + 25056\beta^2 + 29952 + 1728\beta^3)\lambda_i^2, \\
\xi_{30}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1824768 + 10368\beta^2 + 311040\beta)\lambda_j \\
&\quad +(864\beta^3 + 38880\beta^2 - 573696 + 205632\beta)\lambda_i, \\
\xi_{30}^{(4)}(\lambda_j;\lambda_i;\beta) &= 10368(\beta + 22)(8 + \beta) \geq 0 \text{ for all } \beta \geq -8.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{30}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i^4(\beta + 3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -0.7085, \\
\xi_{30}^{(1)}(\lambda_i;\lambda_i;\beta) &= 192\lambda_i^3(416 + 267\beta^2 + 758\beta + 15\beta^3) > 0 \text{ for all } \beta > -0.7277, \\
\xi_{30}^{(2)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i^2(1280 + 240\beta^2 + 1124\beta + 9\beta^3) > 0 \text{ for all } \beta > -1.7494, \\
\xi_{30}^{(3)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i(1448 + 57\beta^2 + 598\beta + \beta^3) > 0 \text{ for all } \beta > -3.5444.
\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &= (8064\beta - 16128)\lambda_i\lambda_j\lambda_l^4 + (10944 + 3168\beta)(\lambda_i^2 + \lambda_j^2)\lambda_l^4 \\
&\quad +(10944 + 3168\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l^2 \\
&\quad +(5184\beta + 2592\beta^2 - 20736)(\lambda_i^3\lambda_j + \lambda_i\lambda_j^3)\lambda_l^2 \\
&\quad +(7776\beta^2 + 31104 - 31104\beta)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&\quad +(4416\beta + 288\beta^2 + 36096)(\lambda_j^3 + \lambda_i^3)\lambda_l^3 \\
&\quad +(5184\beta + 2592\beta^2 - 20736)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^3 \\
&\quad +(10944 + 3168\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2) \\
&\quad +(8064\beta - 16128)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l \\
&\quad +(5184\beta + 2592\beta^2 - 20736)(\lambda_i^3\lambda_j^2 + \lambda_i^2\lambda_j^3)\lambda_l \\
&\quad +(4416\beta + 288\beta^2 + 36096)\lambda_i^3\lambda_j^3.
\end{aligned}$$

Define $\xi_{44}(\lambda_j;\lambda_i;\beta)=\xi_4^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{44} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{44} with respect to λ_j gives

$$\begin{aligned}
\xi_{44}(\lambda_j;\lambda_i;\beta) &= (76032\beta+262656)\lambda_j^2+(193536\beta-387072)\lambda_i\lambda_j\\
&\quad +(76032\beta+262656)\lambda_i^2,\\
\xi_{44}^{(1)}(\lambda_j;\lambda_i;\beta) &= (152064\beta+525312)\lambda_j+(193536\beta-387072)\lambda_i,\\
\xi_{44}^{(2)}(\lambda_j;\lambda_i;\beta) &= 152064\beta+525312\geq 0 \text{ for all } \beta\geq -3.4545.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{44}(\lambda_i;\lambda_i;\beta) &= 69120\lambda_i^2(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5},\\
\xi_{44}^{(1)}(\lambda_i;\lambda_i;\beta) &= 69120\lambda_i(2+5\beta)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}$$

Hence, $\xi_4^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{43}(\lambda_j;\lambda_i;\beta)=\xi_4^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{43} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{43} with respect to λ_j gives

$$\begin{aligned}
\xi_{43}(\lambda_j;\lambda_i;\beta) &= (102528\beta+479232+1728\beta^2)\lambda_j^3+(224640\beta+15552\beta^2-511488)\lambda_i\lambda_j^2\\
&\quad +(107136\beta+138240+15552\beta^2)\lambda_i^2\lambda_j+(26496\beta+216576+1728\beta^2)\lambda_i^3,\\
\xi_{43}^{(1)}(\lambda_j;\lambda_i;\beta) &= (307584\beta+1437696+5184\beta^2)\lambda_j^2+(449280\beta+31104\beta^2-1022976)\lambda_i\lambda_j\\
&\quad +(107136\beta+138240+15552\beta^2)\lambda_i^2,\\
\xi_{43}^{(2)}(\lambda_j;\lambda_i;\beta) &= (615168\beta+2875392+10368\beta^2)\lambda_j+(449280\beta+31104\beta^2-1022976)\lambda_i,\\
\xi_{43}^{(3)}(\lambda_j;\lambda_i;\beta) &= 615168\beta+2875392+10368\beta^2\geq 0 \text{ for all } \beta\geq -5.1151.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{43}(\lambda_i;\lambda_i;\beta) &= 11520\lambda_i^3(28+40\beta+3\beta^2)\geq 0 \text{ for all } \beta\geq -0.7412,\\
\xi_{43}^{(1)}(\lambda_i;\lambda_i;\beta) &= 17280\lambda_i^2(\beta+16)(3\beta+2)\geq 0 \text{ for all } \beta\geq -\frac{2}{3},\\
\xi_{43}^{(2)}(\lambda_i;\lambda_i;\beta) &= 13824\lambda_i(77\beta+134+3\beta^2)\geq 0 \text{ for all } \beta\geq -1.8776,\\
\xi_{43}^{(3)}(\lambda_i;\lambda_i;\beta) &= 615168\beta+2875392+10368\beta^2\geq 0 \text{ for all } \beta\geq -5.1151.
\end{aligned}$$

Hence, $\xi_4^{(3)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{42}(\lambda_j;\lambda_i;\beta)=\xi_4^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{42} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned}
\xi_{42}(\lambda_j;\lambda_i;\beta) &= (369792 + 70848\beta + 1728\beta^2)\lambda_j^4 \\
&\quad +(138240\beta + 20736\beta^2 - 359424)\lambda_i\lambda_j^3 \\
&\quad +(69120 + 6912\beta + 31104\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(175104 + 36864\beta + 6912\beta^2)\lambda_i^3\lambda_j \\
&\quad +(21888 + 6336\beta)\lambda_i^4, \\
\xi_{42}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1479168 + 283392\beta + 6912\beta^2)\lambda_j^3 \\
&\quad +(414720\beta + 62208\beta^2 - 1078272)\lambda_i\lambda_j^2 \\
&\quad +(138240 + 13824\beta + 62208\beta^2)\lambda_i^2\lambda_j \\
&\quad +(175104 + 36864\beta + 6912\beta^2)\lambda_i^3, \\
\xi_{42}^{(2)}(\lambda_j;\lambda_i;\beta) &= (4437504 + 850176\beta + 20736\beta^2)\lambda_j^2 \\
&\quad +(829440\beta + 124416\beta^2 - 2156544)\lambda_i\lambda_j \\
&\quad +(138240 + 13824\beta + 62208\beta^2)\lambda_i^2, \\
\xi_{42}^{(3)}(\lambda_j;\lambda_i;\beta) &= (8875008 + 1700352\beta + 41472\beta^2)\lambda_j \\
&\quad +(829440\beta + 124416\beta^2 - 2156544)\lambda_i, \\
\xi_{42}^{(4)}(\lambda_j;\lambda_i;\beta) &= 8875008 + 1700352\beta + 41472\beta^2 \geq 0 \text{ for all } \beta \geq -6.1386.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{42}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^4(7\beta+16)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{42}^{(1)}(\lambda_i;\lambda_i;\beta) &= 11520\lambda_i^3(62 + 65\beta + 12\beta^2) \geq 0 \text{ for all } \beta \geq -1.2358, \\
\xi_{42}^{(2)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i^2(70 + 49\beta + 6\beta^2) \geq 0 \text{ for all } \beta \geq -1.8457, \\
\xi_{42}^{(3)}(\lambda_i;\lambda_i;\beta) &= 41472\lambda_i(162 + 61\beta + 4\beta^2) \geq 0 \text{ for all } \beta \geq -3.4249.
\end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{41}(\lambda_j;\lambda_i;\beta) = \xi_4^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{41} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$
\begin{aligned}
\xi_{41}(\lambda_j;\lambda_i;\beta) &= (32256\beta + 173952 + 864\beta^2)\lambda_j^4 \\
&\quad +(-184320 + 66240\beta + 12960\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(25920\beta^2 + 23040 - 28800\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(28800\beta + 8640\beta^2 + 46080)\lambda_i^3\lambda_j \\
&\quad +(14400\beta + 5760)\lambda_i^4, \\
\xi_{41}^{(1)}(\lambda_j;\lambda_i;\beta) &= (129024\beta + 695808 + 3456\beta^2)\lambda_j^3 \\
&\quad +(-552960 + 198720\beta + 38880\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(51840\beta^2 + 46080 - 57600\beta)\lambda_i^2\lambda_j + (28800\beta + 8640\beta^2 + 46080)\lambda_i^3, \\
\xi_{41}^{(2)}(\lambda_j;\lambda_i;\beta) &= (387072\beta + 2087424 + 10368\beta^2)\lambda_j^2 \\
&\quad +(-1105920 + 397440\beta + 77760\beta^2)\lambda_i\lambda_j \\
&\quad +(51840\beta^2 + 46080 - 57600\beta)\lambda_i^2, \\
\xi_{41}^{(3)}(\lambda_j;\lambda_i;\beta) &= (774144\beta + 4174848 + 20736\beta^2)\lambda_j \\
&\quad +(-1105920 + 397440\beta + 77760\beta^2)\lambda_i, \\
\xi_{41}^{(4)}(\lambda_j;\lambda_i;\beta) &= 774144\beta + 4174848 + 20736\beta^2 \geq 0 \text{ for all } \beta \geq -6.5377.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{41}(\lambda_i;\lambda_i;\beta) &= 16128\lambda_i^4(3\beta+4)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{41}^{(1)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^3(119\beta^2 + 272 + 346\beta) \geq 0 \text{ for all } \beta, \\
\xi_{41}^{(2)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i^2(1262\beta + 1784 + 243\beta^2) \geq 0 \text{ for all } \beta, \\
\xi_{41}^{(3)}(\lambda_i;\lambda_i;\beta) &= 5184\lambda_i(8+\beta)(19\beta+74) \geq 0 \text{ for all } \beta \geq -\frac{74}{19}.
\end{aligned}
$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j;\lambda_i;\beta) = \xi_4(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{40} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}
\xi_{40}(\lambda_j;\lambda_i;\beta) &= (57984+10752\beta+288\beta^2)\lambda_j^4+(5184\beta^2-73728+26496\beta)\lambda_i\lambda_j^3\\
&\quad+(11520-14400\beta+12960\beta^2)\lambda_i^2\lambda_j^2\\
&\quad+(5760\beta^2+19200\beta+30720)\lambda_i^3\lambda_j\\
&\quad+(14400\beta+5760)\lambda_i^4,\\
\xi_{40}^{(1)}(\lambda_j;\lambda_i;\beta) &= (231936+43008\beta+1152\beta^2)\lambda_j^3\\
&\quad+(15552\beta^2-221184+79488\beta)\lambda_i\lambda_j^2\\
&\quad+(23040-28800\beta+25920\beta^2)\lambda_i^2\lambda_j\\
&\quad+(5760\beta^2+19200\beta+30720)\lambda_i^3,\\
\xi_{40}^{(2)}(\lambda_j;\lambda_i;\beta) &= (695808+129024\beta+3456\beta^2)\lambda_j^2\\
&\quad+(31104\beta^2-442368+158976\beta)\lambda_i\lambda_j\\
&\quad+(23040-28800\beta+25920\beta^2)\lambda_i^2,\\
\xi_{40}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1391616+258048\beta+6912\beta^2)\lambda_j\\
&\quad+(31104\beta^2-442368+158976\beta)\lambda_i,\\
\xi_{40}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1391616+258048\beta+6912\beta^2 \geq 0 \text{ for all } \beta \geq -6.5377.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{40}(\lambda_i;\lambda_i;\beta) &= 8064\lambda_i^4(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1,\\
\xi_{40}^{(1)}(\lambda_i;\lambda_i;\beta) &= 16128\lambda_i^3(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1,\\
\xi_{40}^{(2)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^2(7\beta+16)(\beta+2) > 0 \text{ for all } \beta > -2,\\
\xi_{40}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1152\lambda_i(824+362\beta+33\beta^2) > 0 \text{ for all } \beta > -3.2235.
\end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

$\square$

Lemma B.14 Let

$$\begin{aligned}\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (648\beta^3+15552-10368\beta)t(2,2,2,3)\\ &+(2208\beta+432\beta^2-6144)t(1,1,3,4)\\ &+(1296\beta^2-1296\beta-2592)t(1,2,2,4)\\ &+(576\beta^2+3168\beta+216\beta^3-10368)t(1,2,3,3)\\ &+(144\beta^2+1296\beta+5472)t(0,2,3,4)\\ &+(240+600\beta)t(0,1,4,4)\\ &+(144\beta^2+3072\beta+16128+24\beta^3)t(0,3,3,3),\end{aligned}$$

where

$$t(a,b,c,d)=\sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\mathrm{perm}(a,b,c,d)}}\lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_k^{n_4}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k\geq\lambda_l\geq\lambda_j\geq\lambda_i$, and $\beta\geq-\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a,b,c)=\sum_{\substack{(n_1,n_2,n_3)\\ \in\mathrm{perm}(a,b,c)}}\lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) \;=\;& (1728\beta^2-24576+8832\beta)\lambda_k^3\hat{s}(1,1,3)\\
&+(2400\beta+960)\lambda_k^3\hat{s}(0,1,4)\\
&+(-5184\beta+5184\beta^2-10368)\lambda_k^3\hat{s}(1,2,2)\\
&+(21888+5184\beta+576\beta^2)\lambda_k^3\hat{s}(0,2,3)\\
&+(-31104\beta+46656+1944\beta^3)\lambda_k^2\hat{s}(2,2,2)\\
&+(1296\beta^2+6624\beta-18432)\lambda_k^2\hat{s}(1,1,4)\\
&+(-31104+648\beta^3+1728\beta^2+9504\beta)\lambda_k^2\hat{s}(1,2,3)\\
&+(3888\beta+16416+432\beta^2)\lambda_k^2\hat{s}(0,2,4)\\
&+(432\beta^2+72\beta^3+48384+9216\beta)\lambda_k^2\hat{s}(0,3,3)\\
&+(-20736\beta+1296\beta^3+31104)\lambda_k\hat{s}(2,2,3)\\
&+(6336\beta+432\beta^3-20736+1152\beta^2)\lambda_k\hat{s}(1,3,3)\\
&+(-5184+2592\beta^2-2592\beta)\lambda_k\hat{s}(1,2,4)\\
&+(10944+288\beta^2+2592\beta)\lambda_k\hat{s}(0,3,4)\\
&+(240+600\beta)\hat{s}(0,4,4)\\
&+(-2592+1296\beta^2-1296\beta)\hat{s}(2,2,4)\\
&+(576\beta^2+3168\beta+216\beta^3-10368)\hat{s}(2,3,3)\\
&+(2208\beta+432\beta^2-6144)\hat{s}(1,3,4),\\
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) \;=\;& 3(1728\beta^2-24576+8832\beta)\lambda_k^2\hat{s}(1,1,3)\\
&+3(2400\beta+960)\lambda_k^2\hat{s}(0,1,4)\\
&+3(-5184\beta+5184\beta^2-10368)\lambda_k^2\hat{s}(1,2,2)\\
&+3(21888+5184\beta+576\beta^2)\lambda_k^2\hat{s}(0,2,3)\\
&+2(-31104\beta+46656+1944\beta^3)\lambda_k\hat{s}(2,2,2)\\
&+2(1296\beta^2+6624\beta-18432)\lambda_k\hat{s}(1,1,4)\\
&+2(-31104+648\beta^3+1728\beta^2+9504\beta)\lambda_k\hat{s}(1,2,3)\\
&+2(3888\beta+16416+432\beta^2)\lambda_k\hat{s}(0,2,4)\\
&+2(432\beta^2+72\beta^3+48384+9216\beta)\lambda_k\hat{s}(0,3,3)\\
&+(-20736\beta+1296\beta^3+31104)\hat{s}(2,2,3)\\
&+(6336\beta+432\beta^3-20736+1152\beta^2)\hat{s}(1,3,3)\\
&+(-5184+2592\beta^2-2592\beta)\hat{s}(1,2,4)\\
&+(10944+288\beta^2+2592\beta)\hat{s}(0,3,4),
\end{aligned}$$

$$
\begin{aligned}
\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(1728\beta^2-24576+8832\beta)\lambda_k\hat{s}(1,1,3)\\
&+6(2400\beta+960)\lambda_k\hat{s}(0,1,4)\\
&+6(-5184\beta+5184\beta^2-10368)\lambda_k\hat{s}(1,2,2)\\
&+6(21888+5184\beta+576\beta^2)\lambda_k\hat{s}(0,2,3)\\
&+2(-31104\beta+46656+1944\beta^3)\hat{s}(2,2,2)\\
&+2(1296\beta^2+6624\beta-18432)\hat{s}(1,1,4)\\
&+2(-31104+648\beta^3+1728\beta^2+9504\beta)\hat{s}(1,2,3)\\
&+2(3888\beta+16416+432\beta^2)\hat{s}(0,2,4)\\
&+2(432\beta^2+72\beta^3+48384+9216\beta)\hat{s}(0,3,3),\\
\xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(1728\beta^2-24576+8832\beta)\hat{s}(1,1,3)+6(2400\beta+960)\hat{s}(0,1,4)\\
&+6(-5184\beta+5184\beta^2-10368)\hat{s}(1,2,2)\\
&+6(21888+5184\beta+576\beta^2)\hat{s}(0,2,3).
\end{aligned}
$$

Define

$$
\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)/\lambda_l,\\
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta).
\end{aligned}
$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= ((240+600\beta)\lambda_i + (240+600\beta)\lambda_j)\lambda_l^7 \\
&\quad +(288\beta^2+2592\beta+10944)(\lambda_j^2+\lambda_i^2)\lambda_l^6 \\
&\quad +(4416\beta-12288+864\beta^2)\lambda_i\lambda_j\lambda_l^6 \\
&\quad +(-15552+576\beta+216\beta^3+3168\beta^2)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^5 \\
&\quad +(432\beta^2+24\beta^3+27072+5664\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^5 \\
&\quad +(10752\beta-33024+2016\beta^2+432\beta^3)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^4 \\
&\quad +(11424+3792\beta+288\beta^2)(\lambda_j^4+\lambda_i^4)\lambda_l^4 \\
&\quad +(2592\beta^2+25920+1296\beta^3-23328\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&\quad +(1440\beta^2+5760-1440\beta+1080\beta^3)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^3 \\
&\quad +(2160\beta^2-14400+4320\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^3 \\
&\quad +(2880\beta^2+5760)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l^2 \\
&\quad +(11520+12480\beta+1440\beta^2+480\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 \\
&\quad +(4800+4800\beta+720\beta^2)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\lambda_l \\
&\quad +(480+1200\beta)\lambda_i^4\lambda_j^4.
\end{aligned}$$

Define $\xi_{07}(\lambda_j;\lambda_i;\beta) = \xi_0^{(7)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{07} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{07} with respect to λ_j gives

$$\begin{aligned}
\xi_{07}(\lambda_j;\lambda_i;\beta) &= (3024000\beta+1209600)\lambda_j + (3024000\beta+1209600)\lambda_i, \\
\xi_{07}^{(1)}(\lambda_j;\lambda_i;\beta) &= 3024000\beta+1209600\beta \geq -\frac{2}{5}.
\end{aligned}$$

Now

$$\xi_{07}(\lambda_i;\lambda_i;\beta) = 1209600(2+5\beta)\lambda_i \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.$$

Hence, $\xi_0^{(7)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{06}(\lambda_j;\lambda_i;\beta) = \xi_0^{(6)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{06} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{06} with respect to λ_j gives

$$\begin{aligned}
\xi_{06}(\lambda_j;\lambda_i;\beta) &= (4890240\beta+9089280+207360\beta^2)\lambda_j^2 \\
&\quad +(6203520\beta-7637760+622080\beta^2)\lambda_i\lambda_j \\
&\quad +(7879680+1866240\beta+207360\beta^2)\lambda_i^2, \\
\xi_{06}^{(1)}(\lambda_j;\lambda_i;\beta) &= (9780480\beta+18178560+414720\beta^2)\lambda_j \\
&\quad +(6203520\beta-7637760+622080\beta^2)\lambda_i, \\
\xi_{06}^{(2)}(\lambda_j;\lambda_i;\beta) &= 9780480\beta+18178560+414720\beta^2. \geq 0 \text{ for all } \beta \geq -2.0341.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{06}(\lambda_i;\lambda_i;\beta) &= 518400\lambda_i^2(25\beta+18+2\beta^2) \geq 0 \text{ for all } \beta \geq -0.7671, \\
\xi_{06}^{(1)}(\lambda_i;\lambda_i;\beta) &= 86400\lambda_i(185\beta+122+12\beta^2) \geq 0 \text{ for all } \beta \geq -0.6904.
\end{aligned}$$

Hence, $\xi_0^{(6)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{05}(\lambda_j;\lambda_i;\beta) = \xi_0^{(5)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{05} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{05} with respect to λ_j gives

$$\begin{aligned}
\xi_{05}(\lambda_j;\lambda_i;\beta) &= (4057920\beta + 11733120 + 259200\beta^2 + 2880\beta^3)\lambda_j^3 \\
&\quad +(4760640\beta - 10108800 + 1002240\beta^2 + 25920\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(6013440 + 1935360\beta + 587520\beta^2 + 25920\beta^3)\lambda_i^2\lambda_j \\
&\quad +(679680\beta + 3248640 + 51840\beta^2 + 2880\beta^3)\lambda_i^3, \\
\xi_{05}^{(1)}(\lambda_j;\lambda_i;\beta) &= (12173760\beta + 35199360 + 777600\beta^2 + 8640\beta^3)\lambda_j^2 \\
&\quad +(9521280\beta - 20217600 + 2004480\beta^2 + 51840\beta^3)\lambda_i\lambda_j \\
&\quad +(6013440 + 1935360\beta + 587520\beta^2 + 25920\beta^3)\lambda_i^2, \\
\xi_{05}^{(2)}(\lambda_j;\lambda_i;\beta) &= (24347520\beta + 70398720 + 1555200\beta^2 + 17280\beta^3)\lambda_j \\
&\quad +(9521280\beta - 20217600 + 2004480\beta^2 + 51840\beta^3)\lambda_i, \\
\xi_{05}^{(3)}(\lambda_j;\lambda_i;\beta) &= 24347520\beta + 70398720 + 1555200\beta^2 + 17280\beta^3 \geq 0 \text{ for all } \beta \geq -3.7540.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{05}(\lambda_i;\lambda_i;\beta) &= 28800\lambda_i^3(397\beta + 378 + 66\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1726, \\
\xi_{05}^{(1)}(\lambda_i;\lambda_i;\beta) &= 43200\lambda_i^2(547\beta + 486 + 78\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.0380, \\
\xi_{05}^{(2)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i(980\beta + 1452 + 103\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.8160.
\end{aligned}$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{04}(\lambda_j;\lambda_i;\beta) = \xi_0^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{04} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned}
\xi_{04}(\lambda_j;\lambda_i;\beta) &= (7664256 + 2207808\beta + 162432\beta^2 + 2880\beta^3)\lambda_j^4 \\
&\quad +(-6880896 + 2420928\beta + 739584\beta^2 + 36288\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(442368\beta + 2695680 + 546048\beta^2 + 57024\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(937728\beta + 2456064 + 100224\beta^2 + 13248\beta^3)\lambda_i^3\lambda_j \\
&\quad +(91008\beta + 6912\beta^2 + 274176)\lambda_i^4, \\
\xi_{04}^{(1)}(\lambda_j;\lambda_i;\beta) &= (30657024 + 8831232\beta + 649728\beta^2 + 11520\beta^3)\lambda_j^3 \\
&\quad +(-20642688 + 7262784\beta + 2218752\beta^2 + 108864\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(884736\beta + 5391360 + 1092096\beta^2 + 114048\beta^3)\lambda_i^2\lambda_j \\
&\quad +(937728\beta + 2456064 + 100224\beta^2 + 13248\beta^3)\lambda_i^3, \\
\xi_{04}^{(2)}(\lambda_j;\lambda_i;\beta) &= (91971072 + 26493696\beta + 1949184\beta^2 + 34560\beta^3)\lambda_j^2 \\
&\quad +(-41285376 + 14525568\beta + 4437504\beta^2 + 217728\beta^3)\lambda_i\lambda_j \\
&\quad +(884736\beta + 5391360 + 1092096\beta^2 + 114048\beta^3)\lambda_i^2, \\
\xi_{04}^{(3)}(\lambda_j;\lambda_i;\beta) &= (183942144 + 52987392\beta + 3898368\beta^2 + 69120\beta^3)\lambda_j \\
&\quad +(-41285376 + 14525568\beta + 4437504\beta^2 + 217728\beta^3)\lambda_i, \\
\xi_{04}^{(4)}(\lambda_j;\lambda_i;\beta) &= 183942144 + 52987392\beta + 3898368\beta^2 + 69120\beta^3 \geq 0 \text{ for all } \beta \geq -5.4358.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{04}(\lambda_i;\lambda_i;\beta) &= 5760\lambda_i^4(1078+1059\beta+270\beta^2+19\beta^3)\geq 0 \text{ for all } \beta\geq -1.5912,\\
\xi_{04}^{(1)}(\lambda_i;\lambda_i;\beta) &= 2880\lambda_i^3(6202+6221\beta+1410\beta^2+86\beta^3)\geq 0 \text{ for all } \beta\geq -1.4073,\\
\xi_{04}^{(2)}(\lambda_i;\lambda_i;\beta) &= 3456\lambda_i^2(16226+12125\beta+2164\beta^2+106\beta^3)\geq 0 \text{ for all } \beta\geq -1.9551,\\
\xi_{04}^{(3)}(\lambda_i;\lambda_i;\beta) &= 3456\lambda_i(41278+19535\beta+2412\beta^2+83\beta^3)\geq 0 \text{ for all } \beta\geq -3.3153.
\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j;\lambda_i;\beta) = \xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{03} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}
\xi_{03}(\lambda_j;\lambda_i;\beta) &= (3262176+867888\beta+67392\beta^2+1440\beta^3)\lambda_j^4\\
&\quad +(974448\beta+355104\beta^2-3236256+23328\beta^3)\lambda_i\lambda_j^3\\
&\quad +(295488\beta^2+1036800-222912\beta+50544\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(82944\beta^2+589248\beta+18288\beta^3+866304)\lambda_i^3\lambda_j\\
&\quad +(116928\beta+19872\beta^2+187776)\lambda_i^4,\\
\xi_{03}^{(1)}(\lambda_j;\lambda_i;\beta) &= (13048704+3471552\beta+269568\beta^2+5760\beta^3)\lambda_j^3\\
&\quad +(2923344\beta+1065312\beta^2-9708768+69984\beta^3)\lambda_i\lambda_j^2\\
&\quad +(590976\beta^2+2073600-445824\beta+101088\beta^3)\lambda_i^2\lambda_j\\
&\quad +(82944\beta^2+589248\beta+18288\beta^3+866304)\lambda_i^3,\\
\xi_{03}^{(2)}(\lambda_j;\lambda_i;\beta) &= (39146112+10414656\beta+808704\beta^2+17280\beta^3)\lambda_j^2\\
&\quad +(5846688\beta+2130624\beta^2-19417536+139968\beta^3)\lambda_i\lambda_j\\
&\quad +(590976\beta^2+2073600-445824\beta+101088\beta^3)\lambda_i^2,\\
\xi_{03}^{(3)}(\lambda_j;\lambda_i;\beta) &= (78292224+20829312\beta+1617408\beta^2+34560\beta^3)\lambda_j\\
&\quad +(5846688\beta+2130624\beta^2-19417536+139968\beta^3)\lambda_i,\\
\xi_{03}^{(4)}(\lambda_j;\lambda_i;\beta) &= 78292224+20829312\beta+1617408\beta^2+34560\beta^3\geq 0 \text{ for all } \beta\geq -6.9654.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{03}(\lambda_i;\lambda_i;\beta) &= 7200\lambda_i^4(13\beta+49)(\beta+3)(\beta+2)\geq 0 \text{ for all } \beta\geq -2,\\
\xi_{03}^{(1)}(\lambda_i;\lambda_i;\beta) &= 720\lambda_i^3(8722+9081\beta+2790\beta^2+271\beta^3)\geq 0 \text{ for all } \beta\geq -1.7083,\\
\xi_{03}^{(2)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^2(25234+18305\beta+4086\beta^2+299\beta^3)\geq 0 \text{ for all } \beta\geq -2.6023,\\
\xi_{03}^{(3)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i(68142+30875\beta+4338\beta^2+202\beta^3)\geq 0 \text{ for all } \beta\geq -4.2046.
\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j;\lambda_i;\beta) = \xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{02} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned}
\xi_{02}(\lambda_j;\lambda_i;\beta) &= (261744\beta + 1016928 + 480\beta^3 + 20736\beta^2)\lambda_j^4 \\
&\quad +(126432\beta^2 - 1152288 + 324144\beta + 9504\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(117504\beta^2 - 199296\beta + 374400 + 26352\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(13104\beta^3 + 44352\beta^2 + 258624\beta + 202752)\lambda_i^3\lambda_j \\
&\quad +(22176\beta^2 + 62208 + 71424\beta)\lambda_i^4, \\
\xi_{02}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1046976\beta + 4067712 + 1920\beta^3 + 82944\beta^2)\lambda_j^3 \\
&\quad +(379296\beta^2 - 3456864 + 972432\beta + 28512\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(235008\beta^2 - 398592\beta + 748800 + 52704\beta^3)\lambda_i^2\lambda_j \\
&\quad +(13104\beta^3 + 44352\beta^2 + 258624\beta + 202752)\lambda_i^3, \\
\xi_{02}^{(2)}(\lambda_j;\lambda_i;\beta) &= (3140928\beta + 12203136 + 5760\beta^3 + 248832\beta^2)\lambda_j^2 \\
&\quad +(758592\beta^2 - 6913728 + 1944864\beta + 57024\beta^3)\lambda_i\lambda_j \\
&\quad +(235008\beta^2 - 398592\beta + 748800 + 52704\beta^3)\lambda_i^2, \\
\xi_{02}^{(3)}(\lambda_j;\lambda_i;\beta) &= (6281856\beta + 24406272 + 11520\beta^3 + 497664\beta^2)\lambda_j \\
&\quad +(758592\beta^2 - 6913728 + 1944864\beta + 57024\beta^3)\lambda_i, \\
\xi_{02}^{(4)}(\lambda_j;\lambda_i;\beta) &= 6281856\beta + 24406272 + 11520\beta^3 + 497664\beta^2 \geq 0 \text{ for all } \beta \geq -8.2141.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{02}(\lambda_i;\lambda_i;\beta) &= 480\lambda_i^4(103\beta + 175)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{175}{103}, \\
\xi_{02}^{(1)}(\lambda_i;\lambda_i;\beta) &= 240\lambda_i^3(401\beta + 1085)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{1085}{401}, \\
\xi_{02}^{(2)}(\lambda_i;\lambda_i;\beta) &= 288\lambda_i^2(16275\beta + 20966 + 401\beta^3 + 4314\beta^2) \geq 0 \text{ for all } \beta \geq -3.1231, \\
\xi_{02}^{(3)}(\lambda_i;\lambda_i;\beta) &= \geq 0 \text{ for all } \beta \geq -4.2173.
\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{01} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j;\lambda_i;\beta) &= (248400 + 63240\beta + 120\beta^3 + 5040\beta^2)\lambda_j^4 \\
&\quad +(35568\beta^2 - 325104 + 89544\beta + 2808\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(38016\beta^2 - 79200\beta + 120384 + 9504\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(96768\beta + 6048\beta^3 + 18144\beta^2 + 48384)\lambda_i^3\lambda_j \\
&\quad +(14112\beta^2 + 32928\beta + 18816)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= (993600 + 252960\beta + 480\beta^3 + 20160\beta^2)\lambda_j^3 \\
&\quad +(106704\beta^2 - 975312 + 268632\beta + 8424\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(76032\beta^2 - 158400\beta + 240768 + 19008\beta^3)\lambda_i^2\lambda_j \\
&\quad +(96768\beta + 6048\beta^3 + 18144\beta^2 + 48384)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= (2980800 + 758880\beta + 1440\beta^3 + 60480\beta^2)\lambda_j^2 \\
&\quad +(213408\beta^2 - 1950624 + 537264\beta + 16848\beta^3)\lambda_i\lambda_j \\
&\quad +(76032\beta^2 - 158400\beta + 240768 + 19008\beta^3)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j;\lambda_i;\beta) &= (5961600 + 1517760\beta + 2880\beta^3 + 120960\beta^2)\lambda_j \\
&\quad +(213408\beta^2 - 1950624 + 537264\beta + 16848\beta^3)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2880(\beta+10)(\beta+9)(\beta+23) \geq 0 \text{ for all } \beta \geq -9.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 18480\lambda_i^4(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 120\lambda_i^3(283\beta+427)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{427}{283}, \\
\xi_{01}^{(2)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i^2(\beta+3)(259\beta^2+1653\beta+2942) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(27854 + 14271\beta + 137\beta^3 + 2322\beta^2) \geq 0 \text{ for all } \beta \geq -3.6636.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= (49680+12648\beta+1008\beta^2+24\beta^3)\lambda_j^4 \\
&\quad +(20664\beta-75024+8208\beta^2+648\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(10368\beta^2-21600\beta+32832+2592\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(6048\beta^2+2016\beta^3+16128+32256\beta)\lambda_i^3\lambda_j \\
&\quad +(8064+14112\beta+6048\beta^2)\lambda_i^4, \\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= (198720+50592\beta+4032\beta^2+96\beta^3)\lambda_j^3 \\
&\quad +(61992\beta-225072+24624\beta^2+1944\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(20736\beta^2-43200\beta+65664+5184\beta^3)\lambda_i^2\lambda_j \\
&\quad +(6048\beta^2+2016\beta^3+16128+32256\beta)\lambda_i^3, \\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= (596160+151776\beta+12096\beta^2+288\beta^3)\lambda_j^2 \\
&\quad +(123984\beta-450144+49248\beta^2+3888\beta^3)\lambda_i\lambda_j \\
&\quad +(20736\beta^2-43200\beta+65664+5184\beta^3)\lambda_i^2, \\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1192320+303552\beta+24192\beta^2+576\beta^3)\lambda_j \\
&\quad +(123984\beta-450144+49248\beta^2+3888\beta^3)\lambda_i, \\
\xi_{00}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1192320+303552\beta+24192\beta^2+576\beta^3 \geq 0 \text{ for all } \beta \geq -9.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 5280\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 9240\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 720\lambda_i^2(13\beta+49)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -2, \\
\xi_{00}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(\beta+3)(31\beta^2+417\beta+1718) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$
\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (2400\beta+960)(\lambda_j+\lambda_i)\lambda_l^7 \\
&\quad +(38304+9072\beta+1008\beta^2)(\lambda_j^2+\lambda_i^2)\lambda_l^6 \\
&\quad +(-43008+3024\beta^2+15456\beta)\lambda_i\lambda_j\lambda_l^6 \\
&\quad +(1728\beta+9504\beta^2-46656+648\beta^3)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^5 \\
&\quad +(1296\beta^2+81216+72\beta^3+16992\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^5 \\
&\quad +(1080\beta^3+5040\beta^2-82560+26880\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^4 \\
&\quad +(720\beta^2+28560+9480\beta)(\lambda_j^4+\lambda_i^4)\lambda_l^4 \\
&\quad +(6480\beta^2-58320\beta+64800+3240\beta^3)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&\quad +(2160\beta^3+11520-2880\beta+2880\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^3 \\
&\quad +(-28800+4320\beta^2+8640\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^3 \\
&\quad +(4320\beta^2+8640)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l^2 \\
&\quad +(18720\beta+17280+2160\beta^2+720\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 \\
&\quad +(4800+4800\beta+720\beta^2)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\lambda_l \\
&\quad +(240+600\beta)\lambda_i^4\lambda_j^4,
\end{aligned}
$$

and, hence,

$$
\xi_1^{(7)}(\lambda_l;\lambda_j;\lambda_i;\beta) = (12096000\beta+4838400)(\lambda_j+\lambda_i) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
$$

Define $\xi_{16}(\lambda_j;\lambda_i;\beta) = \xi_1^{(6)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{16} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{16} with respect to λ_j gives

$$
\begin{aligned}
\xi_{16}(\lambda_j;\lambda_i;\beta) &= (18627840\beta+32417280+725760\beta^2)\lambda_j^2 \\
&\quad +(23224320\beta-26127360+2177280\beta^2)\lambda_i\lambda_j \\
&\quad +(27578880+725760\beta^2+6531840\beta)\lambda_i^2, \\
\xi_{16}^{(1)}(\lambda_j;\lambda_i;\beta) &= (37255680\beta+64834560+1451520\beta^2)\lambda_j \\
&\quad +(23224320\beta-26127360+2177280\beta^2)\lambda_i, \\
\xi_{16}^{(2)}(\lambda_j;\lambda_i;\beta) &= 37255680\beta+64834560+1451520\beta^2 \geq 0 \text{ for all } \beta \geq -1.8776.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{16}(\lambda_i;\lambda_i;\beta) &= 1209600\lambda_i^2(28+40\beta+3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\
\xi_{16}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1209600\lambda_i(\beta+16)(3\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}.
\end{aligned}
$$

Hence, $\xi_1^{(6)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{15}(\lambda_j;\lambda_i;\beta)=\xi_1^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{15} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{15} with respect to λ_j gives

$$\begin{aligned}
\xi_{15}(\lambda_j;\lambda_i;\beta) &= (14618880\beta+39744000+881280\beta^2+8640\beta^3)\lambda_j^3\\
&\quad+(17383680\beta-34145280+77760\beta^3+3317760\beta^2)\lambda_i\lambda_j^2\\
&\quad+(21980160+1866240\beta^2+6739200\beta+77760\beta^3)\lambda_i^2\lambda_j\\
&\quad+(9745920+8640\beta^3+155520\beta^2+2039040\beta)\lambda_i^3,\\
\xi_{15}^{(1)}(\lambda_j;\lambda_i;\beta) &= (43856640\beta+119232000+2643840\beta^2+25920\beta^3)\lambda_j^2\\
&\quad+(34767360\beta-68290560+155520\beta^3+6635520\beta^2)\lambda_i\lambda_j\\
&\quad+(21980160+1866240\beta^2+6739200\beta+77760\beta^3)\lambda_i^2,\\
\xi_{15}^{(2)}(\lambda_j;\lambda_i;\beta) &= (87713280\beta+238464000+5287680\beta^2+51840\beta^3)\lambda_j\\
&\quad+(34767360\beta-68290560+155520\beta^3+6635520\beta^2)\lambda_i,\\
\xi_{15}^{(3)}(\lambda_j;\lambda_i;\beta) &= 87713280\beta+238464000+5287680\beta^2+51840\beta^3\geq 0 \text{ for all } \beta\geq-3.3874.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{15}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i^3(236\beta+216+36\beta^2+\beta^3)\geq 0 \text{ for all } \beta\geq-1.0915,\\
\xi_{15}^{(1)}(\lambda_i;\lambda_i;\beta) &= 86400\lambda_i^2(988\beta+844+129\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq-0.9757,\\
\xi_{15}^{(2)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i(3544\beta+4924+345\beta^2+6\beta^3)\geq 0 \text{ for all } \beta\geq-1.6454.
\end{aligned}$$

Hence, $\xi_1^{(5)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{14}(\lambda_j;\lambda_i;\beta)=\xi_1^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{14} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned}
\xi_{14}(\lambda_j;\lambda_i;\beta) &= (25027200+7548480\beta+8640\beta^3+535680\beta^2)\lambda_j^4\\
&\quad+(-22256640+8432640\beta+103680\beta^3+2350080\beta^2)\lambda_i\lambda_j^3\\
&\quad+(2073600\beta+9745920+1658880\beta^2+155520\beta^3)\lambda_i^2\lambda_j^2\\
&\quad+(7764480+34560\beta^3+276480\beta^2+2684160\beta)\lambda_i^3\lambda_j\\
&\quad+(227520\beta+17280\beta^2+685440)\lambda_i^4,\\
\xi_{14}^{(1)}(\lambda_j;\lambda_i;\beta) &= (100108800+30193920\beta+34560\beta^3+2142720\beta^2)\lambda_j^3\\
&\quad+(-66769920+25297920\beta+311040\beta^3+7050240\beta^2)\lambda_i\lambda_j^2\\
&\quad+(4147200\beta+19491840+3317760\beta^2+311040\beta^3)\lambda_i^2\lambda_j\\
&\quad+(7764480+34560\beta^3+276480\beta^2+2684160\beta)\lambda_i^3,\\
\xi_{14}^{(2)}(\lambda_j;\lambda_i;\beta) &= (300326400+90581760\beta+103680\beta^3+6428160\beta^2)\lambda_j^2\\
&\quad+(-133539840+50595840\beta+622080\beta^3+14100480\beta^2)\lambda_i\lambda_j\\
&\quad+(4147200\beta+19491840+3317760\beta^2+311040\beta^3)\lambda_i^2,\\
\xi_{14}^{(3)}(\lambda_j;\lambda_i;\beta) &= (600652800+181163520\beta+207360\beta^3+12856320\beta^2)\lambda_j\\
&\quad+(-133539840+50595840\beta+622080\beta^3+14100480\beta^2)\lambda_i,\\
\xi_{14}^{(4)}(\lambda_j;\lambda_i;\beta) &= 600652800+181163520\beta+207360\beta^3+12856320\beta^2\geq 0 \text{ for all } \beta\geq-4.8608.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{14}(\lambda_i;\lambda_i;\beta) &= 100800\lambda_i^4(208+208\beta+3\beta^3+48\beta^2) \geq 0 \text{ for all } \beta \geq -1.4294,\\
\xi_{14}^{(1)}(\lambda_i;\lambda_i;\beta) &= 115200\lambda_i^3(526+541\beta+6\beta^3+111\beta^2) \geq 0 \text{ for all } \beta \geq -1.2898,\\
\xi_{14}^{(2)}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i^2(1078+841\beta+6\beta^3+138\beta^2) \geq 0 \text{ for all } \beta \geq -1.7421,\\
\xi_{14}^{(3)}(\lambda_i;\lambda_i;\beta) &= 69120\lambda_i(6758+3353\beta+12\beta^3+390\beta^2) \geq 0 \text{ for all } \beta \geq -2.9155.
\end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j;\lambda_i;\beta) = \xi_1^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}
\xi_{13}(\lambda_j;\lambda_i;\beta) &= (10356480+2839680\beta+216000\beta^2+4320\beta^3)\lambda_j^4\\
&\quad +(1080000\beta^2-9912960+3159360\beta+64800\beta^3)\lambda_i\lambda_j^3\\
&\quad +(3421440+864000\beta^2-224640\beta+129600\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(216000\beta^2+2960640+43200\beta^3+1647360\beta)\lambda_i^3\lambda_j\\
&\quad +(279360\beta+43200\beta^2+512640)\lambda_i^4,\\
\xi_{13}^{(1)}(\lambda_j;\lambda_i;\beta) &= (41425920+11358720\beta+864000\beta^2+17280\beta^3)\lambda_j^3\\
&\quad +(3240000\beta^2-29738880+9478080\beta+194400\beta^3)\lambda_i\lambda_j^2\\
&\quad +(6842880+1728000\beta^2-449280\beta+259200\beta^3)\lambda_i^2\lambda_j\\
&\quad +(216000\beta^2+2960640+43200\beta^3+1647360\beta)\lambda_i^3,\\
\xi_{13}^{(2)}(\lambda_j;\lambda_i;\beta) &= (124277760+34076160\beta+2592000\beta^2+51840\beta^3)\lambda_j^2\\
&\quad +(6480000\beta^2-59477760+18956160\beta+388800\beta^3)\lambda_i\lambda_j\\
&\quad +(6842880+1728000\beta^2-449280\beta+259200\beta^3)\lambda_i^2,\\
\xi_{13}^{(3)}(\lambda_j;\lambda_i;\beta) &= (248555520+68152320\beta+5184000\beta^2+103680\beta^3)\lambda_j\\
&\quad +(6480000\beta^2-59477760+18956160\beta+388800\beta^3)\lambda_i,\\
\xi_{13}^{(4)}(\lambda_j;\lambda_i;\beta) &= 248555520+68152320\beta+5184000\beta^2+103680\beta^3 \geq 0 \text{ for all } \beta \geq -6.2366.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{13}(\lambda_i;\lambda_i;\beta) &= 40320\lambda_i^4(182+191\beta+60\beta^2+6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321,\\
\xi_{13}^{(1)}(\lambda_i;\lambda_i;\beta) &= 10080\lambda_i^3(2132+2186\beta+600\beta^2+51\beta^3) \geq 0 \text{ for all } \beta \geq -1.5432,\\
\xi_{13}^{(2)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^2(8292+6086\beta+1250\beta^2+81\beta^3) \geq 0 \text{ for all } \beta \geq -2.2524,\\
\xi_{13}^{(3)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i(21884+10082\beta+1350\beta^2+57\beta^3) \geq 0 \text{ for all } \beta \geq -3.7735.
\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j;\lambda_i;\beta) = \xi_1^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{12} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$
\begin{aligned}
\xi_{12}(\lambda_j;\lambda_i;\beta) &= (826560\beta + 3156480 + 1440\beta^3 + 64800\beta^2)\lambda_j^4 \\
&\quad +(367200\beta^2 + 973440\beta - 3346560 + 25920\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(1080000 - 410400\beta + 324000\beta^2 + 64800\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(737280 + 682560\beta + 28800\beta^3 + 108000\beta^2)\lambda_i^3\lambda_j \\
&\quad +(187200 + 43200\beta^2 + 165600\beta)\lambda_i^4, \\
\xi_{12}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3306240\beta + 12625920 + 5760\beta^3 + 259200\beta^2)\lambda_j^3 \\
&\quad +(1101600\beta^2 + 2920320\beta - 10039680 + 77760\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(2160000 - 820800\beta + 648000\beta^2 + 129600\beta^3)\lambda_i^2\lambda_j \\
&\quad +(737280 + 682560\beta + 28800\beta^3 + 108000\beta^2)\lambda_i^3, \\
\xi_{12}^{(2)}(\lambda_j;\lambda_i;\beta) &= (9918720\beta + 37877760 + 17280\beta^3 + 777600\beta^2)\lambda_j^2 \\
&\quad +(2203200\beta^2 + 5840640\beta - 20079360 + 155520\beta^3)\lambda_i\lambda_j \\
&\quad +(2160000 - 820800\beta + 648000\beta^2 + 129600\beta^3)\lambda_i^2, \\
\xi_{12}^{(3)}(\lambda_j;\lambda_i;\beta) &= (19837440\beta + 75755520 + 34560\beta^3 + 1555200\beta^2)\lambda_j \\
&\quad +(2203200\beta^2 + 5840640\beta - 20079360 + 155520\beta^3)\lambda_i, \\
\xi_{12}^{(4)}(\lambda_j;\lambda_i;\beta) &= 19837440\beta + 75755520 + 34560\beta^3 + 1555200\beta^2 \geq 0 \text{ for all } \beta \geq -7.4474.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{12}(\lambda_i;\lambda_i;\beta) &= 60480\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{12}^{(1)}(\lambda_i;\lambda_i;\beta) &= 20160\lambda_i^3(302\beta + 272 + 12\beta^3 + 105\beta^2) \geq 0 \text{ for all } \beta \geq -1.7670, \\
\xi_{12}^{(2)}(\lambda_i;\lambda_i;\beta) &= 60480\lambda_i^2(247\beta + 330 + 5\beta^3 + 60\beta^2) \geq 0 \text{ for all } \beta \geq -2.7671, \\
\xi_{12}^{(3)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i(2972\beta + 6444 + 22\beta^3 + 435\beta^2) \geq 0 \text{ for all } \beta \geq -4.2105.
\end{aligned}
$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j;\lambda_i;\beta) = \xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{11} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$
\begin{aligned}
\xi_{11}(\lambda_j;\lambda_i;\beta) &= (756864+194112\beta+360\beta^3+15408\beta^2)\lambda_j^4 \\
&\quad +(251616\beta-901248+98784\beta^2+7560\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(307584+96768\beta^2-178848\beta+22680\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(226080\beta+149760+12600\beta^3+40320\beta^2)\lambda_i^3\lambda_j \\
&\quad +(49920+68640\beta+25200\beta^2)\lambda_i^4, \\
\xi_{11}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3027456+776448\beta+1440\beta^3+61632\beta^2)\lambda_j^3 \\
&\quad +(754848\beta-2703744+296352\beta^2+22680\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(615168+193536\beta^2-357696\beta+45360\beta^3)\lambda_i^2\lambda_j \\
&\quad +(226080\beta+149760+12600\beta^3+40320\beta^2)\lambda_i^3, \\
\xi_{11}^{(2)}(\lambda_j;\lambda_i;\beta) &= (9082368+2329344\beta+4320\beta^3+184896\beta^2)\lambda_j^2 \\
&\quad +(1509696\beta-5407488+592704\beta^2+45360\beta^3)\lambda_i\lambda_j \\
&\quad +(615168+193536\beta^2-357696\beta+45360\beta^3)\lambda_i^2, \\
\xi_{11}^{(3)}(\lambda_j;\lambda_i;\beta) &= (18164736+4658688\beta+8640\beta^3+369792\beta^2)\lambda_j \\
&\quad +(1509696\beta-5407488+592704\beta^2+45360\beta^3)\lambda_i, \\
\xi_{11}^{(4)}(\lambda_j;\lambda_i;\beta) &= 18164736+4658688\beta+8640\beta^3+369792\beta^2 \geq 0 \text{ for all } \beta \geq -8.4121.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{11}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^4(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{7}{4}, \\
\xi_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= 4320\lambda_i^3(19\beta+42)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(2)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i^2(7448+6044\beta+165\beta^3+1686\beta^2) \geq 0 \text{ for all } \beta \geq -3.0958, \\
\xi_{11}^{(3)}(\lambda_i;\lambda_i;\beta) &= 144\lambda_i(88592+42836\beta+375\beta^3+6684\beta^2) \geq 0 \text{ for all } \beta \geq -4.0263.
\end{aligned}
$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j;\lambda_i;\beta) = \xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{10} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$
\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= (37944\beta + 149040 + 3024\beta^2 + 72\beta^3)\lambda_j^4 \\
&\quad +(-200064 + 21888\beta^2 + 55104\beta + 1728\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(76608 - 50400\beta + 24192\beta^2 + 6048\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(12096\beta^2 + 32256 + 4032\beta^3 + 64512\beta)\lambda_i^3\lambda_j \\
&\quad +(10080\beta^2 + 13440 + 23520\beta)\lambda_i^4, \\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= (151776\beta + 596160 + 12096\beta^2 + 288\beta^3)\lambda_j^3 \\
&\quad +(-600192 + 65664\beta^2 + 165312\beta + 5184\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(153216 - 100800\beta + 48384\beta^2 + 12096\beta^3)\lambda_i^2\lambda_j \\
&\quad +(12096\beta^2 + 32256 + 4032\beta^3 + 64512\beta)\lambda_i^3, \\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= (455328\beta + 1788480 + 36288\beta^2 + 864\beta^3)\lambda_j^2 \\
&\quad +(-1200384 + 131328\beta^2 + 330624\beta + 10368\beta^3)\lambda_i\lambda_j \\
&\quad +(153216 - 100800\beta + 48384\beta^2 + 12096\beta^3)\lambda_i^2, \\
\xi_{10}^{(3)}(\lambda_j;\lambda_i;\beta) &= (910656\beta + 3576960 + 72576\beta^2 + 1728\beta^3)\lambda_j \\
&\quad +(-1200384 + 131328\beta^2 + 330624\beta + 10368\beta^3)\lambda_i, \\
\xi_{10}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1728(\beta+10)(\beta+9)(\beta+23) \geq 0 \text{ for all } \beta \geq -9.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 11880\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 4320\lambda_i^3(5\beta+7)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -\frac{7}{5}, \\
\xi_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^2(\beta+3)(27\beta^2 + 169\beta + 286) > 0 \text{ for all } \beta > -3, \\
\xi_{10}^{(3)}(\lambda_i;\lambda_i;\beta) &= 576\lambda_i(2155\beta + 4126 + 354\beta^2 + 21\beta^3) > 0 \text{ for all } \beta > -3.5443.
\end{aligned}
$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$
\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) \;=\;& (7200\beta+2880)(\lambda_j+\lambda_i)\lambda_l^6 \\
&+(98496+2592\beta^2+23328\beta)(\lambda_j^2+\lambda_i^2)\lambda_l^5 \\
&+(39744\beta+7776\beta^2-110592)\lambda_i\lambda_j\lambda_l^5 \\
&+(21600\beta^2+864\beta-98496+1296\beta^3)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^4 \\
&+(36576\beta+144\beta^3+173376+2880\beta^2)(\lambda_j^3+\lambda_i^3)\lambda_l^4 \\
&+(-156672+1728\beta^3+51840\beta+9792\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^3 \\
&+(1152\beta^2+46656+17568\beta)(\lambda_j^4+\lambda_i^4)\lambda_l^3 \\
&+(-98496\beta+15552\beta^2+5184\beta^3+93312)\lambda_i^2\lambda_j^2\lambda_l^3 \\
&+(13824\beta+2592\beta^3+34560+5184\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^2 \\
&+(5184\beta^2+17856\beta-39168)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^2 \\
&+(27648+5184\beta+3456\beta^2)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l \\
&+(576\beta^3+24768\beta+76032+2016\beta^2)\lambda_i^3\lambda_j^3\lambda_l \\
&+(288\beta^2+2592\beta+10944)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3),
\end{aligned}
$$

and, hence,

$$
\xi_2^{(6)}(\lambda_l;\lambda_j;\lambda_i;\beta) \;=\; (5184000\beta+2073600)(\lambda_j+\lambda_i)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
$$

Define $\xi_{25}(\lambda_j;\lambda_i;\beta)=\xi_2^{(5)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{25} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{25} with respect to λ_j gives

$$
\begin{aligned}
\xi_{25}(\lambda_j;\lambda_i;\beta) \;=\;& (7983360\beta+13893120+311040\beta^2)\lambda_j^2 \\
&+(9953280\beta-11197440+933120\beta^2)\lambda_i\lambda_j \\
&+(11819520+311040\beta^2+2799360\beta)\lambda_i^2, \\
\xi_{25}^{(1)}(\lambda_j;\lambda_i;\beta) \;=\;& (15966720\beta+27786240+622080\beta^2)\lambda_j \\
&+(9953280\beta-11197440+933120\beta^2)\lambda_i, \\
\xi_{25}^{(2)}(\lambda_j;\lambda_i;\beta) \;=\;& 15966720\beta+27786240+622080\beta^2\geq 0 \text{ for all } \beta\geq -1.8776.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{25}(\lambda_i;\lambda_i;\beta) \;=\;& 518400\lambda_i^2(28+40\beta+3\beta^2)\geq 0 \text{ for all } \beta\geq -0.7412, \\
\xi_{25}^{(1)}(\lambda_i;\lambda_i;\beta) \;=\;& 518400\lambda_i(\beta+16)(3\beta+2)\geq 0 \text{ for all } \beta\geq -\frac{2}{3}.
\end{aligned}
$$

Hence, $\xi_2^{(5)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{24}(\lambda_j;\lambda_i;\beta)=\xi_2^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{24} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{24} with respect to λ_j gives

$$\begin{aligned}
\xi_{24}(\lambda_j;\lambda_i;\beta) &= (6269184\beta + 17017344 + 380160\beta^2 + 3456\beta^3)\lambda_j^3 \\
&\quad +(7382016\beta - 14598144 + 1451520\beta^2 + 31104\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(9455616 + 829440\beta^2 + 2820096\beta + 31104\beta^3)\lambda_i^2\lambda_j \\
&\quad +(877824\beta + 3456\beta^3 + 4161024 + 69120\beta^2)\lambda_i^3, \\
\xi_{24}^{(1)}(\lambda_j;\lambda_i;\beta) &= (18807552\beta + 51052032 + 1140480\beta^2 + 10368\beta^3)\lambda_j^2 \\
&\quad +(14764032\beta - 29196288 + 2903040\beta^2 + 62208\beta^3)\lambda_i\lambda_j \\
&\quad +(9455616 + 829440\beta^2 + 2820096\beta + 31104\beta^3)\lambda_i^2, \\
\xi_{24}^{(2)}(\lambda_j;\lambda_i;\beta) &= (37615104\beta + 102104064 + 2280960\beta^2 + 20736\beta^3)\lambda_j \\
&\quad +(14764032\beta - 29196288 + 2903040\beta^2 + 62208\beta^3)\lambda_i, \\
\xi_{24}^{(3)}(\lambda_j;\lambda_i;\beta) &= 37615104\beta + 102104064 + 2280960\beta^2 + 20736\beta^3 \geq 0 \text{ for all } \beta \geq -3.3897.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{24}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i^3(502\beta + 464 + 79\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1141, \\
\xi_{24}^{(1)}(\lambda_i;\lambda_i;\beta) &= 103680\lambda_i^2(351\beta + 302 + 47\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -0.9885, \\
\xi_{24}^{(2)}(\lambda_i;\lambda_i;\beta) &= 41472\lambda_i(1263\beta + 1758 + 125\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.6562.
\end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{23}(\lambda_j;\lambda_i;\beta) = \xi_2^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{23} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned}
\xi_{23}(\lambda_j;\lambda_i;\beta) &= (10696320 + 3246912\beta + 3456\beta^3 + 231552\beta^2)\lambda_j^4 \\
&\quad +(-9593856 + 3580416\beta + 1043712\beta^2 + 41472\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(4105728 + 767232\beta^2 + 829440\beta + 62208\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(1188864\beta + 13824\beta^3 + 3220992 + 127872\beta^2)\lambda_i^3\lambda_j \\
&\quad +(6912\beta^2 + 105408\beta + 279936)\lambda_i^4, \\
\xi_{23}^{(1)}(\lambda_j;\lambda_i;\beta) &= (42785280 + 12987648\beta + 13824\beta^3 + 926208\beta^2)\lambda_j^3 \\
&\quad +(-28781568 + 10741248\beta + 3131136\beta^2 + 124416\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(8211456 + 1534464\beta^2 + 1658880\beta + 124416\beta^3)\lambda_i^2\lambda_j \\
&\quad +(1188864\beta + 13824\beta^3 + 3220992 + 127872\beta^2)\lambda_i^3, \\
\xi_{23}^{(2)}(\lambda_j;\lambda_i;\beta) &= (128355840 + 38962944\beta + 41472\beta^3 + 2778624\beta^2)\lambda_j^2 \\
&\quad +(-57563136 + 21482496\beta + 6262272\beta^2 + 248832\beta^3)\lambda_i\lambda_j \\
&\quad +(8211456 + 1534464\beta^2 + 1658880\beta + 124416\beta^3)\lambda_i^2, \\
\xi_{23}^{(3)}(\lambda_j;\lambda_i;\beta) &= (256711680 + 77925888\beta + 82944\beta^3 + 5557248\beta^2)\lambda_j \\
&\quad +(-57563136 + 21482496\beta + 6262272\beta^2 + 248832\beta^3)\lambda_i, \\
\xi_{23}^{(4)}(\lambda_j;\lambda_i;\beta) &= 256711680 + 77925888\beta + 82944\beta^3 + 5557248\beta^2 \geq 0 \text{ for all } \beta \geq -4.8510.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{23}(\lambda_i;\lambda_i;\beta) &= 120960\lambda_i^4(\beta+4)(\beta^2+14\beta+18) \geq 0 \text{ for all } \beta \geq -1.4322,\\
\xi_{23}^{(1)}(\lambda_i;\lambda_i;\beta) &= 17280\lambda_i^3(1472+1538\beta+16\beta^3+331\beta^2) \geq 0 \text{ for all } \beta \geq -1.2958,\\
\xi_{23}^{(2)}(\lambda_i;\lambda_i;\beta) &= 103680\lambda_i^2(762+599\beta+4\beta^3+102\beta^2) \geq 0 \text{ for all } \beta \geq -1.7669,\\
\xi_{23}^{(3)}(\lambda_i;\lambda_i;\beta) &= 41472\lambda_i(4802+2397\beta+8\beta^3+285\beta^2) \geq 0 \text{ for all } \beta \geq -2.9562.
\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j;\lambda_i;\beta) = \xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}
\xi_{22}(\lambda_j;\lambda_i;\beta) &= (4416768+1226880\beta+93312\beta^2+1728\beta^3)\lambda_j^4\\
&\quad +(1368000\beta-4325760+483840\beta^2+25920\beta^3)\lambda_i\lambda_j^3\\
&\quad +(1416960+414720\beta^2-86400\beta+51840\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(1209600+103680\beta^2+17280\beta^3+777600\beta)\lambda_i^3\lambda_j\\
&\quad +(17280\beta^2+141120\beta+201600)\lambda_i^4,\\
\xi_{22}^{(1)}(\lambda_j;\lambda_i;\beta) &= (17667072+4907520\beta+373248\beta^2+6912\beta^3)\lambda_j^3\\
&\quad +(4104000\beta-12977280+1451520\beta^2+77760\beta^3)\lambda_i\lambda_j^2\\
&\quad +(2833920+829440\beta^2-172800\beta+103680\beta^3)\lambda_i^2\lambda_j\\
&\quad +(1209600+103680\beta^2+17280\beta^3+777600\beta)\lambda_i^3,\\
\xi_{22}^{(2)}(\lambda_j;\lambda_i;\beta) &= (53001216+14722560\beta+1119744\beta^2+20736\beta^3)\lambda_j^2\\
&\quad +(8208000\beta-25954560+2903040\beta^2+155520\beta^3)\lambda_i\lambda_j\\
&\quad +(2833920+829440\beta^2-172800\beta+103680\beta^3)\lambda_i^2,\\
\xi_{22}^{(3)}(\lambda_j;\lambda_i;\beta) &= (106002432+29445120\beta+2239488\beta^2+41472\beta^3)\lambda_j\\
&\quad +(8208000\beta-25954560+2903040\beta^2+155520\beta^3)\lambda_i,\\
\xi_{22}^{(4)}(\lambda_j;\lambda_i;\beta) &= 106002432+29445120\beta+2239488\beta^2+41472\beta^3 \geq 0 \text{ for all } \beta \geq -6.1457.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{22}(\lambda_i;\lambda_i;\beta) &= 8064\lambda_i^4(362+425\beta+138\beta^2+12\beta^3) \geq 0 \text{ for all } \beta \geq -1.4432,\\
\xi_{22}^{(1)}(\lambda_i;\lambda_i;\beta) &= 12096\lambda_i^3(722+795\beta+228\beta^2+17\beta^3) \geq 0 \text{ for all } \beta \geq -1.4368,\\
\xi_{22}^{(2)}(\lambda_i;\lambda_i;\beta) &= 10368\lambda_i^2(2882+2195\beta+468\beta^2+27\beta^3) \geq 0 \text{ for all } \beta \geq -2.2600,\\
\xi_{22}^{(3)}(\lambda_i;\lambda_i;\beta) &= 3456\lambda_i(23162+10895\beta+1488\beta^2+57\beta^3) \geq 0 \text{ for all } \beta \geq -3.8598.
\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j;\lambda_i;\beta) = \xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{21} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}
\xi_{21}(\lambda_j;\lambda_i;\beta) &= (358848\beta + 1343232 + 27936\beta^2 + 576\beta^3)\lambda_j^4 \\
&\quad +(436608\beta - 1478016 + 165024\beta^2 + 10368\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(-142560\beta + 159840\beta^2 + 475200 + 25920\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(368640 + 53280\beta^2 + 354240\beta + 11520\beta^3)\lambda_i^3\lambda_j \\
&\quad +(89280 + 17280\beta^2 + 93600\beta)\lambda_i^4, \\
\xi_{21}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1435392\beta + 5372928 + 111744\beta^2 + 2304\beta^3)\lambda_j^3 \\
&\quad +(1309824\beta - 4434048 + 495072\beta^2 + 31104\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(-285120\beta + 319680\beta^2 + 950400 + 51840\beta^3)\lambda_i^2\lambda_j \\
&\quad +(368640 + 53280\beta^2 + 354240\beta + 11520\beta^3)\lambda_i^3, \\
\xi_{21}^{(2)}(\lambda_j;\lambda_i;\beta) &= (4306176\beta + 16118784 + 335232\beta^2 + 6912\beta^3)\lambda_j^2 \\
&\quad +(2619648\beta - 8868096 + 990144\beta^2 + 62208\beta^3)\lambda_i\lambda_j \\
&\quad +(-285120\beta + 319680\beta^2 + 950400 + 51840\beta^3)\lambda_i^2, \\
\xi_{21}^{(3)}(\lambda_j;\lambda_i;\beta) &= (8612352\beta + 32237568 + 670464\beta^2 + 13824\beta^3)\lambda_j \\
&\quad +(2619648\beta - 8868096 + 990144\beta^2 + 62208\beta^3)\lambda_i, \\
\xi_{21}^{(4)}(\lambda_j;\lambda_i;\beta) &= 8612352\beta + 32237568 + 670464\beta^2 + 13824\beta^3 \geq 0 \text{ for all } \beta \geq -7.0502.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{21}(\lambda_i;\lambda_i;\beta) &= 12096\lambda_i^4(\beta+3)(4\beta^2 + 23\beta + 22) \geq 0 \text{ for all } \beta \geq -1.2120, \\
\xi_{21}^{(1)}(\lambda_i;\lambda_i;\beta) &= 4032\lambda_i^3(698\beta + 560 + 243\beta^2 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.3588, \\
\xi_{21}^{(2)}(\lambda_i;\lambda_i;\beta) &= 12096\lambda_i^2(549\beta + 678 + 136\beta^2 + 10\beta^3) \geq 0 \text{ for all } \beta \geq -2.4869, \\
\xi_{21}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i(6500\beta + 13524 + 961\beta^2 + 44\beta^3) \geq 0 \text{ for all } \beta \geq -4.0951.
\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j;\lambda_i;\beta) = \xi_2(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{20} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}
\xi_{20}(\lambda_j;\lambda_i;\beta) &= (84672\beta + 321408 + 144\beta^3 + 6624\beta^2)\lambda_j^4 \\
&\quad +(117504\beta + 44352\beta^2 - 402048 + 3024\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(155520 + 48384\beta^2 - 55296\beta + 9072\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(129600\beta + 5040\beta^3 + 138240 + 20160\beta^2)\lambda_i^3\lambda_j \\
&\quad +(10080\beta^2 + 46080 + 43200\beta)\lambda_i^4, \\
\xi_{20}^{(1)}(\lambda_j;\lambda_i;\beta) &= (338688\beta + 1285632 + 576\beta^3 + 26496\beta^2)\lambda_j^3 \\
&\quad +(352512\beta + 133056\beta^2 - 1206144 + 9072\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(311040 + 96768\beta^2 - 110592\beta + 18144\beta^3)\lambda_i^2\lambda_j \\
&\quad +(129600\beta + 5040\beta^3 + 138240 + 20160\beta^2)\lambda_i^3, \\
\xi_{20}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1016064\beta + 3856896 + 1728\beta^3 + 79488\beta^2)\lambda_j^2 \\
&\quad +(705024\beta + 266112\beta^2 - 2412288 + 18144\beta^3)\lambda_i\lambda_j \\
&\quad +(311040 + 96768\beta^2 - 110592\beta + 18144\beta^3)\lambda_i^2, \\
\xi_{20}^{(3)}(\lambda_j;\lambda_i;\beta) &= (2032128\beta + 7713792 + 3456\beta^3 + 158976\beta^2)\lambda_j \\
&\quad +(705024\beta + 266112\beta^2 - 2412288 + 18144\beta^3)\lambda_i, \\
\xi_{20}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2032128\beta + 7713792 + 3456\beta^3 + 158976\beta^2 \geq 0 \text{ for all } \beta \geq -7.3254.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{20}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -2, \\
\xi_{20}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i^3(\beta+3)(19\beta^2+103\beta+102) > 0 \text{ for all } \beta > -1.3039, \\
\xi_{20}^{(2)}(\lambda_i;\lambda_i;\beta) &= 3456\lambda_i^2(\beta+2)(11\beta^2+106\beta+254) > 0 \text{ for all } \beta > -2, \\
\xi_{20}^{(3)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i(3168\beta + 6136 + 25\beta^3 + 492\beta^2) > 0 \text{ for all } \beta > -3.5049.
\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &= (14400\beta+5760)(\lambda_j+\lambda_i)\lambda_l^5 \\
&+(4320\beta^2+38880\beta+164160)(\lambda_j^2+\lambda_i^2)\lambda_l^4 \\
&+(12960\beta^2-184320+66240\beta)\lambda_i\lambda_j\lambda_l^4 \\
&+(-124416+34560\beta^2-12096\beta+1296\beta^3)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^3 \\
&+(4320\beta^2+228096+144\beta^3+49536\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^3 \\
&+(1296\beta^3+13824\beta^2-209664+72000\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^2 \\
&+(38592+864\beta^2+22176\beta)(\lambda_j^4+\lambda_i^4)\lambda_l^2 \\
&+(3888\beta^3-93312\beta+31104+31104\beta^2)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&+(50112\beta+6912\beta^2+69120+1296\beta^3)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l \\
&+(2592\beta^2-31104+27648\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l \\
&+(144\beta^3+18432\beta+96768+864\beta^2)\lambda_i^3\lambda_j^3 \\
&+(864\beta^2+7776\beta+32832)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2),
\end{aligned}$$

and, hence,

$$\xi_3^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) = (691200+1728000\beta)(\lambda_j+\lambda_i)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.$$

Define $\xi_{34}(\lambda_j;\lambda_i;\beta)=\xi_3^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{34} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{34} with respect to λ_j gives

$$\begin{aligned}
\xi_{34}(\lambda_j;\lambda_i;\beta) &= (4631040+2661120\beta+103680\beta^2)\lambda_j^2 \\
&+(-3732480+3317760\beta+311040\beta^2)\lambda_i\lambda_j \\
&+(3939840+103680\beta^2+933120\beta)\lambda_i^2, \\
\xi_{34}^{(1)}(\lambda_j;\lambda_i;\beta) &= (9262080+5322240\beta+207360\beta^2)\lambda_j \\
&+(-3732480+3317760\beta+311040\beta^2)\lambda_i, \\
\xi_{34}^{(2)}(\lambda_j;\lambda_i;\beta) &= 9262080+5322240\beta+207360\beta^2\geq 0 \text{ for all } \beta\geq -1.8776.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{34}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i^2(28+40\beta+3\beta^2)\geq 0 \text{ for all } \beta\geq -0.7412, \\
\xi_{34}^{(1)}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i(\beta+16)(3\beta+2)\geq 0 \text{ for all } \beta\geq -\frac{2}{3}.
\end{aligned}$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{33}(\lambda_j;\lambda_i;\beta)=\xi_3^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{33} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{33} with respect to λ_j gives

$$\begin{aligned}
\xi_{33}(\lambda_j;\lambda_i;\beta) &= (5654016 + 2094336\beta + 129600\beta^2 + 864\beta^3)\lambda_j^3 \\
&\quad +(-4824576 + 2381184\beta + 7776\beta^3 + 518400\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(3193344 + 311040\beta^2 + 860544\beta + 7776\beta^3)\lambda_i^2\lambda_j \\
&\quad +(864\beta^3 + 1368576 + 25920\beta^2 + 297216\beta)\lambda_i^3, \\
\xi_{33}^{(1)}(\lambda_j;\lambda_i;\beta) &= (16962048 + 6283008\beta + 388800\beta^2 + 2592\beta^3)\lambda_j^2 \\
&\quad +(-9649152 + 4762368\beta + 15552\beta^3 + 1036800\beta^2)\lambda_i\lambda_j \\
&\quad +(3193344 + 311040\beta^2 + 860544\beta + 7776\beta^3)\lambda_i^2, \\
\xi_{33}^{(2)}(\lambda_j;\lambda_i;\beta) &= (33924096 + 12566016\beta + 777600\beta^2 + 5184\beta^3)\lambda_j \\
&\quad +(-9649152 + 4762368\beta + 15552\beta^3 + 1036800\beta^2)\lambda_i, \\
\xi_{33}^{(3)}(\lambda_j;\lambda_i;\beta) &= 33924096 + 12566016\beta + 777600\beta^2 + 5184\beta^3 \geq 0 \text{ for all } \beta \geq -3.40.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{33}(\lambda_i;\lambda_i;\beta) &= 17280\lambda_i^3(312 + 326\beta + 57\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -1.2060, \\
\xi_{33}^{(1)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^2(1216 + 1378\beta + 201\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.0368, \\
\xi_{33}^{(2)}(\lambda_i;\lambda_i;\beta) &= 3456\lambda_i(7024 + 5014\beta + 525\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.6963.
\end{aligned}$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{32}(\lambda_j;\lambda_i;\beta) = \xi_3^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{32} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}
\xi_{32}(\lambda_j;\lambda_i;\beta) &= (3530880 + 1096128\beta + 864\beta^3 + 79488\beta^2)\lambda_j^4 \\
&\quad +(-3262464 + 1154304\beta + 10368\beta^3 + 390528\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(207360\beta + 1285632 + 321408\beta^2 + 15552\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(3456\beta^3 + 949248 + 53568\beta^2 + 441216\beta)\lambda_i^3\lambda_j \\
&\quad +(44352\beta + 77184 + 1728\beta^2)\lambda_i^4, \\
\xi_{32}^{(1)}(\lambda_j;\lambda_i;\beta) &= (14123520 + 4384512\beta + 3456\beta^3 + 317952\beta^2)\lambda_j^3 \\
&\quad +(-9787392 + 3462912\beta + 31104\beta^3 + 1171584\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(414720\beta + 2571264 + 642816\beta^2 + 31104\beta^3)\lambda_i^2\lambda_j \\
&\quad +(3456\beta^3 + 949248 + 53568\beta^2 + 441216\beta)\lambda_i^3, \\
\xi_{32}^{(2)}(\lambda_j;\lambda_i;\beta) &= (42370560 + 13153536\beta + 10368\beta^3 + 953856\beta^2)\lambda_j^2 \\
&\quad +(-19574784 + 6925824\beta + 62208\beta^3 + 2343168\beta^2)\lambda_i\lambda_j \\
&\quad +(414720\beta + 2571264 + 642816\beta^2 + 31104\beta^3)\lambda_i^2, \\
\xi_{32}^{(3)}(\lambda_j;\lambda_i;\beta) &= (84741120 + 26307072\beta + 20736\beta^3 + 1907712\beta^2)\lambda_j \\
&\quad +(-19574784 + 6925824\beta + 62208\beta^3 + 2343168\beta^2)\lambda_i, \\
\xi_{32}^{(4)}(\lambda_j;\lambda_i;\beta) &= 84741120 + 26307072\beta + 20736\beta^3 + 1907712\beta^2 \geq 0 \text{ for all } \beta \geq -4.8136.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i;\lambda_i;\beta) &= 10080\lambda_i^4(256+292\beta+3\beta^3+84\beta^2)\geq 0 \text{ for all } \beta\geq -1.4510,\\
\xi_{32}^{(1)}(\lambda_i;\lambda_i;\beta) &= 2880\lambda_i^3(2728+3022\beta+24\beta^3+759\beta^2)\geq 0 \text{ for all } \beta\geq -1.3255,\\
\xi_{32}^{(2)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i^2(734+593\beta+3\beta^3+114\beta^2)\geq 0 \text{ for all } \beta\geq -1.8910,\\
\xi_{32}^{(3)}(\lambda_i;\lambda_i;\beta) &= 6912\lambda_i(9428+4808\beta+12\beta^3+615\beta^2)\geq 0 \text{ for all } \beta\geq -3.1578.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j;\lambda_i;\beta) = \xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{31} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j;\lambda_i;\beta) &= (420480\beta+1446912+31968\beta^2+432\beta^3)\lambda_j^4\\
&\quad +(472320\beta-1532160+185760\beta^2+6480\beta^3)\lambda_i\lambda_j^3\\
&\quad +(-17280\beta+414720+190080\beta^2+12960\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(334080+47520\beta^2+4320\beta^3+342720\beta)\lambda_i^3\lambda_j\\
&\quad +(72000\beta+46080+4320\beta^2)\lambda_i^4,\\
\xi_{31}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1681920\beta+5787648+127872\beta^2+1728\beta^3)\lambda_j^3\\
&\quad +(1416960\beta-4596480+557280\beta^2+19440\beta^3)\lambda_i\lambda_j^2\\
&\quad +(-34560\beta+829440+380160\beta^2+25920\beta^3)\lambda_i^2\lambda_j\\
&\quad +(334080+47520\beta^2+4320\beta^3+342720\beta)\lambda_i^3,\\
\xi_{31}^{(2)}(\lambda_j;\lambda_i;\beta) &= (5045760\beta+17362944+383616\beta^2+5184\beta^3)\lambda_j^2\\
&\quad +(2833920\beta-9192960+1114560\beta^2+38880\beta^3)\lambda_i\lambda_j\\
&\quad +(-34560\beta+829440+380160\beta^2+25920\beta^3)\lambda_i^2,\\
\xi_{31}^{(3)}(\lambda_j;\lambda_i;\beta) &= (10091520\beta+34725888+767232\beta^2+10368\beta^3)\lambda_j\\
&\quad +(2833920\beta-9192960+1114560\beta^2+38880\beta^3)\lambda_i,\\
\xi_{31}^{(4)}(\lambda_j;\lambda_i;\beta) &= 10091520\beta+34725888+767232\beta^2+10368\beta^3\geq 0 \text{ for all } \beta\geq -5.7919.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{31}(\lambda_i;\lambda_i;\beta) &= 8064\lambda_i^4(160\beta+88+57\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq -0.7350,\\
\xi_{31}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1008\lambda_i^3(3380\beta+2336+1104\beta^2+51\beta^3)\geq 0 \text{ for all } \beta\geq -1.0068,\\
\xi_{31}^{(2)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i^2(9080\beta+10416+2174\beta^2+81\beta^3)\geq 0 \text{ for all } \beta\geq -22.0124,\\
\xi_{31}^{(3)}(\lambda_i;\lambda_i;\beta) &= 864\lambda_i(14960\beta+29552+2178\beta^2+57\beta^3)\geq 0 \text{ for all } \beta\geq -30.0509.
\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j;\lambda_i;\beta) = \xi_3(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{30} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}
\xi_{30}(\lambda_j;\lambda_i;\beta) &= (124992\beta + 436608 + 9504\beta^2 + 144\beta^3)\lambda_j^4 \\
&\quad +(63936\beta^2 - 543744 + 168192\beta + 2592\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(77760\beta^2 - 8640\beta + 172800 + 6480\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(25920\beta^2 + 184320 + 2880\beta^3 + 190080\beta)\lambda_i^3\lambda_j \\
&\quad +(40320 + 4320\beta^2 + 57600\beta)\lambda_i^4, \\
\xi_{30}^{(1)}(\lambda_j;\lambda_i;\beta) &= (499968\beta + 1746432 + 38016\beta^2 + 576\beta^3)\lambda_j^3 \\
&\quad +(191808\beta^2 - 1631232 + 504576\beta + 7776\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(155520\beta^2 - 17280\beta + 345600 + 12960\beta^3)\lambda_i^2\lambda_j \\
&\quad +(25920\beta^2 + 184320 + 2880\beta^3 + 190080\beta)\lambda_i^3, \\
\xi_{30}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1499904\beta + 5239296 + 114048\beta^2 + 1728\beta^3)\lambda_j^2 \\
&\quad +(383616\beta^2 - 3262464 + 1009152\beta + 15552\beta^3)\lambda_i\lambda_j \\
&\quad +(155520\beta^2 - 17280\beta + 345600 + 12960\beta^3)\lambda_i^2, \\
\xi_{30}^{(3)}(\lambda_j;\lambda_i;\beta) &= (2999808\beta + 10478592 + 228096\beta^2 + 3456\beta^3)\lambda_j \\
&\quad +(383616\beta^2 - 3262464 + 1009152\beta + 15552\beta^3)\lambda_i, \\
\xi_{30}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2999808\beta + 10478592 + 228096\beta^2 + 3456\beta^3 \geq 0 \text{ for all } \beta \geq -5.9149.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{30}(\lambda_i;\lambda_i;\beta) &= 12096\lambda_i^4(\beta+3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -0.7085, \\
\xi_{30}^{(1)}(\lambda_i;\lambda_i;\beta) &= 8064\lambda_i^3(146\beta + 80 + 51\beta^2 + 3\beta^3) > 0 \text{ for all } \beta > -0.7226, \\
\xi_{30}^{(2)}(\lambda_i;\lambda_i;\beta) &= 6048\lambda_i^2(412\beta + 384 + 108\beta^2 + 5\beta^3) > 0 \text{ for all } \beta > -1.4380, \\
\xi_{30}^{(3)}(\lambda_i;\lambda_i;\beta) &= 1728\lambda_i(2320\beta + 4176 + 354\beta^2 + 11\beta^3) > 0 \text{ for all } \beta > -3.2433.
\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &= (131328 + 31104\beta + 3456\beta^2)(\lambda_j^2 + \lambda_i^2)\lambda_l^3 \\
&\quad +(52992\beta - 147456 + 10368\beta^2)\lambda_i\lambda_j\lambda_l^3 \\
&\quad +(14400\beta + 5760)(\lambda_j + \lambda_i)\lambda_l^4 \\
&\quad +(52992\beta - 147456 + 10368\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l \\
&\quad +(14400\beta + 5760)(\lambda_j^4 + \lambda_i^4)\lambda_l \\
&\quad +(-62208 + 31104\beta^2 - 31104\beta)\lambda_i^2\lambda_j^2\lambda_l \\
&\quad +(-62208 + 31104\beta^2 - 31104\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^2 \\
&\quad +(131328 + 31104\beta + 3456\beta^2)(\lambda_i^3 + \lambda_j^3)\lambda_l^2 \\
&\quad +(14400\beta + 5760)(\lambda_i^4\lambda_j + \lambda_i\lambda_j^4) \\
&\quad +(131328 + 31104\beta + 3456\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2),
\end{aligned}$$

and, hence,

$$\xi_4^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) \quad = \quad (345600\beta+138240)(\lambda_j+\lambda_i) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.$$

Define $\xi_{43}(\lambda_j;\lambda_i;\beta) = \xi_4^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{43} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{43} with respect to λ_j gives

$$\begin{aligned}
\xi_{43}(\lambda_j;\lambda_i;\beta) &= (532224\beta+926208+20736\beta^2)\lambda_j^2\\
&\quad +(663552\beta-746496+62208\beta^2)\lambda_i\lambda_j\\
&\quad +(186624\beta+20736\beta^2+787968)\lambda_i^2,\\
\xi_{43}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1064448\beta+1852416+41472\beta^2)\lambda_j\\
&\quad +(663552\beta-746496+62208\beta^2)\lambda_i,\\
\xi_{43}^{(2)}(\lambda_j;\lambda_i;\beta) &= 1064448\beta+1852416+41472\beta^2 \geq 0 \text{ for all } \beta \geq -1.8776.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{43}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i^2(28+40\beta+3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412,\\
\xi_{43}^{(1)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i(\beta+16)(3\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}.
\end{aligned}$$

Hence, $\xi_4^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{42}(\lambda_j;\lambda_i;\beta) = \xi_4^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{42} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned}
\xi_{42}(\lambda_j;\lambda_i;\beta) &= (421632\beta+1119744+27648\beta^2)\lambda_j^3\\
&\quad +(428544\beta-940032+124416\beta^2)\lambda_i\lambda_j^2\\
&\quad +(124416\beta+82944\beta^2+663552)\lambda_i^2\lambda_j\\
&\quad +(6912\beta^2+262656+62208\beta)\lambda_i^3,\\
\xi_{42}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1264896\beta+3359232+82944\beta^2)\lambda_j^2\\
&\quad +(857088\beta-1880064+248832\beta^2)\lambda_i\lambda_j\\
&\quad +(124416\beta+82944\beta^2+663552)\lambda_i^2,\\
\xi_{42}^{(2)}(\lambda_j;\lambda_i;\beta) &= (2529792\beta+6718464+165888\beta^2)\lambda_j\\
&\quad +(857088\beta-1880064+248832\beta^2)\lambda_i,\\
\xi_{42}^{(3)}(\lambda_j;\lambda_i;\beta) &= 2529792\beta+6718464+165888\beta^2 \geq 0 \text{ for all } \beta \geq -3.4249.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{42}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i^3(7\beta+16)(\beta+2) \geq 0 \text{ for all } \beta \geq -2,\\
\xi_{42}^{(1)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i^2(65\beta+62+12\beta^2) \geq 0 \text{ for all } \beta \geq -1.2358,\\
\xi_{42}^{(2)}(\lambda_i;\lambda_i;\beta) &= 69120\lambda_i(49\beta+70+6\beta^2) \geq 0 \text{ for all } \beta \geq -1.8457.
\end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{41}(\lambda_j;\lambda_i;\beta) = \xi_4^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}
\xi_{41}(\lambda_j;\lambda_i;\beta) &= (685440 + 227520\beta + 17280\beta^2)\lambda_j^4 \\
&\quad +(103680\beta^2 + 207360\beta - 691200)\lambda_i\lambda_j^3 \\
&\quad +(207360 + 103680\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(17280\beta^2 + 115200 + 115200\beta)\lambda_i^3\lambda_j \\
&\quad +(14400\beta + 5760)\lambda_i^4, \\
\xi_{41}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2741760 + 910080\beta + 69120\beta^2)\lambda_j^3 \\
&\quad +(311040\beta^2 + 622080\beta - 2073600)\lambda_i\lambda_j^2 \\
&\quad +(414720 + 207360\beta^2)\lambda_i^2\lambda_j + (17280\beta^2 + 115200 + 115200\beta)\lambda_i^3, \\
\xi_{41}^{(2)}(\lambda_j;\lambda_i;\beta) &= (8225280 + 2730240\beta + 207360\beta^2)\lambda_j^2 \\
&\quad +(622080\beta^2 + 1244160\beta - 4147200)\lambda_i\lambda_j \\
&\quad +(414720 + 207360\beta^2)\lambda_i^2, \\
\xi_{41}^{(3)}(\lambda_j;\lambda_i;\beta) &= (16450560 + 5460480\beta + 414720\beta^2)\lambda_j \\
&\quad +(622080\beta^2 + 1244160\beta - 4147200)\lambda_i, \\
\xi_{41}^{(4)}(\lambda_j;\lambda_i;\beta) &= 69120(2\beta + 17)(3\beta + 14) \geq 0 \text{ for all } \beta \geq -\frac{14}{3}.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{41}(\lambda_i;\lambda_i;\beta) &= 80640\lambda_i^4(3\beta + 4)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{41}^{(1)}(\lambda_i;\lambda_i;\beta) &= 5760\lambda_i^3(105\beta^2 + 208 + 286\beta) \geq 0 \text{ for all } \beta, \\
\xi_{41}^{(2)}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i^2(26 + 23\beta + 6\beta^2) \geq 0 \text{ for all } \beta, \\
\xi_{41}^{(3)}(\lambda_i;\lambda_i;\beta) &= 69120\lambda_i(178 + 97\beta + 15\beta^2) \geq 0 \text{ for all } \beta.
\end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j;\lambda_i;\beta) = \xi_4(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{40} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}
\xi_{40}(\lambda_j;\lambda_i;\beta) &= (274176 + 91008\beta + 6912\beta^2)\lambda_j^4 \\
&\quad +(103680\beta - 345600 + 51840\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(138240 + 69120\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(115200 + 115200\beta + 17280\beta^2)\lambda_i^3\lambda_j \\
&\quad +(28800\beta + 11520)\lambda_i^4, \\
\xi_{40}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1096704 + 364032\beta + 27648\beta^2)\lambda_j^3 \\
&\quad +(311040\beta - 1036800 + 155520\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(276480 + 138240\beta^2)\lambda_i^2\lambda_j \\
&\quad +(115200 + 115200\beta + 17280\beta^2)\lambda_i^3, \\
\xi_{40}^{(2)}(\lambda_j;\lambda_i;\beta) &= (3290112 + 1092096\beta + 82944\beta^2)\lambda_j^2 \\
&\quad +(622080\beta - 2073600 + 311040\beta^2)\lambda_i\lambda_j \\
&\quad +(276480 + 138240\beta^2)\lambda_i^2, \\
\xi_{40}^{(3)}(\lambda_j;\lambda_i;\beta) &= (6580224 + 2184192\beta + 165888\beta^2)\lambda_j \\
&\quad +(622080\beta - 2073600 + 311040\beta^2)\lambda_i, \\
\xi_{40}^{(4)}(\lambda_j;\lambda_i;\beta) &= 27648(2\beta + 17)(3\beta + 14) \geq 0 \text{ for all } \beta \geq -\frac{14}{3}.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{40}(\lambda_i;\lambda_i;\beta) &= 48384\lambda_i^4(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\
\xi_{40}^{(1)}(\lambda_i;\lambda_i;\beta) &= 112896\lambda_i^3(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\
\xi_{40}^{(2)}(\lambda_i;\lambda_i;\beta) &= 6912\lambda_i^2(216 + 248\beta + 77\beta^2) > 0 \text{ for all } \beta, \\
\xi_{40}^{(3)}(\lambda_i;\lambda_i;\beta) &= 6912\lambda_i(652 + 406\beta + 69\beta^2) > 0 \text{ for all } \beta.
\end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.15 Let

$$\begin{aligned}\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (1200\beta+480)t(0,0,4,4)+(7200\beta+720\beta^2+5760)t(0,1,3,4)\\ &\quad +(30240+6480\beta+2160\beta^2)t(0,2,2,4)\\ &\quad +(-28800+6480\beta^2+1440\beta)t(1,1,2,4)\\ &\quad +(2160\beta^2+288\beta^3+81792+22752\beta)t(0,2,3,3)\\ &\quad +(30336\beta+864\beta^3+5760\beta^2-90624)t(1,1,3,3)\\ &\quad +(2592\beta^3+8640\beta^2-41472-6912\beta)t(1,2,2,3)\\ &\quad +(7776\beta^3+186624-124416\beta)t(2,2,2,2),\end{aligned}$$

where

$$t(a,b,c,d) = \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\text{perm}(a,b,c,d)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_k^{n_4}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in\text{perm}(a,b,c)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) \;=\;& 4(1200\beta+480)\lambda_k^3\hat{s}(0,0,4)\\
&+4(7200\beta+720\beta^2+5760)\lambda_k^3\hat{s}(0,1,3)\\
&+4(-28800+6480\beta^2+1440\beta)\lambda_k^3\hat{s}(1,1,2)\\
&+4(30240+6480\beta+2160\beta^2)\lambda_k^3\hat{s}(0,2,2)\\
&+3(2160\beta^2+288\beta^3+81792+22752\beta)\lambda_k^2\hat{s}(0,2,3)\\
&+3(30336\beta+864\beta^3+5760\beta^2-90624)\lambda_k^2\hat{s}(1,1,3)\\
&+3(7200\beta+720\beta^2+5760)\lambda_k^2\hat{s}(0,1,4)\\
&+3(2592\beta^3+8640\beta^2-41472-6912\beta)\lambda_k^2\hat{s}(1,2,2)\\
&+2(2160\beta^2+288\beta^3+81792+22752\beta)\lambda_k\hat{s}(0,3,3)\\
&+2(7776\beta^3+186624-124416\beta)\lambda_k\hat{s}(2,2,2)\\
&+2(-28800+6480\beta^2+1440\beta)\lambda_k\hat{s}(1,1,4)\\
&+2(2592\beta^3+8640\beta^2-41472-6912\beta)\lambda_k\hat{s}(1,2,3)\\
&+2(30240+6480\beta+2160\beta^2)\lambda_k\hat{s}(0,2,4)\\
&+(7200\beta+720\beta^2+5760)\hat{s}(0,3,4)\\
&+(2592\beta^3+8640\beta^2-41472-6912\beta)\hat{s}(2,2,3)\\
&+(30336\beta+864\beta^3+5760\beta^2-90624)\hat{s}(1,3,3)\\
&+(-28800+6480\beta^2+1440\beta)\hat{s}(1,2,4),
\end{aligned}$$

$$\begin{aligned}
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) \;=\;& 12(1200\beta+480)\lambda_k^2\hat{s}(0,0,4)\\
&+12(7200\beta+720\beta^2+5760)\lambda_k^2\hat{s}(0,1,3)\\
&+12(-28800+6480\beta^2+1440\beta)\lambda_k^2\hat{s}(1,1,2)\\
&+12(30240+6480\beta+2160\beta^2)\lambda_k^2\hat{s}(0,2,2)\\
&+6(2160\beta^2+288\beta^3+81792+22752\beta)\lambda_k\hat{s}(0,2,3)\\
&+6(30336\beta+864\beta^3+5760\beta^2-90624)\lambda_k\hat{s}(1,1,3)\\
&+6(7200\beta+720\beta^2+5760)\lambda_k\hat{s}(0,1,4)\\
&+6(2592\beta^3+8640\beta^2-41472-6912\beta)\lambda_k\hat{s}(1,2,2)\\
&+2(2160\beta^2+288\beta^3+81792+22752\beta)\hat{s}(0,3,3)\\
&+2(7776\beta^3+186624-124416\beta)\hat{s}(2,2,2)\\
&+2(-28800+6480\beta^2+1440\beta)\hat{s}(1,1,4)\\
&+2(2592\beta^3+8640\beta^2-41472-6912\beta)\hat{s}(1,2,3)\\
&+2(30240+6480\beta+2160\beta^2)\hat{s}(0,2,4),
\end{aligned}$$

$$\begin{aligned}\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(1200\beta+480)\lambda_k\hat{s}(0,0,4)\\ &\quad +24(7200\beta+720\beta^2+5760)\lambda_k\hat{s}(0,1,3)\\ &\quad +24(-28800+6480\beta^2+1440\beta)\lambda_k\hat{s}(1,1,2)\\ &\quad +24(30240+6480\beta+2160\beta^2)\lambda_k\hat{s}(0,2,2)\\ &\quad +6(2160\beta^2+288\beta^3+81792+22752\beta)\hat{s}(0,2,3)\\ &\quad +6(30336\beta+864\beta^3+5760\beta^2-90624)\hat{s}(1,1,3)\\ &\quad +6(7200\beta+720\beta^2+5760)\hat{s}(0,1,4)\\ &\quad +6(2592\beta^3+8640\beta^2-41472-6912\beta)\hat{s}(1,2,2),\end{aligned}$$

$$\begin{aligned}\xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(1200\beta+480)\hat{s}(0,0,4)\\ &\quad +24(7200\beta+720\beta^2+5760)\hat{s}(0,1,3)\\ &\quad +24(-28800+6480\beta^2+1440\beta)\hat{s}(1,1,2)\\ &\quad +24(30240+6480\beta+2160\beta^2)\hat{s}(0,2,2).\end{aligned}$$

Define

$$\begin{aligned}\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta).\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (1200\beta+480)\lambda_l^8\\
&+(14400\beta+11520+1440\beta^2)(\lambda_j+\lambda_i)\lambda_l^7\\
&+(142272+6480\beta^2+288\beta^3+35712\beta)(\lambda_j^2+\lambda_i^2)\lambda_l^6\\
&+(-148224+33216\beta+18720\beta^2+864\beta^3)\lambda_i\lambda_j\lambda_l^6\\
&+(-140544+5184\beta^3-10944\beta+30240\beta^2)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^5\\
&+(175104+576\beta^3+59904\beta+5760\beta^2)(\lambda_j^3+\lambda_i^3)\lambda_l^5\\
&+(-211200+21600\beta^2+68160\beta+4320\beta^3)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^4\\
&+(23280\beta+42720+3600\beta^2)(\lambda_j^4+\lambda_i^4)\lambda_l^4\\
&+(-125280\beta+12960\beta^3+21600\beta^2+164160)\lambda_i^2\lambda_j^2\lambda_l^4\\
&+(5760\beta^3+31680\beta+80640+21600\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^3\\
&+(-46080+14400\beta^2+17280\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^3\\
&+(31680+14400\beta+10800\beta^2)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l^2\\
&+(10080\beta^2+72960+75840\beta+1440\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2\\
&+(14400\beta+11520+1440\beta^2)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3)\lambda_l\\
&+(1200\beta+480)\lambda_i^4\lambda_j^4,
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_0^{(7)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (48384000\beta+19353600)\lambda_l\\
&+(72576000\beta+58060800+7257600\beta^2)(\lambda_i+\lambda_j)\geq 0 \text{ for all } \beta\geq-\frac{2}{5}.
\end{aligned}$$

Define $\xi_{06}(\lambda_j;\lambda_i;\beta)=\xi_0^{(6)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{06} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{06} with respect to λ_j gives

$$\begin{aligned}
\xi_{06}(\lambda_j;\lambda_i;\beta) &= (122480640\beta+170173440+11923200\beta^2+207360\beta^3)\lambda_j^2\\
&+(96491520\beta-48660480+20736000\beta^2+622080\beta^3)\lambda_i\lambda_j\\
&+(102435840+4665600\beta^2+207360\beta^3+25712640\beta)\lambda_i^2,\\
\xi_{06}^{(1)}(\lambda_j;\lambda_i;\beta) &= (244961280\beta+340346880+23846400\beta^2+414720\beta^3)\lambda_j\\
&+(96491520\beta-48660480+20736000\beta^2+622080\beta^3)\lambda_i,\\
\xi_{06}^{(2)}(\lambda_j;\lambda_i;\beta) &= 244961280\beta+340346880+23846400\beta^2+414720\beta^3\\
&\geq 0 \text{ for all } \beta\geq-1.6454.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{06}(\lambda_i;\lambda_i;\beta) &= 1036800\lambda_i^2(236\beta+216+36\beta^2\beta^3)\geq 0 \text{ for all } \beta\geq-1.0915,\\
\xi_{06}^{(1)}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i(988\beta+844+129\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq-0.9757,\\
\xi_{06}^{(2)}(\lambda_i;\lambda_i;\beta) &= \geq 0 \text{ for all } \beta\geq-1.6454.
\end{aligned}$$

Hence, $\xi_0^{(6)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{05}(\lambda_j;\lambda_i;\beta) = \xi_0^{(5)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{05} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{05} with respect to λ_j gives

$$\begin{aligned}
\xi_{05}(\lambda_j;\lambda_i;\beta) &= (77253120\beta + 155704320 + 8985600\beta^2 + 276480\beta^3)\lambda_j^3 \\
&\quad +(-94556160 + 58890240\beta + 20736000\beta^2 + 1244160\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(85570560 + 8294400\beta^2 + 829440\beta^3 + 24399360\beta)\lambda_i^2\lambda_j \\
&\quad +(21012480 + 69120\beta^3 + 7188480\beta + 691200\beta^2)\lambda_i^3, \\
\xi_{05}^{(1)}(\lambda_j;\lambda_i;\beta) &= (231759360\beta + 467112960 + 26956800\beta^2 + 829440\beta^3)\lambda_j^2 \\
&\quad +(-189112320 + 117780480\beta + 41472000\beta^2 + 2488320\beta^3)\lambda_i\lambda_j \\
&\quad +(85570560 + 8294400\beta^2 + 829440\beta^3 + 24399360\beta)\lambda_i^2, \\
\xi_{05}^{(2)}(\lambda_j;\lambda_i;\beta) &= (463518720\beta + 934225920 + 53913600\beta^2 + 1658880\beta^3)\lambda_j \\
&\quad +(-189112320 + 117780480\beta + 41472000\beta^2 + 2488320\beta^3)\lambda_i, \\
\xi_{05}^{(3)}(\lambda_j;\lambda_i;\beta) &= 463518720\beta + 934225920 + 53913600\beta^2 + 1658880\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -2.9155.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{05}(\lambda_i;\lambda_i;\beta) &= 806400\lambda_i^3(208\beta + 208 + 48\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4294, \\
\xi_{05}^{(1)}(\lambda_i;\lambda_i;\beta) &= 691200\lambda_i^2(541\beta + 526 + 111\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.2898, \\
\xi_{05}^{(2)}(\lambda_i;\lambda_i;\beta) &= 691200\lambda_i(841\beta + 1078 + 138\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7421.
\end{aligned}$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{04}(\lambda_j;\lambda_i;\beta) = \xi_0^{(4)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{04} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$
\begin{aligned}
\xi_{04}(\lambda_j;\lambda_i;\beta) &= (34715520\beta + 83738880 + 4320000\beta^2 + 172800\beta^3)\lambda_j^4 \\
&\quad +(-65617920 + 24376320\beta + 12096000\beta^2 + 1036800\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(38292480 + 6480000\beta^2 + 1036800\beta^3 + 8536320\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(15943680 + 1209600\beta^2 + 8824320\beta + 172800\beta^3)\lambda_i^3\lambda_j \\
&\quad +(558720\beta + 1025280 + 86400\beta^2)\lambda_i^4, \\
\xi_{04}^{(1)}(\lambda_j;\lambda_i;\beta) &= (138862080\beta + 334955520 + 17280000\beta^2 + 691200\beta^3)\lambda_j^3 \\
&\quad +(-196853760 + 73128960\beta + 36288000\beta^2 + 3110400\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(76584960 + 12960000\beta^2 + 2073600\beta^3 + 17072640\beta)\lambda_i^2\lambda_j \\
&\quad +(15943680 + 1209600\beta^2 + 8824320\beta + 172800\beta^3)\lambda_i^3, \\
\xi_{04}^{(2)}(\lambda_j;\lambda_i;\beta) &= (416586240\beta + 1004866560 + 51840000\beta^2 + 2073600\beta^3)\lambda_j^2 \\
&\quad +(-393707520 + 146257920\beta + 72576000\beta^2 + 6220800\beta^3)\lambda_i\lambda_j \\
&\quad +(76584960 + 12960000\beta^2 + 2073600\beta^3 + 17072640\beta)\lambda_i^2, \\
\xi_{04}^{(3)}(\lambda_j;\lambda_i;\beta) &= (833172480\beta + 2009733120 + 103680000\beta^2 + 4147200\beta^3)\lambda_j \\
&\quad +(-393707520 + 146257920\beta + 72576000\beta^2 + 6220800\beta^3)\lambda_i, \\
\xi_{04}^{(4)}(\lambda_j;\lambda_i;\beta) &= 833172480\beta + 2009733120 + 103680000\beta^2 + 4147200\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.3861.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{04}(\lambda_i;\lambda_i;\beta) &= 403200\lambda_i^4(191\beta + 182 + 60\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\
\xi_{04}^{(1)}(\lambda_i;\lambda_i;\beta) &= 403200\lambda_i^3(590\beta + 572 + 168\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.5802, \\
\xi_{04}^{(2)}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i^2(3356\beta + 3980 + 795\beta^2 + 60\beta^3) \geq 0 \text{ for all } \beta \geq -1.9649, \\
\xi_{04}^{(3)}(\lambda_i;\lambda_i;\beta) &= 691200\lambda_i(1417\beta + 2338 + 255\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -2.9237.
\end{aligned}
$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j;\lambda_i;\beta) = \xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{03} is non-negative

for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}
\xi_{03}(\lambda_j;\lambda_i;\beta) &= (11865600\beta + 31184640 + 1512000\beta^2 + 69120\beta^3)\lambda_j^4 \\
&\quad +(-29145600 + 8092800\beta + 4968000\beta^2 + 518400\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(13063680 + 3240000\beta^2 + 691200\beta^3 + 812160\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(5921280 + 172800\beta^3 + 5420160\beta + 993600\beta^2)\lambda_i^3\lambda_j \\
&\quad +(662400\beta + 748800 + 172800\beta^2)\lambda_i^4, \\
\xi_{03}^{(1)}(\lambda_j;\lambda_i;\beta) &= (47462400\beta + 124738560 + 6048000\beta^2 + 276480\beta^3)\lambda_j^3 \\
&\quad +(-87436800 + 24278400\beta + 14904000\beta^2 + 1555200\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(26127360 + 6480000\beta^2 + 1382400\beta^3 + 1624320\beta)\lambda_i^2\lambda_j \\
&\quad +(5921280 + 172800\beta^3 + 5420160\beta + 993600\beta^2)\lambda_i^3, \\
\xi_{03}^{(2)}(\lambda_j;\lambda_i;\beta) &= (142387200\beta + 374215680 + 18144000\beta^2 + 829440\beta^3)\lambda_j^2 \\
&\quad +(-174873600 + 48556800\beta + 29808000\beta^2 + 3110400\beta^3)\lambda_i\lambda_j \\
&\quad +(26127360 + 6480000\beta^2 + 1382400\beta^3 + 1624320\beta)\lambda_i^2, \\
\xi_{03}^{(3)}(\lambda_j;\lambda_i;\beta) &= (284774400\beta + 748431360 + 36288000\beta^2 + 1658880\beta^3)\lambda_j \\
&\quad +(-174873600 + 48556800\beta + 29808000\beta^2 + 3110400\beta^3)\lambda_i, \\
\xi_{03}^{(4)}(\lambda_j;\lambda_i;\beta) &= 284774400\beta + 748431360 + 36288000\beta^2 + 1658880\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -5.6105.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{03}(\lambda_i;\lambda_i;\beta) &= 725760\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{03}^{(1)}(\lambda_i;\lambda_i;\beta) &= 40320\lambda_i^3(1954\beta + 1720 + 705\beta^2 + 84\beta^3) \geq 0 \text{ for all } \beta \geq -1.7877, \\
\xi_{03}^{(2)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i^2(796\beta + 932 + 225\beta^2 + 22\beta^3) \geq 0 \text{ for all } \beta \geq -2.5561, \\
\xi_{03}^{(3)}(\lambda_i;\lambda_i;\beta) &= 51840\lambda_i(6430\beta + 11064 + 1275\beta^2 + 92\beta^3) \geq 0 \text{ for all } \beta \geq -3.7901.
\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j;\lambda_i;\beta) = \xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{02} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned}
\xi_{02}(\lambda_j;\lambda_i;\beta) &= (3220800\beta + 8793600 + 413280\beta^2 + 20160\beta^3)\lambda_j^4 \\
&\quad +(-9584640 + 2304000\beta + 1572480\beta^2 + 181440\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(3974400 + 1209600\beta^2 + 302400\beta^3 - 432000\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(1597440 + 100800\beta^3 + 2357760\beta + 524160\beta^2)\lambda_i^3\lambda_j \\
&\quad +(411840\beta + 299520 + 151200\beta^2)\lambda_i^4, \\
\xi_{02}^{(1)}(\lambda_j;\lambda_i;\beta) &= (12883200\beta + 35174400 + 1653120\beta^2 + 80640\beta^3)\lambda_j^3 \\
&\quad +(-28753920 + 6912000\beta + 4717440\beta^2 + 544320\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(7948800 + 2419200\beta^2 + 604800\beta^3 - 864000\beta)\lambda_i^2\lambda_j \\
&\quad +(1597440 + 100800\beta^3 + 2357760\beta + 524160\beta^2)\lambda_i^3, \\
\xi_{02}^{(2)}(\lambda_j;\lambda_i;\beta) &= (38649600\beta + 105523200 + 4959360\beta^2 + 241920\beta^3)\lambda_j^2 \\
&\quad +(-57507840 + 13824000\beta + 9434880\beta^2 + 1088640\beta^3)\lambda_i\lambda_j \\
&\quad +(7948800 + 2419200\beta^2 + 604800\beta^3 - 864000\beta)\lambda_i^2, \\
\xi_{02}^{(3)}(\lambda_j;\lambda_i;\beta) &= (77299200\beta + 211046400 + 9918720\beta^2 + 483840\beta^3)\lambda_j \\
&\quad +(-57507840 + 13824000\beta + 9434880\beta^2 + 1088640\beta^3)\lambda_i, \\
\xi_{02}^{(4)}(\lambda_j;\lambda_i;\beta) &= 77299200\beta + 211046400 + 9918720\beta^2 + 483840\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -5.9825.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{02}(\lambda_i;\lambda_i;\beta) &= 120960\lambda_i^4(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1330560\lambda_i^3(\beta+3)(\beta+2)^2 \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(2)}(\lambda_i;\lambda_i;\beta) &= 40320\lambda_i^2(1280\beta + 1388 + 417\beta^2 + 48\beta^3) \geq 0 \text{ for all } \beta \geq -3.0665, \\
\xi_{02}^{(3)}(\lambda_i;\lambda_i;\beta) &= 40320\lambda_i(2260\beta + 3808 + 480\beta^2 + 39\beta^3) \geq 0 \text{ for all } \beta \geq -3.8373.
\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{01} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j;\lambda_i;\beta) &= (717312\beta + 1984512 + 92160\beta^2 + 4608\beta^3)\lambda_j^4 \\
&\quad +(-2494464 + 569856\beta + 403200\beta^2 + 48384\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(1112832 + 362880\beta^2 + 96768\beta^3 - 217728\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(430080 + 40320\beta^3 + 833280\beta + 201600\beta^2)\lambda_i^3\lambda_j \\
&\quad +(188160\beta + 107520 + 80640\beta^2)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2869248\beta + 7938048 + 368640\beta^2 + 18432\beta^3)\lambda_j^3 \\
&\quad +(-7483392 + 1709568\beta + 1209600\beta^2 + 145152\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(2225664 + 725760\beta^2 + 193536\beta^3 - 435456\beta)\lambda_i^2\lambda_j \\
&\quad +(430080 + 40320\beta^3 + 833280\beta + 201600\beta^2)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= (8607744\beta + 23814144 + 1105920\beta^2 + 55296\beta^3)\lambda_j^2 \\
&\quad +(-14966784 + 3419136\beta + 2419200\beta^2 + 290304\beta^3)\lambda_i\lambda_j \\
&\quad +(2225664 + 725760\beta^2 + 193536\beta^3 - 435456\beta)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j;\lambda_i;\beta) &= (17215488\beta + 47628288 + 2211840\beta^2 + 110592\beta^3)\lambda_j \\
&\quad +(-14966784 + 3419136\beta + 2419200\beta^2 + 290304\beta^3)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j;\lambda_i;\beta) &= 17215488\beta + 47628288 + 2211840\beta^2 + 110592\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -6.0282.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 190080\lambda_i^4(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 17280\lambda_i^3(23\beta+30)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{30}{23}, \\
\xi_{01}^{(2)}(\lambda_i;\lambda_i;\beta) &= 20736\lambda_i^2(\beta+3)(26\beta^2+127\beta+178) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i;\lambda_i;\beta) &= 4608\lambda_i(4478\beta + 7088 + 1005\beta^2 + 87\beta^3) \\
&\geq 0 \text{ for all } \beta \geq -3.4940.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^4$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= (134496\beta + 372096 + 17280\beta^2 + 864\beta^3)\lambda_j^4 \\
&\quad +(-534528 + 122112\beta + 86400\beta^2 + 10368\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(278208 + 90720\beta^2 + 24192\beta^3 - 54432\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(129024 + 12096\beta^3 + 249984\beta + 60480\beta^2)\lambda_i^3\lambda_j \\
&\quad +(70560\beta + 40320 + 30240\beta^2)\lambda_i^4, \\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= (537984\beta + 1488384 + 69120\beta^2 + 3456\beta^3)\lambda_j^3 \\
&\quad +(-1603584 + 366336\beta + 259200\beta^2 + 31104\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(556416 + 181440\beta^2 + 48384\beta^3 - 108864\beta)\lambda_i^2\lambda_j \\
&\quad +(129024 + 12096\beta^3 + 249984\beta + 60480\beta^2)\lambda_i^3, \\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1613952\beta + 4465152 + 207360\beta^2 + 10368\beta^3)\lambda_j^2 \\
&\quad +(-3207168 + 732672\beta + 518400\beta^2 + 62208\beta^3)\lambda_i\lambda_j \\
&\quad +(556416 + 181440\beta^2 + 48384\beta^3 - 108864\beta)\lambda_i^2, \\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= (3227904\beta + 8930304 + 414720\beta^2 + 20736\beta^3)\lambda_j \\
&\quad +(-3207168 + 732672\beta + 518400\beta^2 + 62208\beta^3)\lambda_i, \\
\xi_{00}^{(4)}(\lambda_j;\lambda_i;\beta) &= 3227904\beta + 8930304 + 414720\beta^2 + 20736\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -6.0282.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 47520\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 95040\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 60480\lambda_i^2(2\beta+5)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -2, \\
\xi_{00}^{(3)}(\lambda_i;\lambda_i;\beta) &= 20736\lambda_i(\beta+3)(4\beta^2+33\beta+92) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (4800\beta+1920)\lambda_l^7\\
&+(50400\beta+40320+5040\beta^2)(\lambda_i+\lambda_j)\lambda_l^6\\
&+(19440\beta^2+107136\beta+864\beta^3+426816)(\lambda_j^2+\lambda_i^2)\lambda_l^5\\
&+(56160\beta^2-444672+99648\beta+2592\beta^3)\lambda_i\lambda_j\lambda_l^5\\
&+(-27360\beta-351360+12960\beta^3+75600\beta^2)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^4\\
&+(437760+14400\beta^2+1440\beta^3+149760\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^4\\
&+(8640\beta^3+43200\beta^2-422400+136320\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^3\\
&+(7200\beta^2+46560\beta+85440)(\lambda_j^4+\lambda_i^4)\lambda_l^3\\
&+(-250560\beta+25920\beta^3+328320+43200\beta^2)\lambda_i^2\lambda_j^2\lambda_l^3\\
&+(120960+32400\beta^2+47520\beta+8640\beta^3)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^2\\
&+(25920\beta+21600\beta^2-69120)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^2\\
&+(31680+10800\beta^2+14400\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l\\
&+(72960+75840\beta+1440\beta^3+10080\beta^2)\lambda_i^3\lambda_j^3\lambda_l\\
&+(7200\beta+720\beta^2+5760)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_1^{(6)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (24192000\beta+9676800)\lambda_l+(36288000\beta+29030400+3628800\beta^2)(\lambda_i+\lambda_j)\\
&\geq 0 \text{ for all } \beta\geq-\frac{2}{5}.
\end{aligned}$$

Define $\xi_{15}(\lambda_j;\lambda_i;\beta)=\xi_1^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{15} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{15} with respect to λ_j gives

$$\begin{aligned}
\xi_{15}(\lambda_j;\lambda_i;\beta) &= (61240320\beta+85086720+5961600\beta^2+103680\beta^3)\lambda_j^2\\
&+(48245760\beta-24330240+10368000\beta^2+311040\beta^3)\lambda_i\lambda_j\\
&+(2332800\beta^2+12856320\beta+103680\beta^3+51217920)\lambda_i^2,\\
\xi_{15}^{(1)}(\lambda_j;\lambda_i;\beta) &= (122480640\beta+170173440+11923200\beta^2+207360\beta^3)\lambda_j\\
&+(48245760\beta-24330240+10368000\beta^2+311040\beta^3)\lambda_i,\\
\xi_{15}^{(2)}(\lambda_j;\lambda_i;\beta) &= 122480640\beta+170173440+11923200\beta^2+207360\beta^3\geq 0 \text{ for all } \beta\geq-1.6454.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{15}(\lambda_i;\lambda_i;\beta) &= 518400\lambda_i^2(236\beta+216+36\beta^2\beta^3)\geq 0 \text{ for all } \beta\geq-1.0915,\\
\xi_{15}^{(1)}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i(988\beta+844+129\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq-0.9757.
\end{aligned}$$

Hence, $\xi_1^{(5)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{14}(\lambda_j;\lambda_i;\beta) = \xi_1^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{14} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned}
\xi_{14}(\lambda_j;\lambda_i;\beta) &= (38626560\beta + 77852160 + 4492800\beta^2 + 138240\beta^3)\lambda_j^3 \\
&\quad +(10368000\beta^2 - 47278080 + 29445120\beta + 622080\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(4147200\beta^2 + 12199680\beta + 414720\beta^3 + 42785280)\lambda_i^2\lambda_j \\
&\quad +(10506240 + 345600\beta^2 + 34560\beta^3 + 3594240\beta)\lambda_i^3, \\
\xi_{14}^{(1)}(\lambda_j;\lambda_i;\beta) &= (115879680\beta + 233556480 + 13478400\beta^2 + 414720\beta^3)\lambda_j^2 \\
&\quad +(20736000\beta^2 - 94556160 + 58890240\beta + 1244160\beta^3)\lambda_i\lambda_j \\
&\quad +(4147200\beta^2 + 12199680\beta + 414720\beta^3 + 42785280)\lambda_i^2, \\
\xi_{14}^{(2)}(\lambda_j;\lambda_i;\beta) &= (231759360\beta + 467112960 + 26956800\beta^2 + 829440\beta^3)\lambda_j \\
&\quad +(20736000\beta^2 - 94556160 + 58890240\beta + 1244160\beta^3)\lambda_i, \\
\xi_{14}^{(3)}(\lambda_j;\lambda_i;\beta) &= 231759360\beta + 467112960 + 26956800\beta^2 + 829440\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -2.9155.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{14}(\lambda_i;\lambda_i;\beta) &= 403200\lambda_i^3(208\beta + 208 + 48\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4294, \\
\xi_{14}^{(1)}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i^2(541\beta + 526 + 111\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.2898, \\
\xi_{14}^{(2)}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i(841\beta + 1078 + 138\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7421.
\end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j;\lambda_i;\beta) = \xi_1^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{13} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}
\xi_{13}(\lambda_j;\lambda_i;\beta) &= (17357760\beta + 41869440 + 2160000\beta^2 + 86400\beta^3)\lambda_j^4 \\
&\quad +(6048000\beta^2 - 32808960 + 12188160\beta + 518400\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(3240000\beta^2 + 4268160\beta + 518400\beta^3 + 19146240)\lambda_i^2\lambda_j^2 \\
&\quad +(86400\beta^3 + 604800\beta^2 + 7971840 + 4412160\beta)\lambda_i^3\lambda_j \\
&\quad +(43200\beta^2 + 279360\beta + 512640)\lambda_i^4, \\
\xi_{13}^{(1)}(\lambda_j;\lambda_i;\beta) &= (69431040\beta + 167477760 + 8640000\beta^2 + 345600\beta^3)\lambda_j^3 \\
&\quad +(18144000\beta^2 - 98426880 + 36564480\beta + 1555200\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(6480000\beta^2 + 8536320\beta + 1036800\beta^3 + 38292480)\lambda_i^2\lambda_j \\
&\quad +(86400\beta^3 + 604800\beta^2 + 7971840 + 4412160\beta)\lambda_i^3, \\
\xi_{13}^{(2)}(\lambda_j;\lambda_i;\beta) &= (208293120\beta + 502433280 + 25920000\beta^2 + 1036800\beta^3)\lambda_j^2 \\
&\quad +(36288000\beta^2 - 196853760 + 73128960\beta + 3110400\beta^3)\lambda_i\lambda_j \\
&\quad +(6480000\beta^2 + 8536320\beta + 1036800\beta^3 + 38292480)\lambda_i^2, \\
\xi_{13}^{(3)}(\lambda_j;\lambda_i;\beta) &= (416586240\beta + 1004866560 + 51840000\beta^2 + 2073600\beta^3)\lambda_j \\
&\quad +(36288000\beta^2 - 196853760 + 73128960\beta + 3110400\beta^3)\lambda_i, \\
\xi_{13}^{(4)}(\lambda_j;\lambda_i;\beta) &= 416586240\beta + 1004866560 + 51840000\beta^2 + 2073600\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.3861.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{13}(\lambda_i;\lambda_i;\beta) &= 201600\lambda_i^4(191\beta + 182 + 60\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\
\xi_{13}^{(1)}(\lambda_i;\lambda_i;\beta) &= 201600\lambda_i^3(590\beta + 572 + 168\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.5802, \\
\xi_{13}^{(2)}(\lambda_i;\lambda_i;\beta) &= 86400\lambda_i^2(3356\beta + 3980 + 795\beta^2 + 60\beta^3) \geq 0 \text{ for all } \beta \geq -1.9649, \\
\xi_{13}^{(3)}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i(1417\beta + 2338 + 255\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -2.9237.
\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j;\lambda_i;\beta) = \xi_1^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{12} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$
\begin{aligned}
\xi_{12}(\lambda_j;\lambda_i;\beta) &= (5932800\beta + 15592320 + 756000\beta^2 + 34560\beta^3)\lambda_j^4 \\
&\quad +(2484000\beta^2 - 14572800 + 4046400\beta + 259200\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(1620000\beta^2 + 406080\beta + 345600\beta^3 + 6531840)\lambda_i^2\lambda_j^2 \\
&\quad +(2960640 + 496800\beta^2 + 86400\beta^3 + 2710080\beta)\lambda_i^3\lambda_j \\
&\quad +(86400\beta^2 + 331200\beta + 374400)\lambda_i^4, \\
\xi_{12}^{(1)}(\lambda_j;\lambda_i;\beta) &= (23731200\beta + 62369280 + 3024000\beta^2 + 138240\beta^3)\lambda_j^3 \\
&\quad +(7452000\beta^2 - 43718400 + 12139200\beta + 777600\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(3240000\beta^2 + 812160\beta + 691200\beta^3 + 13063680)\lambda_i^2\lambda_j \\
&\quad +(2960640 + 496800\beta^2 + 86400\beta^3 + 2710080\beta)\lambda_i^3, \\
\xi_{12}^{(2)}(\lambda_j;\lambda_i;\beta) &= (71193600\beta + 187107840 + 9072000\beta^2 + 414720\beta^3)\lambda_j^2 \\
&\quad +(14904000\beta^2 - 87436800 + 24278400\beta + 1555200\beta^3)\lambda_i\lambda_j \\
&\quad +(3240000\beta^2 + 812160\beta + 691200\beta^3 + 13063680)\lambda_i^2, \\
\xi_{12}^{(3)}(\lambda_j;\lambda_i;\beta) &= (142387200\beta + 374215680 + 18144000\beta^2 + 829440\beta^3)\lambda_j \\
&\quad +(14904000\beta^2 - 87436800 + 24278400\beta + 1555200\beta^3)\lambda_i, \\
\xi_{12}^{(4)}(\lambda_j;\lambda_i;\beta) &= 142387200\beta + 374215680 + 18144000\beta^2 + 829440\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -5.6105.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{12}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{12}^{(1)}(\lambda_i;\lambda_i;\beta) &= 20160\lambda_i^3(1954\beta + 1720 + 705\beta^2 + 84\beta^3) \geq 0 \text{ for all } \beta \geq -1.7877, \\
\xi_{12}^{(2)}(\lambda_i;\lambda_i;\beta) &= 120960\lambda_i^2(796\beta + 932 + 225\beta^2 + 22\beta^3) \geq 0 \text{ for all } \beta \geq -2.5561, \\
\xi_{12}^{(3)}(\lambda_i;\lambda_i;\beta) &= 25920\lambda_i(6430\beta + 11064 + 1275\beta^2 + 92\beta^3) \geq 0 \text{ for all } \beta \geq -3.7901.
\end{aligned}
$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j;\lambda_i;\beta) = \xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{11} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j;\lambda_i;\beta) &= (1610400\beta + 4396800 + 206640\beta^2 + 10080\beta^3)\lambda_j^4 \\
&\quad +(786240\beta^2 - 4792320 + 1152000\beta + 90720\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(604800\beta^2 - 216000\beta + 151200\beta^3 + 1987200)\lambda_i^2\lambda_j^2 \\
&\quad +(798720 + 262080\beta^2 + 50400\beta^3 + 1178880\beta)\lambda_i^3\lambda_j \\
&\quad +(75600\beta^2 + 205920\beta + 149760)\lambda_i^4, \\
\xi_{11}^{(1)}(\lambda_j;\lambda_i;\beta) &= (6441600\beta + 17587200 + 826560\beta^2 + 40320\beta^3)\lambda_j^3 \\
&\quad +(2358720\beta^2 - 14376960 + 3456000\beta + 272160\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(1209600\beta^2 - 432000\beta + 302400\beta^3 + 3974400)\lambda_i^2\lambda_j \\
&\quad +(798720 + 262080\beta^2 + 50400\beta^3 + 1178880\beta)\lambda_i^3, \\
\xi_{11}^{(2)}(\lambda_j;\lambda_i;\beta) &= (19324800\beta + 52761600 + 2479680\beta^2 + 120960\beta^3)\lambda_j^2 \\
&\quad +(4717440\beta^2 - 28753920 + 6912000\beta + 544320\beta^3)\lambda_i\lambda_j \\
&\quad +(1209600\beta^2 - 432000\beta + 302400\beta^3 + 3974400)\lambda_i^2, \\
\xi_{11}^{(3)}(\lambda_j;\lambda_i;\beta) &= (38649600\beta + 105523200 + 4959360\beta^2 + 241920\beta^3)\lambda_j \\
&\quad +(4717440\beta^2 - 28753920 + 6912000\beta + 544320\beta^3)\lambda_i, \\
\xi_{11}^{(4)}(\lambda_j;\lambda_i;\beta) &= 38649600\beta + 105523200 + 4959360\beta^2 + 241920\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -5.9825.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i;\lambda_i;\beta) &= 60480\lambda_i^4(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{7}{4}, \\
\xi_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= 665280\lambda_i^3(\beta+3)(\beta+2)^2 \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(2)}(\lambda_i;\lambda_i;\beta) &= 20160\lambda_i^2(1280\beta + 1388 + 417\beta^2 + 48\beta^3) \geq 0 \text{ for all } \beta \geq -3.0665, \\
\xi_{11}^{(3)}(\lambda_i;\lambda_i;\beta) &= 20160\lambda_i(2260\beta + 3808 + 480\beta^2 + 39\beta^3) \geq 0 \text{ for all } \beta \geq -3.8373.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j;\lambda_i;\beta) = \xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{10} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= (358656\beta + 992256 + 46080\beta^2 + 2304\beta^3)\lambda_j^4 \\
&\quad +(201600\beta^2 - 1247232 + 284928\beta + 24192\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(181440\beta^2 - 108864\beta + 48384\beta^3 + 556416)\lambda_i^2\lambda_j^2 \\
&\quad +(215040 + 100800\beta^2 + 20160\beta^3 + 416640\beta)\lambda_i^3\lambda_j \\
&\quad +(40320\beta^2 + 94080\beta + 53760)\lambda_i^4, \\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1434624\beta + 3969024 + 184320\beta^2 + 9216\beta^3)\lambda_j^3 \\
&\quad +(604800\beta^2 - 3741696 + 854784\beta + 72576\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(362880\beta^2 - 217728\beta + 96768\beta^3 + 1112832)\lambda_i^2\lambda_j \\
&\quad +(215040 + 100800\beta^2 + 20160\beta^3 + 416640\beta)\lambda_i^3, \\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= (4303872\beta + 11907072 + 552960\beta^2 + 27648\beta^3)\lambda_j^2 \\
&\quad +(1209600\beta^2 - 7483392 + 1709568\beta + 145152\beta^3)\lambda_i\lambda_j \\
&\quad +(362880\beta^2 - 217728\beta + 96768\beta^3 + 1112832)\lambda_i^2, \\
\xi_{10}^{(3)}(\lambda_j;\lambda_i;\beta) &= (8607744\beta + 23814144 + 1105920\beta^2 + 55296\beta^3)\lambda_j \\
&\quad +(1209600\beta^2 - 7483392 + 1709568\beta + 145152\beta^3)\lambda_i, \\
\xi_{10}^{(4)}(\lambda_j;\lambda_i;\beta) &= 8607744\beta + 23814144 + 1105920\beta^2 + 55296\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -6.0282.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 95040\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^3(23\beta+30)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -\frac{30}{23}, \\
\xi_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= 10368\lambda_i^2(\beta+3)(26\beta^2+127\beta+178) > 0 \text{ for all } \beta > -3, \\
\xi_{10}^{(3)}(\lambda_i;\lambda_i;\beta) &= 2304\lambda_i(4478\beta + 7088 + 1005\beta^2 + 87\beta^3) > 0 \text{ for all } \beta > -3.4940.
\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= (14400\beta+5760)\lambda_l^6\\
&\quad+(129600\beta+12960\beta^2+103680)(\lambda_j+\lambda_i)\lambda_l^5\\
&\quad+(43200\beta^2+914112+227232\beta+1728\beta^3)(\lambda_j^2+\lambda_i^2)\lambda_l^4\\
&\quad+(125280\beta^2-946944+202176\beta+5184\beta^3)\lambda_i\lambda_j\lambda_l^4\\
&\quad+(-677376+146880\beta^2-38016\beta+20736\beta^3)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^3\\
&\quad+(268416\beta+723456+2304\beta^3+25920\beta^2)(\lambda_j^3+\lambda_i^3)\lambda_l^3\\
&\quad+(10368\beta^3+254592\beta+60480\beta^2-557568)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^2\\
&\quad+(8640\beta^2+70560\beta+100800)(\lambda_j^4+\lambda_i^4)\lambda_l^2\\
&\quad+(487296-212544\beta+31104\beta^3+77760\beta^2)\lambda_i^2\lambda_j^2\lambda_l^2\\
&\quad+(122688\beta+6912\beta^3+407808+30240\beta^2)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l\\
&\quad+(17280\beta^2-23040+46080\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l\\
&\quad+(4320\beta^2+576\beta^3+163584+45504\beta)\lambda_i^3\lambda_j^3\\
&\quad+(60480+12960\beta+4320\beta^2)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_2^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (10368000\beta+4147200)\lambda_l\\
&\quad+(15552000\beta+1555200\beta^2+12441600)(\lambda_i+\lambda_j)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}$$

Define $\xi_{24}(\lambda_j;\lambda_i;\beta)=\xi_2^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{24} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{24} with respect to λ_j gives

$$\begin{aligned}
\xi_{24}(\lambda_j;\lambda_i;\beta) &= (26189568\beta+36453888+2592000\beta^2+41472\beta^3)\lambda_j^2\\
&\quad+(20404224\beta+4561920\beta^2-10285056+124416\beta^3)\lambda_i\lambda_j\\
&\quad+(1036800\beta^2+21938688+5453568\beta+41472\beta^3)\lambda_i^2,\\
\xi_{24}^{(1)}(\lambda_j;\lambda_i;\beta) &= (52379136\beta+72907776+5184000\beta^2+82944\beta^3)\lambda_j\\
&\quad+(20404224\beta+4561920\beta^2-10285056+124416\beta^3)\lambda_i,\\
\xi_{24}^{(2)}(\lambda_j;\lambda_i;\beta) &= 52379136\beta+72907776+5184000\beta^2+82944\beta^3\geq 0 \text{ for all } \beta\geq -1.6562.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{24}(\lambda_i;\lambda_i;\beta) &= 103680\lambda_i^2(502\beta+464+79\beta^2+2\beta^3)\geq 0 \text{ for all } \beta\geq -1.1141,\\
\xi_{24}^{(1)}(\lambda_i;\lambda_i;\beta) &= 207360\lambda_i(351\beta+302+47\beta^2\beta^3)\geq 0 \text{ for all } \beta\geq -0.9885.
\end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{23}(\lambda_j;\lambda_i;\beta)=\xi_2^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{23} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned}
\xi_{23}(\lambda_j;\lambda_i;\beta) &= (16568064\beta + 33191424 + 1969920\beta^2 + 55296\beta^3)\lambda_j^3 \\
&\quad +(4665600\beta^2 - 20570112 + 12400128\beta + 248832\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(1918080\beta^2 + 17874432 + 5225472\beta + 165888\beta^3)\lambda_i^2\lambda_j \\
&\quad +(1610496\beta + 4340736 + 13824\beta^3 + 155520\beta^2)\lambda_i^3, \\
\xi_{23}^{(1)}(\lambda_j;\lambda_i;\beta) &= (49704192\beta + 99574272 + 5909760\beta^2 + 165888\beta^3)\lambda_j^2 \\
&\quad +(9331200\beta^2 - 41140224 + 24800256\beta + 497664\beta^3)\lambda_i\lambda_j \\
&\quad +(1918080\beta^2 + 17874432 + 5225472\beta + 165888\beta^3)\lambda_i^2, \\
\xi_{23}^{(2)}(\lambda_j;\lambda_i;\beta) &= (99408384\beta + 199148544 + 11819520\beta^2 + 331776\beta^3)\lambda_j \\
&\quad +(9331200\beta^2 - 41140224 + 24800256\beta + 497664\beta^3)\lambda_i, \\
\xi_{23}^{(3)}(\lambda_j;\lambda_i;\beta) &= 99408384\beta + 199148544 + 11819520\beta^2 + 331776\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -2.9562.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{23}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i^3(\beta+4)(\beta^2+14\beta+18) \geq 0 \text{ for all } \beta \geq -1.4322, \\
\xi_{23}^{(1)}(\lambda_i;\lambda_i;\beta) &= 51840\lambda_i^2(1538\beta + 1472 + 331\beta^2 + 16\beta^3) \geq 0 \text{ for all } \beta \geq -1.2958, \\
\xi_{23}^{(2)}(\lambda_i;\lambda_i;\beta) &= 207360\lambda_i(599\beta + 762 + 102\beta^2 + 4\beta^3) \geq 0 \text{ for all } \beta \geq -1.7669.
\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j;\lambda_i;\beta) = \xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}
\xi_{22}(\lambda_j;\lambda_i;\beta) &= (7502400\beta + 17758080 + 950400\beta^2 + 34560\beta^3)\lambda_j^4 \\
&\quad +(2764800\beta^2 - 14469120 + 5299200\beta + 207360\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(1555200\beta^2 + 7879680 + 2073600\beta + 207360\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(34560\beta^3 + 2119680\beta + 276480\beta^2 + 3225600)\lambda_i^3\lambda_j \\
&\quad +(17280\beta^2 + 141120\beta + 201600)\lambda_i^4, \\
\xi_{22}^{(1)}(\lambda_j;\lambda_i;\beta) &= (30009600\beta + 71032320 + 3801600\beta^2 + 138240\beta^3)\lambda_j^3 \\
&\quad +(8294400\beta^2 - 43407360 + 15897600\beta + 622080\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(3110400\beta^2 + 15759360 + 4147200\beta + 414720\beta^3)\lambda_i^2\lambda_j \\
&\quad +(34560\beta^3 + 2119680\beta + 276480\beta^2 + 3225600)\lambda_i^3, \\
\xi_{22}^{(2)}(\lambda_j;\lambda_i;\beta) &= (90028800\beta + 213096960 + 11404800\beta^2 + 414720\beta^3)\lambda_j^2 \\
&\quad +(16588800\beta^2 - 86814720 + 31795200\beta + 1244160\beta^3)\lambda_i\lambda_j \\
&\quad +(3110400\beta^2 + 15759360 + 4147200\beta + 414720\beta^3)\lambda_i^2, \\
\xi_{22}^{(3)}(\lambda_j;\lambda_i;\beta) &= (180057600\beta + 426193920 + 22809600\beta^2 + 829440\beta^3)\lambda_j \\
&\quad +(16588800\beta^2 - 86814720 + 31795200\beta + 1244160\beta^3)\lambda_i, \\
\xi_{22}^{(4)}(\lambda_j;\lambda_i;\beta) &= 180057600\beta + 426193920 + 22809600\beta^2 + 829440\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.5933.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{22}(\lambda_i;\lambda_i;\beta) &= 40320\lambda_i^4(425\beta+362+138\beta^2+12\beta^3)\geq 0 \text{ for all } \beta\geq -1.4432,\\
\xi_{22}^{(1)}(\lambda_i;\lambda_i;\beta) &= 80640\lambda_i^3(647\beta+578+192\beta^2+15\beta^3)\geq 0 \text{ for all } \beta\geq -1.4382,\\
\xi_{22}^{(2)}(\lambda_i;\lambda_i;\beta) &= 518400\lambda_i^2(243\beta+274+60\beta^2+4\beta^3)\geq 0 \text{ for all } \beta\geq -1.9255,\\
\xi_{22}^{(3)}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i(613\beta+982+114\beta^2+6\beta^3)\geq 0 \text{ for all } \beta\geq -3.0812.
\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j;\lambda_i;\beta)=\xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{21} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}
\xi_{21}(\lambda_j;\lambda_i;\beta) &= (2589696\beta+6581376+332640\beta^2+13824\beta^3)\lambda_j^4\\
&\quad +(1144800\beta^2-6439680+1897920\beta+103680\beta^3)\lambda_i\lambda_j^3\\
&\quad +(799200\beta^2+3006720+492480\beta+138240\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(1437120\beta+1463040+34560\beta^3+228960\beta^2)\lambda_i^3\lambda_j\\
&\quad +(34560\beta^2+187200\beta+178560)\lambda_i^4,\\
\xi_{21}^{(1)}(\lambda_j;\lambda_i;\beta) &= (10358784\beta+26325504+1330560\beta^2+55296\beta^3)\lambda_j^3\\
&\quad +(3434400\beta^2-19319040+5693760\beta+311040\beta^3)\lambda_i\lambda_j^2\\
&\quad +(1598400\beta^2+6013440+984960\beta+276480\beta^3)\lambda_i^2\lambda_j\\
&\quad +(1437120\beta+1463040+34560\beta^3+228960\beta^2)\lambda_i^3,\\
\xi_{21}^{(2)}(\lambda_j;\lambda_i;\beta) &= (31076352\beta+78976512+3991680\beta^2+165888\beta^3)\lambda_j^2\\
&\quad +(6868800\beta^2-38638080+11387520\beta+622080\beta^3)\lambda_i\lambda_j\\
&\quad +(1598400\beta^2+6013440+984960\beta+276480\beta^3)\lambda_i^2,\\
\xi_{21}^{(3)}(\lambda_j;\lambda_i;\beta) &= (62152704\beta+157953024+7983360\beta^2+331776\beta^3)\lambda_j\\
&\quad +(6868800\beta^2-38638080+11387520\beta+622080\beta^3)\lambda_i,\\
\xi_{21}^{(4)}(\lambda_j;\lambda_i;\beta) &= 62152704\beta+157953024+7983360\beta^2+331776\beta^3\\
&\geq 0 \text{ for all } \beta\geq -10.7897.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{21}(\lambda_i;\lambda_i;\beta) &= 72576\lambda_i^4(\beta+3)(4\beta^2+23\beta+22)\geq 0 \text{ for all } \beta\geq -1.2120,\\
\xi_{21}^{(1)}(\lambda_i;\lambda_i;\beta) &= 4032\lambda_i^3(4582\beta+3592+1635\beta^2+168\beta^3)\geq 0 \text{ for all } \beta\geq -1.3254,\\
\xi_{21}^{(2)}(\lambda_i;\lambda_i;\beta) &= 24192\lambda_i^2(1796\beta+1916+515\beta^2+44\beta^3)\geq 0 \text{ for all } \beta\geq -2.1392,\\
\xi_{21}^{(3)}(\lambda_i;\lambda_i;\beta) &= 5184\lambda_i(14186\beta+23016+2865\beta^2+184\beta^3)\geq 0 \text{ for all } \beta\geq -7.5649.
\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j;\lambda_i;\beta)=\xi_2(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{20} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}
\xi_{20}(\lambda_j;\lambda_i;\beta) &= (710208\beta + 1847808 + 90720\beta^2 + 4032\beta^3)\lambda_j^4 \\
&\quad +(362880\beta^2 - 2101248 + 594432\beta + 36288\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(302400\beta^2 + 1192320 + 112320\beta + 60480\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(691200\beta + 737280 + 20160\beta^3 + 120960\beta^2)\lambda_i^3\lambda_j \\
&\quad +(30240\beta^2 + 129600\beta + 138240)\lambda_i^4, \\
\xi_{20}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2840832\beta + 7391232 + 362880\beta^2 + 16128\beta^3)\lambda_j^3 \\
&\quad +(1088640\beta^2 - 6303744 + 1783296\beta + 108864\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(604800\beta^2 + 2384640 + 224640\beta + 120960\beta^3)\lambda_i^2\lambda_j \\
&\quad +(691200\beta + 737280 + 20160\beta^3 + 120960\beta^2)\lambda_i^3, \\
\xi_{20}^{(2)}(\lambda_j;\lambda_i;\beta) &= (8522496\beta + 22173696 + 1088640\beta^2 + 48384\beta^3)\lambda_j^2 \\
&\quad +(2177280\beta^2 - 12607488 + 3566592\beta + 217728\beta^3)\lambda_i\lambda_j \\
&\quad +(604800\beta^2 + 2384640 + 224640\beta + 120960\beta^3)\lambda_i^2, \\
\xi_{20}^{(3)}(\lambda_j;\lambda_i;\beta) &= (17044992\beta + 44347392 + 2177280\beta^2 + 96768\beta^3)\lambda_j \\
&\quad +(2177280\beta^2 - 12607488 + 3566592\beta + 217728\beta^3)\lambda_i, \\
\xi_{20}^{(4)}(\lambda_j;\lambda_i;\beta) &= 17044992\beta + 44347392 + 2177280\beta^2 + 96768\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -5.7061.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{20}(\lambda_i;\lambda_i;\beta) &= 60480\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -2, \\
\xi_{20}^{(1)}(\lambda_i;\lambda_i;\beta) &= 24192\lambda_i^3(\beta+3)(11\beta^2 + 57\beta + 58) > 0 \text{ for all } \beta > -1.3909, \\
\xi_{20}^{(2)}(\lambda_i;\lambda_i;\beta) &= 24192\lambda_i^2(509\beta + 494 + 160\beta^2 + 16\beta^3) > 0 \text{ for all } \beta > -1.8336, \\
\xi_{20}^{(3)}(\lambda_i;\lambda_i;\beta) &= 24192\lambda_i(852\beta + 1312 + 180\beta^2 + 13\beta^3) > 0 \text{ for all } \beta > -3.2288.
\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$
\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &= (28800\beta+11520)\lambda_l^5\\
&+(172800+21600\beta^2+216000\beta)(\lambda_j+\lambda_i)\lambda_l^4\\
&+(64800\beta^2+1728\beta^3+292032\beta+1216512)(\lambda_j^2+\lambda_i^2)\lambda_l^3\\
&+(-1234944+190080\beta^2+216576\beta+5184\beta^3)\lambda_i\lambda_j\lambda_l^3\\
&+(-6912\beta-940032+207360\beta^2+15552\beta^3)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^2\\
&+(628992+309312\beta+30240\beta^2+1728\beta^3)(\lambda_j^3+\lambda_i^3)\lambda_l^2\\
&+(4320\beta^2+72000\beta+46080)(\lambda_j^4+\lambda_i^4)\lambda_l\\
&+(354816\beta-405504+51840\beta^2+5184\beta^3)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l\\
&+(476928+103680\beta^2+15552\beta^3+114048\beta)\lambda_i^2\lambda_j^2\lambda_l\\
&+(12960\beta^2+1728\beta^3+490752+136512\beta)(\lambda_i^3\lambda_j^2+\lambda_i^2\lambda_j^3)\\
&+(43200\beta+4320\beta^2+34560)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j),
\end{aligned}
$$

and, hence,

$$
\begin{aligned}
\xi_3^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (3456000\beta+1382400)\lambda_l\\
&+(4147200+518400\beta^2+5184000\beta)(\lambda_i+\lambda_j)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}
$$

Define $\xi_{33}(\lambda_j;\lambda_i;\beta)=\xi_3^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{33} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{33} with respect to λ_j gives

$$
\begin{aligned}
\xi_{33}(\lambda_j;\lambda_i;\beta) &= (8664192\beta+12137472+907200\beta^2+10368\beta^3)\lambda_j^2\\
&+(-3262464+1658880\beta^2+6483456\beta+31104\beta^3)\lambda_i\lambda_j\\
&+(388800\beta^2+10368\beta^3+1752192\beta+7299072)\lambda_i^2,\\
\xi_{33}^{(1)}(\lambda_j;\lambda_i;\beta) &= (17328384\beta+24274944+1814400\beta^2+20736\beta^3)\lambda_j\\
&+(-3262464+1658880\beta^2+6483456\beta+31104\beta^3)\lambda_i,\\
\xi_{33}^{(2)}(\lambda_j;\lambda_i;\beta) &= 17328384\beta+24274944+1814400\beta^2+20736\beta^3\geq 0 \text{ for all } \beta\geq -1.6963.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{33}(\lambda_i;\lambda_i;\beta) &= 51840\lambda_i^2(326\beta+312+57\beta^2\beta^3)\geq 0 \text{ for all } \beta\geq -1.2060,\\
\xi_{33}^{(1)}(\lambda_i;\lambda_i;\beta) &= 17280\lambda_i(1378\beta+1216+201\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq -1.0368.
\end{aligned}
$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{32}(\lambda_j;\lambda_i;\beta)=\xi_3^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{32} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}
\xi_{32}(\lambda_j;\lambda_i;\beta) &= (5538816\beta + 10861056 + 708480\beta^2 + 13824\beta^3)\lambda_j^3 \\
&\quad +(-7216128 + 1814400\beta^2 + 3877632\beta + 62208\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(803520\beta^2 + 41472\beta^3 + 1738368\beta + 5419008)\lambda_i^2\lambda_j \\
&\quad +(1257984 + 618624\beta + 60480\beta^2 + 3456\beta^3)\lambda_i^3, \\
\xi_{32}^{(1)}(\lambda_j;\lambda_i;\beta) &= (16616448\beta + 32583168 + 2125440\beta^2 + 41472\beta^3)\lambda_j^2 \\
&\quad +(-14432256 + 3628800\beta^2 + 7755264\beta + 124416\beta^3)\lambda_i\lambda_j \\
&\quad +(803520\beta^2 + 41472\beta^3 + 1738368\beta + 5419008)\lambda_i^2, \\
\xi_{32}^{(2)}(\lambda_j;\lambda_i;\beta) &= (33232896\beta + 65166336 + 4250880\beta^2 + 82944\beta^3)\lambda_j \\
&\quad +(-14432256 + 3628800\beta^2 + 7755264\beta + 124416\beta^3)\lambda_i, \\
\xi_{32}^{(3)}(\lambda_j;\lambda_i;\beta) &= 33232896\beta + 65166336 + 4250880\beta^2 + 82944\beta^3 \geq 0 \text{ for all } \beta \geq -3.1578.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i;\lambda_i;\beta) &= 40320\lambda_i^3(292\beta + 256 + 84\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4510, \\
\xi_{32}^{(1)}(\lambda_i;\lambda_i;\beta) &= 8640\lambda_i^2(3022\beta + 2728 + 759\beta^2 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.3255, \\
\xi_{32}^{(2)}(\lambda_i;\lambda_i;\beta) &= 69120\lambda_i(593\beta + 734 + 114\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.8910.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j;\lambda_i;\beta) = \xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{31} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j;\lambda_i;\beta) &= (2574720\beta + 5702400 + 345600\beta^2 + 8640\beta^3)\lambda_j^4 \\
&\quad +(-5299200 + 1123200\beta^2 + 1854720\beta + 51840\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(712800\beta^2 + 51840\beta^3 + 976320\beta + 2246400)\lambda_i^2\lambda_j^2 \\
&\quad +(852480 + 973440\beta + 112320\beta^2 + 8640\beta^3)\lambda_i^3\lambda_j \\
&\quad +(4320\beta^2 + 72000\beta + 46080)\lambda_i^4, \\
\xi_{31}^{(1)}(\lambda_j;\lambda_i;\beta) &= (10298880\beta + 22809600 + 1382400\beta^2 + 34560\beta^3)\lambda_j^3 \\
&\quad +(-15897600 + 3369600\beta^2 + 5564160\beta + 155520\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(1425600\beta^2 + 103680\beta^3 + 1952640\beta + 4492800)\lambda_i^2\lambda_j \\
&\quad +(852480 + 973440\beta + 112320\beta^2 + 8640\beta^3)\lambda_i^3, \\
\xi_{31}^{(2)}(\lambda_j;\lambda_i;\beta) &= (30896640\beta + 68428800 + 4147200\beta^2 + 103680\beta^3)\lambda_j^2 \\
&\quad +(-31795200 + 6739200\beta^2 + 11128320\beta + 311040\beta^3)\lambda_i\lambda_j \\
&\quad +(1425600\beta^2 + 103680\beta^3 + 1952640\beta + 4492800)\lambda_i^2, \\
\xi_{31}^{(3)}(\lambda_j;\lambda_i;\beta) &= (61793280\beta + 136857600 + 8294400\beta^2 + 207360\beta^3)\lambda_j \\
&\quad +(-31795200 + 6739200\beta^2 + 11128320\beta + 311040\beta^3)\lambda_i, \\
\xi_{31}^{(4)}(\lambda_j;\lambda_i;\beta) &= 61793280\beta + 136857600 + 8294400\beta^2 + 207360\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -31.1005.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{31}(\lambda_i;\lambda_i;\beta) &= 40320\lambda_i^4(160\beta+88+57\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350,\\
\xi_{31}^{(1)}(\lambda_i;\lambda_i;\beta) &= 20160\lambda_i^3(932\beta+608+312\beta^2+15\beta^3) \geq 0 \text{ for all } \beta \geq -0.9275,\\
\xi_{31}^{(2)}(\lambda_i;\lambda_i;\beta) &= 43200\lambda_i^2(1018\beta+952+285\beta^2+12\beta^3) \geq 0 \text{ for all } \beta \geq -1.6188,\\
\xi_{31}^{(3)}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i(422\beta+608+87\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -23.3467.
\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j;\lambda_i;\beta) = \xi_3(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}
\xi_{30}(\lambda_j;\lambda_i;\beta) &= (918144\beta+2075904+120960\beta^2+3456\beta^3)\lambda_j^4\\
&\quad +(-2373120+475200\beta^2+823680\beta+25920\beta^3)\lambda_i\lambda_j^3\\
&\quad +(388800\beta^2+34560\beta^3+535680\beta+1244160)\lambda_i^2\lambda_j^2\\
&\quad +(714240+800640\beta+95040\beta^2+8640\beta^3)\lambda_i^3\lambda_j\\
&\quad +(8640\beta^2+115200\beta+80640)\lambda_i^4,\\
\xi_{30}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3672576\beta+8303616+483840\beta^2+13824\beta^3)\lambda_j^3\\
&\quad +(-7119360+1425600\beta^2+2471040\beta+77760\beta^3)\lambda_i\lambda_j^2\\
&\quad +(777600\beta^2+69120\beta^3+1071360\beta+2488320)\lambda_i^2\lambda_j\\
&\quad +(714240+800640\beta+95040\beta^2+8640\beta^3)\lambda_i^3,\\
\xi_{30}^{(2)}(\lambda_j;\lambda_i;\beta) &= (11017728\beta+24910848+1451520\beta^2+41472\beta^3)\lambda_j^2\\
&\quad +(-14238720+2851200\beta^2+4942080\beta+155520\beta^3)\lambda_i\lambda_j\\
&\quad +(777600\beta^2+69120\beta^3+1071360\beta+2488320)\lambda_i^2,\\
\xi_{30}^{(3)}(\lambda_j;\lambda_i;\beta) &= (22035456\beta+49821696+2903040\beta^2+82944\beta^3)\lambda_j\\
&\quad +(-14238720+2851200\beta^2+4942080\beta+155520\beta^3)\lambda_i,\\
\xi_{30}^{(4)}(\lambda_j;\lambda_i;\beta) &= 22035456\beta+49821696+2903040\beta^2+82944\beta^3\\
&\geq 0 \text{ for all } \beta \geq -25.5082.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{30}(\lambda_i;\lambda_i;\beta) &= 72576\lambda_i^4(\beta+3)(\beta^2+12\beta+8) > 0 \text{ for all } \beta > -1,\\
\xi_{30}^{(1)}(\lambda_i;\lambda_i;\beta) &= 8064\lambda_i^3(994\beta+544+345\beta^2+21\beta^3) > 0 \text{ for all } \beta > -0.7187,\\
\xi_{30}^{(2)}(\lambda_i;\lambda_i;\beta) &= 24192\lambda_i^2(704\beta+544+210\beta^2+11\beta^3) > 0 \text{ for all } \beta > -1.1328,\\
\xi_{30}^{(3)}(\lambda_i;\lambda_i;\beta) &= 10368\lambda_i(\beta+3)(23\beta^2+486\beta+1144) > 0 \text{ for all } \beta > -2.6985.
\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now,

$$\begin{aligned}
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &= (28800\beta+11520)\lambda_l^4 \\
&\quad +(172800\beta+17280\beta^2+138240)(\lambda_i+\lambda_j)\lambda_l^3 \\
&\quad +(725760+155520\beta+51840\beta^2)(\lambda_j^2+\lambda_i^2)\lambda_l^2 \\
&\quad +(-691200+34560\beta+155520\beta^2)\lambda_i\lambda_j\lambda_l^2 \\
&\quad +(-691200+34560\beta+155520\beta^2)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l \\
&\quad +(172800\beta+17280\beta^2+138240)(\lambda_j^3+\lambda_i^3)\lambda_l \\
&\quad +(172800\beta+17280\beta^2+138240)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j) \\
&\quad +(725760+155520\beta+51840\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(28800\beta+11520)(\lambda_j^4+\lambda_i^4),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_4^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (691200\beta+276480)\lambda_l \\
&\quad +(1036800\beta+103680\beta^2+829440)(\lambda_i+\lambda_j)\geq 0 \text{ for all } \beta\geq -\frac{2}{5}.
\end{aligned}$$

Define $\xi_{42}(\lambda_j;\lambda_i;\beta)=\xi_4^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{42} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned}
\xi_{42}(\lambda_j;\lambda_i;\beta) &= (1693440\beta+2419200+207360\beta^2)\lambda_j^2 \\
&\quad +(1105920\beta+414720\beta^2-552960)\lambda_i\lambda_j \\
&\quad +(1451520+311040\beta+103680\beta^2)\lambda_i^2, \\
\xi_{42}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3386880\beta+4838400+414720\beta^2)\lambda_j \\
&\quad +(1105920\beta+414720\beta^2-552960)\lambda_i, \\
\xi_{42}^{(2)}(\lambda_j;\lambda_i;\beta) &= 3386880\beta+4838400+414720\beta^2\geq 0 \text{ for all } \beta\geq -1.8457.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{42}(\lambda_i;\lambda_i;\beta) &= 103680\lambda_i^2(7\beta+16)(\beta+2)\geq 0 \text{ for all } \beta\geq -2, \\
\xi_{42}^{(1)}(\lambda_i;\lambda_i;\beta) &= 69120\lambda_i(65\beta+62+12\beta^2)\geq 0 \text{ for all } \beta\geq -1.2358.
\end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{41}(\lambda_j;\lambda_i;\beta)=\xi_4^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{41} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}
\xi_{41}(\lambda_j;\lambda_i;\beta) &= (1117440\beta + 2050560 + 172800\beta^2)\lambda_j^3 \\
&\quad +(622080\beta + 518400\beta^2 - 1658880)\lambda_i\lambda_j^2 \\
&\quad +(760320 + 345600\beta + 259200\beta^2)\lambda_i^2\lambda_j \\
&\quad +(172800\beta + 17280\beta^2 + 138240)\lambda_i^3, \\
\xi_{41}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3352320\beta + 6151680 + 518400\beta^2)\lambda_j^2 \\
&\quad +(1244160\beta + 1036800\beta^2 - 3317760)\lambda_i\lambda_j \\
&\quad +(760320 + 345600\beta + 259200\beta^2)\lambda_i^2, \\
\xi_{41}^{(2)}(\lambda_j;\lambda_i;\beta) &= (6704640\beta + 12303360 + 1036800\beta^2)\lambda_j \\
&\quad +(1244160\beta + 1036800\beta^2 - 3317760)\lambda_i, \\
\xi_{41}^{(3)}(\lambda_j;\lambda_i;\beta) &= 6704640\beta + 12303360 + 1036800\beta^2 \geq 0 \text{ for all } \beta.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{41}(\lambda_i;\lambda_i;\beta) &= 322560\lambda_i^3(3\beta+4)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{41}^{(1)}(\lambda_i;\lambda_i;\beta) &= 17280\lambda_i^2(105\beta^2 + 208 + 286\beta) \geq 0 \text{ for all } \beta, \\
\xi_{41}^{(2)}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i(23\beta + 26 + 6\beta^2) \geq 0 \text{ for all } \beta.
\end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j;\lambda_i;\beta) = \xi_4(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}
\xi_{40}(\lambda_j;\lambda_i;\beta) &= (558720\beta + 1025280 + 86400\beta^2)\lambda_j^4 \\
&\quad +(-1105920 + 414720\beta + 345600\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(760320 + 345600\beta + 259200\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(345600\beta + 34560\beta^2 + 276480)\lambda_i^3\lambda_j \\
&\quad +(28800\beta + 11520)\lambda_i^4, \\
\xi_{40}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2234880\beta + 4101120 + 345600\beta^2)\lambda_j^3 \\
&\quad +(1244160\beta + 1036800\beta^2 - 3317760)\lambda_i\lambda_j^2 \\
&\quad +(1520640 + 691200\beta + 518400\beta^2)\lambda_i^2\lambda_j \\
&\quad +(345600\beta + 34560\beta^2 + 276480)\lambda_i^3, \\
\xi_{40}^{(2)}(\lambda_j;\lambda_i;\beta) &= (6704640\beta + 12303360 + 1036800\beta^2)\lambda_j^2 \\
&\quad +(2488320\beta + 2073600\beta^2 - 6635520)\lambda_i\lambda_j \\
&\quad +(1520640 + 691200\beta + 518400\beta^2)\lambda_i^2, \\
\xi_{40}^{(3)}(\lambda_j;\lambda_i;\beta) &= (13409280\beta + 24606720 + 2073600\beta^2)\lambda_j \\
&\quad +(2488320\beta + 2073600\beta^2 - 6635520)\lambda_i, \\
\xi_{40}^{(4)}(\lambda_j;\lambda_i;\beta) &= 13409280\beta + 24606720 + 2073600\beta^2 \geq 0 \text{ for all } \beta.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{40}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i^4(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1,\\
\xi_{40}^{(1)}(\lambda_i;\lambda_i;\beta) &= 645120\lambda_i^3(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1,\\
\xi_{40}^{(2)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i^2(105\beta^2+208+286\beta) > 0 \text{ for all } \beta,\\
\xi_{40}^{(3)}(\lambda_i;\lambda_i;\beta) &= 691200\lambda_i(23\beta+26+6\beta^2) > 0 \text{ for all } \beta.
\end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.16 Let

$$\begin{aligned}
\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) = &(8640\beta+38880\beta^2-172800)t(1,1,1,4)\\
&+(12960\beta^2+48960+36000\beta)t(0,1,2,4)\\
&+(1440\beta^2+19200\beta+13440)t(0,0,3,4)\\
&+(15840\beta^2+119040+133440\beta+1440\beta^3)t(0,1,3,3)\\
&+(12960\beta^3-552960+69120\beta+77760\beta^2)t(1,1,2,3)\\
&+(30240\beta^2+4320\beta^3+570240+155520\beta)t(0,2,2,3)\\
&+(-311040\beta+38880\beta^3+77760\beta^2)t(1,2,2,2),
\end{aligned}$$

where

$$t(a,b,c,d) = \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in \mathrm{perm}(a,b,c,d)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_k^{n_4}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) \;=\; & (-691200+155520\beta^2+34560\beta)\lambda_k^3\hat{s}(1,1,1)\\
&+(51840\beta^2+195840+144000\beta)\lambda_k^3\hat{s}(0,1,2)\\
&+(76800\beta+5760\beta^2+53760)\lambda_k^3\hat{s}(0,0,3)\\
&+(233280\beta^2-1658880+38880\beta^3+207360\beta)\lambda_k^2\hat{s}(1,1,2)\\
&+(90720\beta^2+466560\beta+1710720+12960\beta^3)\lambda_k^2\hat{s}(0,2,2)\\
&+(57600\beta+4320\beta^2+40320)\lambda_k^2\hat{s}(0,0,4)\\
&+(400320\beta+4320\beta^3+357120+47520\beta^2)\lambda_k^2\hat{s}(0,1,3)\\
&+(155520\beta^2-1105920+138240\beta+25920\beta^3)\lambda_k\hat{s}(1,1,3)\\
&+(155520\beta^2-622080\beta+77760\beta^3)\lambda_k\hat{s}(1,2,2)\\
&+(25920\beta^2+97920+72000\beta)\lambda_k\hat{s}(0,1,4)\\
&+(60480\beta^2+311040\beta+1140480+8640\beta^3)\lambda_k\hat{s}(0,2,3)\\
&+(133440\beta+1440\beta^3+119040+15840\beta^2)\hat{s}(0,3,3)\\
&+(77760\beta^2-311040\beta+38880\beta^3)\hat{s}(2,2,2)\\
&+(-552960+69120\beta+77760\beta^2+12960\beta^3)\hat{s}(1,2,3)\\
&+(-172800+38880\beta^2+8640\beta)\hat{s}(1,1,4)\\
&+(12960\beta^2+48960+36000\beta)\hat{s}(0,2,4),
\end{aligned}$$

$$\begin{aligned}
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) \;=\; & 3(-691200+155520\beta^2+34560\beta)\lambda_k^2\hat{s}(1,1,1)\\
&+3(51840\beta^2+195840+144000\beta)\lambda_k^2\hat{s}(0,1,2)\\
&+3(76800\beta+5760\beta^2+53760)\lambda_k^2\hat{s}(0,0,3)\\
&+2(233280\beta^2-1658880+38880\beta^3+207360\beta)\lambda_k\hat{s}(1,1,2)\\
&+2(90720\beta^2+466560\beta+1710720+12960\beta^3)\lambda_k\hat{s}(0,2,2)\\
&+2(57600\beta+4320\beta^2+40320)\lambda_k\hat{s}(0,0,4)\\
&+2(400320\beta+4320\beta^3+357120+47520\beta^2)\lambda_k\hat{s}(0,1,3)\\
&+(155520\beta^2-1105920+138240\beta+25920\beta^3)\hat{s}(1,1,3)\\
&+(155520\beta^2-622080\beta+77760\beta^3)\hat{s}(1,2,2)\\
&+(25920\beta^2+97920+72000\beta)\hat{s}(0,1,4)\\
&+(60480\beta^2+311040\beta+1140480+8640\beta^3)\hat{s}(0,2,3),
\end{aligned}$$

$$\begin{aligned}\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(-691200+155520\beta^2+34560\beta)\lambda_k\hat{s}(1,1,1)\\ &+6(51840\beta^2+195840+144000\beta)\lambda_k\hat{s}(0,1,2)\\ &+6(76800\beta+5760\beta^2+53760)\lambda_k\hat{s}(0,0,3)\\ &+2(233280\beta^2-1658880+38880\beta^3+207360\beta)\hat{s}(1,1,2)\\ &+2(90720\beta^2+466560\beta+1710720+12960\beta^3)\hat{s}(0,2,2)\\ &+2(57600\beta+4320\beta^2+40320)\hat{s}(0,0,4)\\ &+2(400320\beta+4320\beta^3+357120+47520\beta^2)\hat{s}(0,1,3),\end{aligned}$$

$$\begin{aligned}\xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(-691200+155520\beta^2+34560\beta)\hat{s}(1,1,1)\\ &+6(51840\beta^2+195840+144000\beta)\hat{s}(0,1,2)\\ &+6(76800\beta+5760\beta^2+53760)\hat{s}(0,0,3).\end{aligned}$$

Define

$$\begin{aligned}\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta).\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (26880+38400\beta+2880\beta^2)\lambda_l^7\\
&\quad+(205440\beta+1440\beta^3+41760\beta^2+216960)(\lambda_j+\lambda_i)\lambda_l^6\\
&\quad+(383040\beta+1238400+8640\beta^3+86400\beta^2)(\lambda_j^2+\lambda_i^2)\lambda_l^5\\
&\quad+(155520\beta+25920\beta^3+233280\beta^2-1451520)\lambda_i\lambda_j\lambda_l^5\\
&\quad+(-100800\beta+259200\beta^2+64800\beta^3-1008000)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^4\\
&\quad+(460800\beta+64800\beta^2+835200+7200\beta^3)(\lambda_j^3+\lambda_i^3)\lambda_l^4\\
&\quad+(187200\beta^2-867840+405120\beta+28800\beta^3)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^3\\
&\quad+(124800+110400\beta+28800\beta^2)(\lambda_j^4+\lambda_i^4)\lambda_l^3\\
&\quad+(216000\beta^2+1140480-311040\beta+86400\beta^3)\lambda_i^2\lambda_j^2\lambda_l^3\\
&\quad+(380160\beta+138240\beta^2+587520+21600\beta^3)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l^2\\
&\quad+(80640\beta-74880+64800\beta^2)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l^2\\
&\quad+(25920\beta^2+97920+72000\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2)\lambda_l\\
&\quad+(266880\beta+238080+31680\beta^2+2880\beta^3)\lambda_i^3\lambda_j^3\lambda_l\\
&\quad+(19200\beta+1440\beta^2+13440)(\lambda_i^3\lambda_j^4+\lambda_i^4\lambda_j^3),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_0^{(6)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (135475200+193536000\beta+14515200\beta^2)\lambda_l\\
&\quad+(147916800\beta+1036800\beta^3+30067200\beta^2+156211200)(\lambda_i+\lambda_j)\\
&\geq 0 \text{ for all } \beta \geq -0.7412.
\end{aligned}$$

Define $\xi_{05}(\lambda_j;\lambda_i;\beta) = \xi_0^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{05} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{05} with respect to λ_j gives

$$\begin{aligned}
\xi_{05}(\lambda_j;\lambda_i;\beta) &= (372556800+290649600\beta+47692800\beta^2+2073600\beta^3)\lambda_j^2\\
&\quad+(166579200\beta+4147200\beta^3+58060800\beta^2-17971200)\lambda_i\lambda_j\\
&\quad+(45964800\beta+148608000+1036800\beta^3+10368000\beta^2)\lambda_i^2,\\
\xi_{05}^{(1)}(\lambda_j;\lambda_i;\beta) &= (745113600+581299200\beta+95385600\beta^2+4147200\beta^3)\lambda_j\\
&\quad+(166579200\beta+4147200\beta^3+58060800\beta^2-17971200)\lambda_i,\\
\xi_{05}^{(2)}(\lambda_j;\lambda_i;\beta) &= 745113600+581299200\beta+95385600\beta^2+4147200\beta^3\\
&\geq 0 \text{ for all } \beta \geq -1.7421.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{05}(\lambda_i;\lambda_i;\beta) &= 2419200\lambda_i^2(208+208\beta+48\beta^2+3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4294,\\
\xi_{05}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1382400\lambda_i(526+541\beta+111\beta^2+6\beta^3) \geq 0 \text{ for all } \beta \geq -1.2898.
\end{aligned}$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{04}(\lambda_j;\lambda_i;\beta) = \xi_0^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{04} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned}
\xi_{04}(\lambda_j;\lambda_i;\beta) &= (269337600 + 163238400\beta + 29376000\beta^2 + 1728000\beta^3)\lambda_j^3 \\
&\quad +(90201600\beta + 5184000\beta^3 + 49248000\beta^2 - 120268800)\lambda_i\lambda_j^2 \\
&\quad +(43545600\beta + 124416000 + 2592000\beta^3 + 16588800\beta^2)\lambda_i^2\lambda_j \\
&\quad +(11059200\beta + 1555200\beta^2 + 20044800 + 172800\beta^3)\lambda_i^3, \\
\xi_{04}^{(1)}(\lambda_j;\lambda_i;\beta) &= (808012800 + 489715200\beta + 88128000\beta^2 + 5184000\beta^3)\lambda_j^2 \\
&\quad +(180403200\beta + 10368000\beta^3 + 98496000\beta^2 - 240537600)\lambda_i\lambda_j \\
&\quad +(43545600\beta + 124416000 + 2592000\beta^3 + 16588800\beta^2)\lambda_i^2, \\
\xi_{04}^{(2)}(\lambda_j;\lambda_i;\beta) &= (979430400\beta + 1616025600 + 176256000\beta^2 + 10368000\beta^3)\lambda_j \\
&\quad +(180403200\beta + 10368000\beta^3 + 98496000\beta^2 - 240537600)\lambda_i, \\
\xi_{04}^{(3)}(\lambda_j;\lambda_i;\beta) &= 979430400\beta + 1616025600 + 176256000\beta^2 + 10368000\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -2.9237.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{04}(\lambda_i;\lambda_i;\beta) &= 1612800\lambda_i^3(182 + 191\beta + 60\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\
\xi_{04}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1209600\lambda_i^2(572 + 590\beta + 168\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.5802, \\
\xi_{04}^{(2)}(\lambda_i;\lambda_i;\beta) &= 345600\lambda_i(3356\beta + 3980 + 795\beta^2 + 60\beta^3) \\
&\geq 0 \text{ for all } \beta \geq -1.9649.
\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j;\lambda_i;\beta) = \xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{03} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$
\begin{aligned}
\xi_{03}(\lambda_j;\lambda_i;\beta) &= (126777600 + 67420800\beta + 12528000\beta^2 + 864000\beta^3)\lambda_j^4 \\
&\quad +(33995520\beta + 3456000\beta^3 + 26352000\beta^2 - 90455040)\lambda_i\lambda_j^3 \\
&\quad +(18696960\beta + 56954880 + 2592000\beta^3 + 12700800\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(2678400\beta^2 + 14837760 + 13489920\beta + 345600\beta^3)\lambda_i^3\lambda_j \\
&\quad +(748800 + 662400\beta + 172800\beta^2)\lambda_i^4, \\
\xi_{03}^{(1)}(\lambda_j;\lambda_i;\beta) &= (507110400 + 269683200\beta + 50112000\beta^2 + 3456000\beta^3)\lambda_j^3 \\
&\quad +(101986560\beta + 10368000\beta^3 + 79056000\beta^2 - 271365120)\lambda_i\lambda_j^2 \\
&\quad +(37393920\beta + 113909760 + 5184000\beta^3 + 25401600\beta^2)\lambda_i^2\lambda_j \\
&\quad +(2678400\beta^2 + 14837760 + 13489920\beta + 345600\beta^3)\lambda_i^3, \\
\xi_{03}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1521331200 + 809049600\beta + 150336000\beta^2 + 10368000\beta^3)\lambda_j^2 \\
&\quad +(203973120\beta + 20736000\beta^3 + 158112000\beta^2 - 542730240)\lambda_i\lambda_j \\
&\quad +(37393920\beta + 113909760 + 5184000\beta^3 + 25401600\beta^2)\lambda_i^2, \\
\xi_{03}^{(3)}(\lambda_j;\lambda_i;\beta) &= (3042662400 + 1618099200\beta + 300672000\beta^2 + 20736000\beta^3)\lambda_j \\
&\quad +(203973120\beta + 20736000\beta^3 + 158112000\beta^2 - 542730240)\lambda_i, \\
\xi_{03}^{(4)}(\lambda_j;\lambda_i;\beta) &= 3042662400 + 1618099200\beta + 300672000\beta^2 + 20736000\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.2760.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{03}(\lambda_i;\lambda_i;\beta) &= 3628800\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{03}^{(1)}(\lambda_i;\lambda_i;\beta) &= 806400\lambda_i^3(452 + 524\beta + 195\beta^2 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.8116, \\
\xi_{03}^{(2)}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i^2(2258 + 2171\beta + 690\beta^2 + 75\beta^3) \geq 0 \text{ for all } \beta \geq -2.3123, \\
\xi_{03}^{(3)}(\lambda_i;\lambda_i;\beta) &= 103680\lambda_i(24112 + 17574\beta + 4425\beta^2 + 400\beta^3) \geq 0 \text{ for all } \beta \geq -3.2529.
\end{aligned}
$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j;\lambda_i;\beta) = \xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{02} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned}
\xi_{02}(\lambda_j;\lambda_i;\beta) &= (43176960 + 21628800\beta + 4052160\beta^2 + 302400\beta^3)\lambda_j^4 \\
&\quad +(10656000\beta + 1512000\beta^3 + 10281600\beta^2 - 39974400)\lambda_i\lambda_j^3 \\
&\quad +(5345280\beta + 20689920 + 1512000\beta^3 + 6410880\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(8720640\beta + 2177280\beta^2 + 5990400 + 302400\beta^3)\lambda_i^3\lambda_j \\
&\quad +(599040 + 823680\beta + 302400\beta^2)\lambda_i^4, \\
\xi_{02}^{(1)}(\lambda_j;\lambda_i;\beta) &= (172707840 + 86515200\beta + 16208640\beta^2 + 1209600\beta^3)\lambda_j^3 \\
&\quad +(31968000\beta + 4536000\beta^3 + 30844800\beta^2 - 119923200)\lambda_i\lambda_j^2 \\
&\quad +(10690560\beta + 41379840 + 3024000\beta^3 + 12821760\beta^2)\lambda_i^2\lambda_j \\
&\quad +(8720640\beta + 2177280\beta^2 + 5990400 + 302400\beta^3)\lambda_i^3, \\
\xi_{02}^{(2)}(\lambda_j;\lambda_i;\beta) &= (518123520 + 259545600\beta + 48625920\beta^2 + 3628800\beta^3)\lambda_j^2 \\
&\quad +(63936000\beta + 9072000\beta^3 + 61689600\beta^2 - 239846400)\lambda_i\lambda_j \\
&\quad +(10690560\beta + 41379840 + 3024000\beta^3 + 12821760\beta^2)\lambda_i^2, \\
\xi_{02}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1036247040 + 519091200\beta + 97251840\beta^2 + 7257600\beta^3)\lambda_j \\
&\quad +(63936000\beta + 9072000\beta^3 + 61689600\beta^2 - 239846400)\lambda_i, \\
\xi_{02}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1036247040 + 519091200\beta + 97251840\beta^2 + 7257600\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.5982.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{02}(\lambda_i;\lambda_i;\beta) &= 725760\lambda_i^4(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(1)}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^3(25\beta+46)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{46}{25}, \\
\xi_{02}^{(2)}(\lambda_i;\lambda_i;\beta) &= 80640\lambda_i^2(3964 + 4144\beta + 1527\beta^2 + 195\beta^3) \geq 0 \text{ for all } \beta \geq -3.0390, \\
\xi_{02}^{(3)}(\lambda_i;\lambda_i;\beta) &= 120960\lambda_i(6584 + 4820\beta + 1314\beta^2 + 135\beta^3) \geq 0 \text{ for all } \beta \geq -3.5381.
\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{01} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$
\begin{aligned}
\xi_{01}(\lambda_j;\lambda_i;\beta) &= (11397120 + 5591040\beta + 1048320\beta^2 + 80640\beta^3)\lambda_j^4 \\
&\quad +(2983680\beta + 483840\beta^3 + 3144960\beta^2 - 12741120)\lambda_i\lambda_j^3 \\
&\quad +(1411200\beta + 6854400 + 604800\beta^3 + 2419200\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(4085760\beta + 1128960\beta^2 + 2150400 + 161280\beta^3)\lambda_i^3\lambda_j \\
&\quad +(322560 + 564480\beta + 241920\beta^2)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= (45588480 + 22364160\beta + 4193280\beta^2 + 322560\beta^3)\lambda_j^3 \\
&\quad +(8951040\beta + 1451520\beta^3 + 9434880\beta^2 - 38223360)\lambda_i\lambda_j^2 \\
&\quad +(2822400\beta + 13708800 + 1209600\beta^3 + 4838400\beta^2)\lambda_i^2\lambda_j \\
&\quad +(4085760\beta + 1128960\beta^2 + 2150400 + 161280\beta^3)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= (136765440 + 67092480\beta + 12579840\beta^2 + 967680\beta^3)\lambda_j^2 \\
&\quad +(17902080\beta + 2903040\beta^3 + 18869760\beta^2 - 76446720)\lambda_i\lambda_j \\
&\quad +(2822400\beta + 13708800 + 1209600\beta^3 + 4838400\beta^2)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j;\lambda_i;\beta) &= (273530880 + 134184960\beta + 25159680\beta^2 + 1935360\beta^3)\lambda_j \\
&\quad +(17902080\beta + 2903040\beta^3 + 18869760\beta^2 - 76446720)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j;\lambda_i;\beta) &= 273530880 + 134184960\beta + 25159680\beta^2 + 1935360\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.6109.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 1330560\lambda_i^4(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i^3(13\beta+16)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{16}{13}, \\
\xi_{01}^{(2)}(\lambda_i;\lambda_i;\beta) &= 725760\lambda_i^2(\beta+3)(7\beta^2+29\beta+34) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i;\lambda_i;\beta) &= 161280\lambda_i(1222 + 943\beta + 273\beta^2 + 30\beta^3) \geq 0 \text{ for all } \beta \geq -3.3405.
\end{aligned}
$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^3$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= (2442240 + 1198080\beta + 224640\beta^2 + 17280\beta^3)\lambda_j^4 \\
&\quad +(745920\beta + 120960\beta^3 + 786240\beta^2 - 3185280)\lambda_i\lambda_j^3 \\
&\quad +(423360\beta + 2056320 + 181440\beta^3 + 725760\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(1532160\beta + 423360\beta^2 + 806400 + 60480\beta^3)\lambda_i^3\lambda_j \\
&\quad +(161280 + 282240\beta + 120960\beta^2)\lambda_i^4, \\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= (9768960 + 4792320\beta + 898560\beta^2 + 69120\beta^3)\lambda_j^3 \\
&\quad +(2237760\beta + 362880\beta^3 + 2358720\beta^2 - 9555840)\lambda_i\lambda_j^2 \\
&\quad +(846720\beta + 4112640 + 362880\beta^3 + 1451520\beta^2)\lambda_i^2\lambda_j \\
&\quad +(1532160\beta + 423360\beta^2 + 806400 + 60480\beta^3)\lambda_i^3, \\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= (29306880 + 14376960\beta + 2695680\beta^2 + 207360\beta^3)\lambda_j^2 \\
&\quad +(4475520\beta + 725760\beta^3 + 4717440\beta^2 - 19111680)\lambda_i\lambda_j \\
&\quad +(846720\beta + 4112640 + 362880\beta^3 + 1451520\beta^2)\lambda_i^2, \\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= (58613760 + 28753920\beta + 5391360\beta^2 + 414720\beta^3)\lambda_j \\
&\quad +(4475520\beta + 725760\beta^3 + 4717440\beta^2 - 19111680)\lambda_i, \\
\xi_{00}^{(4)}(\lambda_j;\lambda_i;\beta) &= 58613760 + 28753920\beta + 5391360\beta^2 + 414720\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.6109.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 380160\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 855360\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 51840\lambda_i^2(25\beta+46)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -\frac{46}{25}, \\
\xi_{00}^{(3)}(\lambda_i;\lambda_i;\beta) &= 51840\lambda_i(\beta+3)(22\beta^2+129\beta+254) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (134400\beta+94080+10080\beta^2)\lambda_l^6\\
&+(4320\beta^3+650880+125280\beta^2+616320\beta)(\lambda_j+\lambda_i)\lambda_l^5\\
&+(216000\beta^2+21600\beta^3+957600\beta+3096000)(\lambda_j^2+\lambda_i^2)\lambda_l^4\\
&+(-3628800+64800\beta^3+388800\beta+583200\beta^2)\lambda_i\lambda_j\lambda_l^4\\
&+(-201600\beta+518400\beta^2+129600\beta^3-2016000)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^3\\
&+(14400\beta^3+921600\beta+1670400+129600\beta^2)(\lambda_j^3+\lambda_i^3)\lambda_l^3\\
&+(-1301760+607680\beta+43200\beta^3+280800\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l^2\\
&+(165600\beta+43200\beta^2+187200)(\lambda_j^4+\lambda_i^4)\lambda_l^2\\
&+(1710720-466560\beta+129600\beta^3+324000\beta^2)\lambda_i^2\lambda_j^2\lambda_l^2\\
&+(380160\beta+138240\beta^2+587520+21600\beta^3)(\lambda_i^2\lambda_j^3+\lambda_i^3\lambda_j^2)\lambda_l\\
&+(80640\beta-74880+64800\beta^2)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\lambda_l\\
&+(133440\beta+1440\beta^3+119040+15840\beta^2)\lambda_i^3\lambda_j^3\\
&+(12960\beta^2+48960+36000\beta)(\lambda_i^2\lambda_j^4+\lambda_i^4\lambda_j^2),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_1^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (96768000\beta+67737600+7257600\beta^2)\lambda_l\\
&\quad+(518400\beta^3+78105600+15033600\beta^2+73958400\beta)(\lambda_i+\lambda_j)\\
&\geq 0 \text{ for all } \beta\geq -0.7412.
\end{aligned}$$

Define $\xi_{14}(\lambda_j;\lambda_i;\beta)=\xi_1^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{14} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned}
\xi_{14}(\lambda_j;\lambda_i;\beta) &= (145324800\beta+186278400+23846400\beta^2+1036800\beta^3)\lambda_j^2\\
&\quad+(2073600\beta^3-8985600+29030400\beta^2+83289600\beta)\lambda_i\lambda_j\\
&\quad+(5184000\beta^2+518400\beta^3+22982400\beta+74304000)\lambda_i^2,\\
\xi_{14}^{(1)}(\lambda_j;\lambda_i;\beta) &= (290649600\beta+372556800+47692800\beta^2+2073600\beta^3)\lambda_j\\
&\quad+(2073600\beta^3-8985600+29030400\beta^2+83289600\beta)\lambda_i,\\
\xi_{14}^{(2)}(\lambda_j;\lambda_i;\beta) &= 290649600\beta+372556800+47692800\beta^2+2073600\beta^3\\
&\geq 0 \text{ for all } \beta\geq -1.7421.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{14}(\lambda_i;\lambda_i;\beta) &= 1209600\lambda_i^2(208\beta+208+48\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq -1.4294,\\
\xi_{14}^{(1)}(\lambda_i;\lambda_i;\beta) &= 691200\lambda_i(541\beta+526+111\beta^2+6\beta^3)\geq 0 \text{ for all } \beta\geq -1.2898.
\end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j;\lambda_i;\beta) = \xi_1^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}
\xi_{13}(\lambda_j;\lambda_i;\beta) &= (81619200\beta + 134668800 + 14688000\beta^2 + 864000\beta^3)\lambda_j^3 \\
&\quad +(-60134400 + 2592000\beta^3 + 45100800\beta + 24624000\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(8294400\beta^2 + 1296000\beta^3 + 21772800\beta + 62208000)\lambda_i^2\lambda_j \\
&\quad +(86400\beta^3 + 5529600\beta + 10022400 + 777600\beta^2)\lambda_i^3, \\
\xi_{13}^{(1)}(\lambda_j;\lambda_i;\beta) &= (244857600\beta + 404006400 + 44064000\beta^2 + 2592000\beta^3)\lambda_j^2 \\
&\quad +(-120268800 + 5184000\beta^3 + 90201600\beta + 49248000\beta^2)\lambda_i\lambda_j \\
&\quad +(8294400\beta^2 + 1296000\beta^3 + 21772800\beta + 62208000)\lambda_i^2, \\
\xi_{13}^{(2)}(\lambda_j;\lambda_i;\beta) &= (489715200\beta + 808012800 + 88128000\beta^2 + 5184000\beta^3)\lambda_j \\
&\quad +(-120268800 + 5184000\beta^3 + 90201600\beta + 49248000\beta^2)\lambda_i, \\
\xi_{13}^{(3)}(\lambda_j;\lambda_i;\beta) &= 489715200\beta + 808012800 + 88128000\beta^2 + 5184000\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -2.9237.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{13}(\lambda_i;\lambda_i;\beta) &= 806400\lambda_i^3(191\beta + 182 + 60\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\
\xi_{13}^{(1)}(\lambda_i;\lambda_i;\beta) &= 604800\lambda_i^2(590\beta + 572 + 168\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.5802, \\
\xi_{13}^{(2)}(\lambda_i;\lambda_i;\beta) &= 172800\lambda_i(3356\beta + 3980 + 795\beta^2 + 60\beta^3) \geq 0 \text{ for all } \beta \geq -1.9649.
\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j;\lambda_i;\beta) = \xi_1^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{12} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned}
\xi_{12}(\lambda_j;\lambda_i;\beta) &= (33710400\beta + 63388800 + 6264000\beta^2 + 432000\beta^3)\lambda_j^4 \\
&\quad +(-45227520 + 1728000\beta^3 + 16997760\beta + 13176000\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(6350400\beta^2 + 1296000\beta^3 + 9348480\beta + 28477440)\lambda_i^2\lambda_j^2 \\
&\quad +(7418880 + 6744960\beta + 172800\beta^3 + 1339200\beta^2)\lambda_i^3\lambda_j \\
&\quad +(331200\beta + 86400\beta^2 + 374400)\lambda_i^4, \\
\xi_{12}^{(1)}(\lambda_j;\lambda_i;\beta) &= (134841600\beta + 253555200 + 25056000\beta^2 + 1728000\beta^3)\lambda_j^3 \\
&\quad +(-135682560 + 5184000\beta^3 + 50993280\beta + 39528000\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(12700800\beta^2 + 2592000\beta^3 + 18696960\beta + 56954880)\lambda_i^2\lambda_j \\
&\quad +(7418880 + 6744960\beta + 172800\beta^3 + 1339200\beta^2)\lambda_i^3, \\
\xi_{12}^{(2)}(\lambda_j;\lambda_i;\beta) &= (404524800\beta + 760665600 + 75168000\beta^2 + 5184000\beta^3)\lambda_j^2 \\
&\quad +(-271365120 + 10368000\beta^3 + 101986560\beta + 79056000\beta^2)\lambda_i\lambda_j \\
&\quad +(12700800\beta^2 + 2592000\beta^3 + 18696960\beta + 56954880)\lambda_i^2, \\
\xi_{12}^{(3)}(\lambda_j;\lambda_i;\beta) &= (809049600\beta + 1521331200 + 150336000\beta^2 + 10368000\beta^3)\lambda_j \\
&\quad +(-271365120 + 10368000\beta^3 + 101986560\beta + 79056000\beta^2)\lambda_i, \\
\xi_{12}^{(4)}(\lambda_j;\lambda_i;\beta) &= 809049600\beta + 1521331200 + 150336000\beta^2 + 10368000\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.2760.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{12}(\lambda_i;\lambda_i;\beta) &= 1814400\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{12}^{(1)}(\lambda_i;\lambda_i;\beta) &= 403200\lambda_i^3(524\beta + 452 + 195\beta^2 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.8116, \\
\xi_{12}^{(2)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i^2(2171\beta + 2258 + 690\beta^2 + 75\beta^3) \geq 0 \text{ for all } \beta \geq -2.3123, \\
\xi_{12}^{(3)}(\lambda_i;\lambda_i;\beta) &= 51840\lambda_i(17574\beta + 24112 + 4425\beta^2 + 400\beta^3) \geq 0 \text{ for all } \beta \geq -3.2529.
\end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j;\lambda_i;\beta) = \xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{11} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j;\lambda_i;\beta) &= (10814400\beta + 21588480 + 2026080\beta^2 + 151200\beta^3)\lambda_j^4 \\
&\quad +(-19987200 + 756000\beta^3 + 5328000\beta + 5140800\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(3205440\beta^2 + 756000\beta^3 + 2672640\beta + 10344960)\lambda_i^2\lambda_j^2 \\
&\quad +(151200\beta^3 + 4360320\beta + 2995200 + 1088640\beta^2)\lambda_i^3\lambda_j \\
&\quad +(411840\beta + 151200\beta^2 + 299520)\lambda_i^4, \\
\xi_{11}^{(1)}(\lambda_j;\lambda_i;\beta) &= (43257600\beta + 86353920 + 8104320\beta^2 + 604800\beta^3)\lambda_j^3 \\
&\quad +(-59961600 + 2268000\beta^3 + 15984000\beta + 15422400\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(6410880\beta^2 + 1512000\beta^3 + 5345280\beta + 20689920)\lambda_i^2\lambda_j \\
&\quad +(151200\beta^3 + 4360320\beta + 2995200 + 1088640\beta^2)\lambda_i^3, \\
\xi_{11}^{(2)}(\lambda_j;\lambda_i;\beta) &= (129772800\beta + 259061760 + 24312960\beta^2 + 1814400\beta^3)\lambda_j^2 \\
&\quad +(-119923200 + 4536000\beta^3 + 31968000\beta + 30844800\beta^2)\lambda_i\lambda_j \\
&\quad +(6410880\beta^2 + 1512000\beta^3 + 5345280\beta + 20689920)\lambda_i^2, \\
\xi_{11}^{(3)}(\lambda_j;\lambda_i;\beta) &= (259545600\beta + 518123520 + 48625920\beta^2 + 3628800\beta^3)\lambda_j \\
&\quad +(-119923200 + 4536000\beta^3 + 31968000\beta + 30844800\beta^2)\lambda_i, \\
\xi_{11}^{(4)}(\lambda_j;\lambda_i;\beta) &= 259545600\beta + 518123520 + 48625920\beta^2 + 3628800\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.5982.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^4(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\
\xi_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= 181440\lambda_i^3(25\beta+46)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{46}{25}, \\
\xi_{11}^{(2)}(\lambda_i;\lambda_i;\beta) &= 40320\lambda_i^2(4144\beta + 3964 + 1527\beta^2 + 195\beta^3) \geq 0 \text{ for all } \beta \geq -3.0390, \\
\xi_{11}^{(3)}(\lambda_i;\lambda_i;\beta) &= 60480\lambda_i(4820\beta + 6584 + 1314\beta^2 + 135\beta^3) \geq 0 \text{ for all } \beta \geq -3.5381.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j;\lambda_i;\beta) = \xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{10} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$
\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= (2795520\beta + 5698560 + 524160\beta^2 + 40320\beta^3)\lambda_j^4 \\
&\quad +(-6370560 + 241920\beta^3 + 1491840\beta + 1572480\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(1209600\beta^2 + 302400\beta^3 + 705600\beta + 3427200)\lambda_i^2\lambda_j^2 \\
&\quad +(80640\beta^3 + 2042880\beta + 1075200 + 564480\beta^2)\lambda_i^3\lambda_j \\
&\quad +(282240\beta + 120960\beta^2 + 161280)\lambda_i^4, \\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= (11182080\beta + 22794240 + 2096640\beta^2 + 161280\beta^3)\lambda_j^3 \\
&\quad +(-19111680 + 725760\beta^3 + 4475520\beta + 4717440\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(2419200\beta^2 + 604800\beta^3 + 1411200\beta + 6854400)\lambda_i^2\lambda_j \\
&\quad +(80640\beta^3 + 2042880\beta + 1075200 + 564480\beta^2)\lambda_i^3, \\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= (33546240\beta + 68382720 + 6289920\beta^2 + 483840\beta^3)\lambda_j^2 \\
&\quad +(-38223360 + 1451520\beta^3 + 8951040\beta + 9434880\beta^2)\lambda_i\lambda_j \\
&\quad +(2419200\beta^2 + 604800\beta^3 + 1411200\beta + 6854400)\lambda_i^2, \\
\xi_{10}^{(3)}(\lambda_j;\lambda_i;\beta) &= (67092480\beta + 136765440 + 12579840\beta^2 + 967680\beta^3)\lambda_j \\
&\quad +(-38223360 + 1451520\beta^3 + 8951040\beta + 9434880\beta^2)\lambda_i, \\
\xi_{10}^{(4)}(\lambda_j;\lambda_i;\beta) &= 67092480\beta + 136765440 + 12579840\beta^2 + 967680\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.6109.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 665280\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 120960\lambda_i^3(13\beta+16)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -\frac{16}{13}, \\
\xi_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^2(\beta+3)(7\beta^2+29\beta+34) > 0 \text{ for all } \beta > -3, \\
\xi_{10}^{(3)}(\lambda_i;\lambda_i;\beta) &= 80640\lambda_i(943\beta + 1222 + 273\beta^2 + 30\beta^3) > 0 \text{ for all } \beta > -3.3405.
\end{aligned}
$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= (25920\beta^2+345600\beta+241920)\lambda_l^5 \\
&+(8640\beta^3+276480\beta^2+1399680+1304640\beta)(\lambda_j+\lambda_i)\lambda_l^4 \\
&+(5149440+1676160\beta+397440\beta^2+34560\beta^3)(\lambda_j^2+\lambda_i^2)\lambda_l^3 \\
&+(-6497280+1088640\beta^2+103680\beta^3+656640\beta)\lambda_i\lambda_j\lambda_l^3 \\
&+(155520\beta^3-2730240+224640\beta+777600\beta^2)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^2 \\
&+(2016000+17280\beta^3+172800\beta^2+1342080\beta)(\lambda_j^3+\lambda_i^3)\lambda_l^2 \\
&+(34560\beta^2+178560+187200\beta)(\lambda_j^4+\lambda_i^4)\lambda_l \\
&+(250560\beta^2+34560\beta^3-391680+938880\beta)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l \\
&+(103680\beta^3+3421440+311040\beta+336960\beta^2)\lambda_i^2\lambda_j^2\lambda_l \\
&+(60480\beta^2+311040\beta+1140480+8640\beta^3)(\lambda_i^3\lambda_j^2+\lambda_i^2\lambda_j^3) \\
&+(25920\beta^2+97920+72000\beta)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_2^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (3110400\beta^2+41472000\beta+29030400)\lambda_l \\
&+(207360\beta^3+6635520\beta^2+33592320+31311360\beta)(\lambda_i+\lambda_j) \\
&\geq 0 \text{ for all } \beta\geq -0.7412.
\end{aligned}$$

Define $\xi_{23}(\lambda_j;\lambda_i;\beta)=\xi_2^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{23} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned}
\xi_{23}(\lambda_j;\lambda_i;\beta) &= (10575360\beta^2+62104320\beta+79004160+414720\beta^3)\lambda_j^2 \\
&+(829440\beta^3+13167360\beta^2-5391360+35251200\beta)\lambda_i\lambda_j \\
&+(30896640+10056960\beta+2384640\beta^2+207360\beta^3)\lambda_i^2, \\
\xi_{23}^{(1)}(\lambda_j;\lambda_i;\beta) &= (21150720\beta^2+124208640\beta+158008320+829440\beta^3)\lambda_j \\
&+(829440\beta^3+13167360\beta^2-5391360+35251200\beta)\lambda_i, \\
\xi_{23}^{(2)}(\lambda_j;\lambda_i;\beta) &= 21150720\beta^2+124208640\beta+158008320+829440\beta^3\geq 0 \text{ for all } \beta\geq -1.7669.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{23}(\lambda_i;\lambda_i;\beta) &= 1451520\lambda_i^2(\beta+4)(\beta^2+14\beta+18)\geq 0 \text{ for all } \beta\geq -1.4322, \\
\xi_{23}^{(1)}(\lambda_i;\lambda_i;\beta) &= 103680\lambda_i(331\beta^2+1538\beta+1472+16\beta^3)\geq 0 \text{ for all } \beta\geq -1.2958.
\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{22}(\lambda_j;\lambda_i;\beta)=\xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{22} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}
\xi_{22}(\lambda_j;\lambda_i;\beta) &= (6566400\beta^2+35308800\beta+56563200+345600\beta^3)\lambda_j^3\\
&\quad+(-27648000+11404800\beta^2+1036800\beta^3+20044800\beta)\lambda_i\lambda_j^2\\
&\quad+(25436160+10506240\beta+3939840\beta^2+518400\beta^3)\lambda_i^2\lambda_j\\
&\quad+(4032000+34560\beta^3+345600\beta^2+2684160\beta)\lambda_i^3,\\
\xi_{22}^{(1)}(\lambda_j;\lambda_i;\beta) &= (19699200\beta^2+105926400\beta+169689600+1036800\beta^3)\lambda_j^2\\
&\quad+(-55296000+22809600\beta^2+2073600\beta^3+40089600\beta)\lambda_i\lambda_j\\
&\quad+(25436160+10506240\beta+3939840\beta^2+518400\beta^3)\lambda_i^2,\\
\xi_{22}^{(2)}(\lambda_j;\lambda_i;\beta) &= (39398400\beta^2+211852800\beta+339379200+2073600\beta^3)\lambda_j\\
&\quad+(-55296000+22809600\beta^2+2073600\beta^3+40089600\beta)\lambda_i,\\
\xi_{22}^{(3)}(\lambda_j;\lambda_i;\beta) &= 39398400\beta^2+211852800\beta+339379200+2073600\beta^3 \geq 0 \text{ for all } \beta \geq -3.0812.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{22}(\lambda_i;\lambda_i;\beta) &= 161280\lambda_i^3(138\beta^2+425\beta+362+12\beta^3) \geq 0 \text{ for all } \beta \geq -1.4432,\\
\xi_{22}^{(1)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i^2(192\beta^2+647\beta+578+15\beta^3) \geq 0 \text{ for all } \beta \geq -1.4382,\\
\xi_{22}^{(2)}(\lambda_i;\lambda_i;\beta) &= 1036800\lambda_i(60\beta^2+243\beta+274+4\beta^3) \geq 0 \text{ for all } \beta \geq -1.9255.
\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j;\lambda_i;\beta) = \xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{21} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}
\xi_{21}(\lambda_j;\lambda_i;\beta) &= (2808000\beta^2+14846400\beta+26467200+172800\beta^3)\lambda_j^4\\
&\quad+(691200\beta^3+6177600\beta^2-19745280+8576640\beta)\lambda_i\lambda_j^3\\
&\quad+(13409280+5788800\beta+3084480\beta^2+518400\beta^3)\lambda_i^2\lambda_j^2\\
&\quad+(3640320+69120\beta^3+596160\beta^2+3623040\beta)\lambda_i^3\lambda_j\\
&\quad+(34560\beta^2+178560+187200\beta)\lambda_i^4,\\
\xi_{21}^{(1)}(\lambda_j;\lambda_i;\beta) &= (11232000\beta^2+59385600\beta+105868800+691200\beta^3)\lambda_j^3\\
&\quad+(2073600\beta^3+18532800\beta^2-59235840+25729920\beta)\lambda_i\lambda_j^2\\
&\quad+(26818560+11577600\beta+6168960\beta^2+1036800\beta^3)\lambda_i^2\lambda_j\\
&\quad+(3640320+69120\beta^3+596160\beta^2+3623040\beta)\lambda_i^3,\\
\xi_{21}^{(2)}(\lambda_j;\lambda_i;\beta) &= (33696000\beta^2+178156800\beta+317606400+2073600\beta^3)\lambda_j^2\\
&\quad+(4147200\beta^3+37065600\beta^2-118471680+51459840\beta)\lambda_i\lambda_j\\
&\quad+(26818560+11577600\beta+6168960\beta^2+1036800\beta^3)\lambda_i^2,\\
\xi_{21}^{(3)}(\lambda_j;\lambda_i;\beta) &= (67392000\beta^2+356313600\beta+635212800+4147200\beta^3)\lambda_j\\
&\quad+(4147200\beta^3+37065600\beta^2-118471680+51459840\beta)\lambda_i,\\
\xi_{21}^{(4)}(\lambda_j;\lambda_i;\beta) &= 67392000\beta^2+356313600\beta+635212800+4147200\beta^3\\
&\geq 0 \text{ for all } \beta \geq -7.5855.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{21}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^4(\beta+3)(4\beta^2+23\beta+22) \geq 0 \text{ for all } \beta \geq -1.2120,\\
\xi_{21}^{(1)}(\lambda_i;\lambda_i;\beta) &= 80640\lambda_i^3(453\beta^2+1244\beta+956+48\beta^3) \geq 0 \text{ for all } \beta \geq -1.2965,\\
\xi_{21}^{(2)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i^2(318\beta^2+997\beta+934+30\beta^3) \geq 0 \text{ for all } \beta \geq -1.7649,\\
\xi_{21}^{(3)}(\lambda_i;\lambda_i;\beta) &= 67392000\beta^2+356313600\beta+635212800+4147200\beta^3 \geq 0 \text{ for all } \beta \geq -7.5855.
\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j;\lambda_i;\beta) = \xi_2(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{20} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}
\xi_{20}(\lambda_j;\lambda_i;\beta) &= (907200\beta^2+4855680\beta+8985600+60480\beta^3)\lambda_j^4\\
&\quad +(-8121600+2419200\beta^2+302400\beta^3+3196800\beta)\lambda_i\lambda_j^3\\
&\quad +(6981120+2522880\beta+1572480\beta^2+302400\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(2764800+60480\beta^3+483840\beta^2+2592000\beta)\lambda_i^3\lambda_j\\
&\quad +(60480\beta^2+276480+259200\beta)\lambda_i^4,\\
\xi_{20}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3628800\beta^2+19422720\beta+35942400+241920\beta^3)\lambda_j^3\\
&\quad +(-24364800+7257600\beta^2+907200\beta^3+9590400\beta)\lambda_i\lambda_j^2\\
&\quad +(13962240+5045760\beta+3144960\beta^2+604800\beta^3)\lambda_i^2\lambda_j\\
&\quad +(2764800+60480\beta^3+483840\beta^2+2592000\beta)\lambda_i^3,\\
\xi_{20}^{(2)}(\lambda_j;\lambda_i;\beta) &= (10886400\beta^2+58268160\beta+107827200+725760\beta^3)\lambda_j^2\\
&\quad +(-48729600+14515200\beta^2+1814400\beta^3+19180800\beta)\lambda_i\lambda_j\\
&\quad +(13962240+5045760\beta+3144960\beta^2+604800\beta^3)\lambda_i^2,\\
\xi_{20}^{(3)}(\lambda_j;\lambda_i;\beta) &= (21772800\beta^2+116536320\beta+215654400+1451520\beta^3)\lambda_j\\
&\quad +(-48729600+14515200\beta^2+1814400\beta^3+19180800\beta)\lambda_i,\\
\xi_{20}^{(4)}(\lambda_j;\lambda_i;\beta) &= 21772800\beta^2+116536320\beta+215654400+1451520\beta^3\\
&\geq 0 \text{ for all } \beta \geq -4.4853.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{20}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -2,\\
\xi_{20}^{(1)}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^3(\beta+3)(5\beta^2+25\beta+26) > 0 \text{ for all } \beta > -1.4753,\\
\xi_{20}^{(2)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i^2(118\beta^2+341\beta+302+13\beta^3) > 0 \text{ for all } \beta > -1.6897,\\
\xi_{20}^{(3)}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i(100\beta^2+374\beta+460+9\beta^3) > 0 \text{ for all } \beta > -2.7709.
\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &= (403200+576000\beta+43200\beta^2)\lambda_l^4\\
&\quad+(1889280+8640\beta^3+1664640\beta+406080\beta^2)(\lambda_i+\lambda_j)\lambda_l^3\\
&\quad+(4596480+1797120\beta+492480\beta^2+25920\beta^3)(\lambda_j^2+\lambda_i^2)\lambda_l^2\\
&\quad+(77760\beta^3+622080\beta-7464960+1399680\beta^2)\lambda_i\lambda_j\lambda_l^2\\
&\quad+(1036800+1261440\beta+8640\beta^3+129600\beta^2)(\lambda_j^3+\lambda_i^3)\lambda_l\\
&\quad+(777600\beta^2-2142720+1278720\beta+77760\beta^3)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l\\
&\quad+(714240+95040\beta^2+800640\beta+8640\beta^3)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\\
&\quad+(25920\beta^3+181440\beta^2+933120\beta+3421440)\lambda_i^2\lambda_j^2\\
&\quad+(8640\beta^2+115200\beta+80640)(\lambda_j^4+\lambda_i^4),
\end{aligned}$$

and,hence,

$$\begin{aligned}
\xi_3^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (9676800+13824000\beta+1036800\beta^2)\lambda_l\\
&\quad+(11335680+51840\beta^3+9987840\beta+2436480\beta^2)(\lambda_j+\lambda_i)\\
&\geq 0 \text{ for all } \beta\geq -0.7412.
\end{aligned}$$

Define $\xi_{32}(\lambda_j;\lambda_i;\beta)=\xi_3^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{32} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}
\xi_{32}(\lambda_j;\lambda_i;\beta) &= (25367040+20494080\beta+3939840\beta^2+103680\beta^3)\lambda_j^2\\
&\quad+(-3594240+207360\beta^3+11232000\beta+5235840\beta^2)\lambda_i\lambda_j\\
&\quad+(9192960+3594240\beta+984960\beta^2+51840\beta^3)\lambda_i^2,\\
\xi_{32}^{(1)}(\lambda_j;\lambda_i;\beta) &= (50734080+40988160\beta+7879680\beta^2+207360\beta^3)\lambda_j\\
&\quad+(-3594240+207360\beta^3+11232000\beta+5235840\beta^2)\lambda_i,\\
\xi_{32}^{(2)}(\lambda_j;\lambda_i;\beta) &= 50734080+40988160\beta+7879680\beta^2+207360\beta^3\\
&\geq 0 \text{ for all } \beta\geq -1.8910.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i;\lambda_i;\beta) &= 120960\lambda_i^2(256+292\beta+84\beta^2+3\beta^3)\geq 0 \text{ for all } \beta\geq -1.4510,\\
\xi_{32}^{(1)}(\lambda_i;\lambda_i;\beta) &= 17280\lambda_i(2728+3022\beta+759\beta^2+24\beta^3)\geq 0 \text{ for all } \beta\geq -1.3255.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j;\lambda_i;\beta)=\xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{31} is non-negative for

all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j;\lambda_i;\beta) &= (17510400 + 12153600\beta + 2505600\beta^2 + 86400\beta^3)\lambda_j^3 \\
&\quad +(-11404800 + 259200\beta^3 + 7516800\beta + 4795200\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(7050240 + 4872960\beta + 1762560\beta^2 + 129600\beta^3)\lambda_i^2\lambda_j \\
&\quad +(1036800 + 1261440\beta + 8640\beta^3 + 129600\beta^2)\lambda_i^3, \\
\xi_{31}^{(1)}(\lambda_j;\lambda_i;\beta) &= (52531200 + 36460800\beta + 7516800\beta^2 + 259200\beta^3)\lambda_j^2 \\
&\quad +(-22809600 + 518400\beta^3 + 15033600\beta + 9590400\beta^2)\lambda_i\lambda_j \\
&\quad +(7050240 + 4872960\beta + 1762560\beta^2 + 129600\beta^3)\lambda_i^2, \\
\xi_{31}^{(2)}(\lambda_j;\lambda_i;\beta) &= (105062400 + 72921600\beta + 15033600\beta^2 + 518400\beta^3)\lambda_j \\
&\quad +(-22809600 + 518400\beta^3 + 15033600\beta + 9590400\beta^2)\lambda_i, \\
\xi_{31}^{(3)}(\lambda_j;\lambda_i;\beta) &= 105062400 + 72921600\beta + 15033600\beta^2 + 518400\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -23.3467.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{31}(\lambda_i;\lambda_i;\beta) &= 161280\lambda_i^3(88 + 160\beta + 57\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350, \\
\xi_{31}^{(1)}(\lambda_i;\lambda_i;\beta) &= 60480\lambda_i^2(608 + 932\beta + 312\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -0.9275, \\
\xi_{31}^{(2)}(\lambda_i;\lambda_i;\beta) &= 86400\lambda_i(952 + 1018\beta + 285\beta^2 + 12\beta^3) \geq 0 \text{ for all } \beta \geq -1.6188.
\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j;\lambda_i;\beta) = \xi_3(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}
\xi_{30}(\lambda_j;\lambda_i;\beta) &= (8006400 + 5414400\beta + 1080000\beta^2 + 43200\beta^3)\lambda_j^4 \\
&\quad +(172800\beta^3 + 4366080\beta - 7004160 + 2678400\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(5875200 + 4008960\beta + 1451520\beta^2 + 129600\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(1751040 + 224640\beta^2 + 2062080\beta + 17280\beta^3)\lambda_i^3\lambda_j \\
&\quad +(8640\beta^2 + 115200\beta + 80640)\lambda_i^4, \\
\xi_{30}^{(1)}(\lambda_j;\lambda_i;\beta) &= (32025600 + 21657600\beta + 4320000\beta^2 + 172800\beta^3)\lambda_j^3 \\
&\quad +(518400\beta^3 + 13098240\beta - 21012480 + 8035200\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(11750400 + 8017920\beta + 2903040\beta^2 + 259200\beta^3)\lambda_i^2\lambda_j \\
&\quad +(1751040 + 224640\beta^2 + 2062080\beta + 17280\beta^3)\lambda_i^3, \\
\xi_{30}^{(2)}(\lambda_j;\lambda_i;\beta) &= (96076800 + 64972800\beta + 12960000\beta^2 + 518400\beta^3)\lambda_j^2 \\
&\quad +(1036800\beta^3 + 26196480\beta - 42024960 + 16070400\beta^2)\lambda_i\lambda_j \\
&\quad +(11750400 + 8017920\beta + 2903040\beta^2 + 259200\beta^3)\lambda_i^2, \\
\xi_{30}^{(3)}(\lambda_j;\lambda_i;\beta) &= (192153600 + 129945600\beta + 25920000\beta^2 + 1036800\beta^3)\lambda_j \\
&\quad +(1036800\beta^3 + 26196480\beta - 42024960 + 16070400\beta^2)\lambda_i, \\
\xi_{30}^{(4)}(\lambda_j;\lambda_i;\beta) &= 192153600 + 129945600\beta + 25920000\beta^2 + 1036800\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -18.8822.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{30}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^4(\beta+3)(\beta^2+12\beta+8) > 0 \text{ for all } \beta > -0.7085,\\
\xi_{30}^{(1)}(\lambda_i;\lambda_i;\beta) &= 322560\lambda_i^3(76+139\beta+48\beta^2+3\beta^3) > 0 \text{ for all } \beta > -0.7158,\\
\xi_{30}^{(2)}(\lambda_i;\lambda_i;\beta) &= 120960\lambda_i^2(544+820\beta+264\beta^2+15\beta^3) > 0 \text{ for all } \beta > -0.9237,\\
\xi_{30}^{(3)}(\lambda_i;\lambda_i;\beta) &= 103680\lambda_i(1448+1506\beta+405\beta^2+20\beta^3) > 0 \text{ for all } \beta > -1.5818.
\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &= (207360\beta-4147200+933120\beta^2)\lambda_i\lambda_j\lambda_l\\
&\quad +(864000\beta+1175040+311040\beta^2)(\lambda_j^2+\lambda_i^2)\lambda_l\\
&\quad +(864000\beta+1175040+311040\beta^2)(\lambda_i+\lambda_j)\lambda_l^2\\
&\quad +(460800\beta+322560+34560\beta^2)(\lambda_i^3+\lambda_j^3)\\
&\quad +(864000\beta+1175040+311040\beta^2)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\\
&\quad +(460800\beta+322560+34560\beta^2)\lambda_l^3,
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_4^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (2764800\beta+1935360+207360\beta^2)\lambda_l\\
&\quad +(1728000\beta+2350080+622080\beta^2)(\lambda_j+\lambda_i)\\
&\geq 0 \text{ for all } \beta \geq -0.7412.
\end{aligned}$$

Define $\xi_{41}(\lambda_j;\lambda_i;\beta) = \xi_4^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}
\xi_{41}(\lambda_j;\lambda_i;\beta) &= (3974400\beta+4492800+1036800\beta^2)\lambda_j^2\\
&\quad +(1935360\beta-1797120+1555200\beta^2)\lambda_i\lambda_j\\
&\quad +(864000\beta+1175040+311040\beta^2)\lambda_i^2,\\
\xi_{41}^{(1)}(\lambda_j;\lambda_i;\beta) &= (7948800\beta+8985600+2073600\beta^2)\lambda_j\\
&\quad +(1935360\beta-1797120+1555200\beta^2)\lambda_i,\\
\xi_{41}^{(2)}(\lambda_j;\lambda_i;\beta) &= 7948800\beta+8985600+2073600\beta^2 \geq 0 \text{ for all } \beta.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{41}(\lambda_i;\lambda_i;\beta) &= 967680\lambda_i^2(3\beta+4)(\beta+1) \geq 0 \text{ for all } \beta \geq -1,\\
\xi_{41}^{(1)}(\lambda_i;\lambda_i;\beta) &= 34560\lambda_i(105\beta^2+208+286\beta) \geq 0 \text{ for all } \beta.
\end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j;\lambda_i;\beta) = \xi_4(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}
\xi_{40}(\lambda_j;\lambda_i;\beta) &= (2649600\beta + 2995200 + 691200\beta^2)\lambda_j^3 \\
&\quad +(1935360\beta - 1797120 + 1555200\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(1728000\beta + 2350080 + 622080\beta^2)\lambda_i^2\lambda_j \\
&\quad +(460800\beta + 322560 + 34560\beta^2)\lambda_i^3, \\
\xi_{40}^{(1)}(\lambda_j;\lambda_i;\beta) &= (7948800\beta + 8985600 + 2073600\beta^2)\lambda_j^2 \\
&\quad +(3870720\beta - 3594240 + 3110400\beta^2)\lambda_i\lambda_j \\
&\quad +(1728000\beta + 2350080 + 622080\beta^2)\lambda_i^2, \\
\xi_{40}^{(2)}(\lambda_j;\lambda_i;\beta) &= (15897600\beta + 17971200 + 4147200\beta^2)\lambda_j \\
&\quad +(3870720\beta - 3594240 + 3110400\beta^2)\lambda_i, \\
\xi_{40}^{(3)}(\lambda_j;\lambda_i;\beta) &= 15897600\beta + 17971200 + 4147200\beta^2 \geq 0 \text{ for all } \beta.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{40}(\lambda_i;\lambda_i;\beta) &= 967680\lambda_i^3(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{40}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1935360\lambda_i^2(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{40}^{(2)}(\lambda_i;\lambda_i;\beta) &= 69120\lambda_i(105\beta^2 + 208 + 286\beta) > 0 \text{ for all } \beta.
\end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.17 Let

$$\begin{aligned}
\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (345600 + 420480\beta + 43200\beta^2 + 2880\beta^3)t(0,0,3,3) \\
&\quad +(1140480 + 924480\beta + 237600\beta^2 + 25920\beta^3)t(0,1,2,3) \\
&\quad +(152640\beta + 23040 + 90720\beta^2)t(0,1,1,4) \\
&\quad +(77760\beta^3 - 4008960 + 449280\beta + 622080\beta^2)t(1,1,1,3) \\
&\quad +(777600\beta^2 - 829440\beta + 233280\beta^3 - 3317760)t(1,1,2,2) \\
&\quad +(129600\beta + 30240\beta^2 + 138240)t(0,0,2,4) \\
&\quad +(349920\beta^2 + 77760\beta^3 + 5132160 + 1088640\beta)t(0,2,2,2),
\end{aligned}$$

where

$$t(a,b,c,d) = \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in \mathrm{perm}(a,b,c,d)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_k^{n_4}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let
$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 4(30240\beta^2+129600\beta+138240)\lambda_k^3\hat{s}(0,0,2)\\
&\quad +4(152640\beta+23040+90720\beta^2)\lambda_k^3\hat{s}(0,1,1)\\
&\quad +3(77760\beta^3-4008960+449280\beta+622080\beta^2)\lambda_k^2\hat{s}(1,1,1)\\
&\quad +3(2880\beta^3+43200\beta^2+345600+420480\beta)\lambda_k^2\hat{s}(0,0,3)\\
&\quad +3(924480\beta+1140480+237600\beta^2+25920\beta^3)\lambda_k^2\hat{s}(0,1,2)\\
&\quad +2(-3317760+233280\beta^3+777600\beta^2-829440\beta)\lambda_k\hat{s}(1,1,2)\\
&\quad +2(5132160+349920\beta^2+77760\beta^3+1088640\beta)\lambda_k\hat{s}(0,2,2)\\
&\quad +(60480\beta^2+259200\beta+276480)\lambda_k\hat{s}(0,0,4)\\
&\quad +2(924480\beta+1140480+237600\beta^2+25920\beta^3)\lambda_k\hat{s}(0,1,3)\\
&\quad +(-3317760+233280\beta^3+777600\beta^2-829440\beta)\hat{s}(1,2,2)\\
&\quad +(77760\beta^3-4008960+449280\beta+622080\beta^2)\hat{s}(1,1,3)\\
&\quad +(924480\beta+1140480+237600\beta^2+25920\beta^3)\hat{s}(0,2,3)\\
&\quad +(152640\beta+23040+90720\beta^2)\hat{s}(0,1,4),\\
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 12(30240\beta^2+129600\beta+138240)\lambda_k^2\hat{s}(0,0,2)\\
&\quad +12(152640\beta+23040+90720\beta^2)\lambda_k^2\hat{s}(0,1,1)\\
&\quad +6(77760\beta^3-4008960+449280\beta+622080\beta^2)\lambda_k\hat{s}(1,1,1)\\
&\quad +6(2880\beta^3+43200\beta^2+345600+420480\beta)\lambda_k\hat{s}(0,0,3)\\
&\quad +6(924480\beta+1140480+237600\beta^2+25920\beta^3)\lambda_k\hat{s}(0,1,2)\\
&\quad +2(-3317760+233280\beta^3+777600\beta^2-829440\beta)\hat{s}(1,1,2)\\
&\quad +2(5132160+349920\beta^2+77760\beta^3+1088640\beta)\hat{s}(0,2,2)\\
&\quad +(60480\beta^2+259200\beta+276480)\hat{s}(0,0,4)\\
&\quad +2(924480\beta+1140480+237600\beta^2+25920\beta^3)\hat{s}(0,1,3),\\
\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(30240\beta^2+129600\beta+138240)\lambda_k\hat{s}(0,0,2)\\
&\quad +24(152640\beta+23040+90720\beta^2)\lambda_k\hat{s}(0,1,1)\\
&\quad +6(77760\beta^3-4008960+449280\beta+622080\beta^2)\hat{s}(1,1,1)\\
&\quad +6(2880\beta^3+43200\beta^2+345600+420480\beta)\hat{s}(0,0,3)\\
&\quad +6(924480\beta+1140480+237600\beta^2+25920\beta^3)\hat{s}(0,1,2),\\
\xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 24(30240\beta^2+129600\beta+138240)\hat{s}(0,0,2)\\
&\quad +24(152640\beta+23040+90720\beta^2)\hat{s}(0,1,1).
\end{aligned}$$

Define

$$
\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta).
\end{aligned}
$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$
\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) = {} & (2880\beta^3 + 103680\beta^2 + 622080 + 679680\beta)\lambda_l^6\\
& +(2154240\beta + 2327040 + 656640\beta^2 + 51840\beta^3)(\lambda_i + \lambda_j)\lambda_l^5\\
& +(3196800\beta + 7689600 + 885600\beta^2 + 129600\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_l^4\\
& +(388800\beta^3 - 11289600 + 374400\beta + 2203200\beta^2)\lambda_i\lambda_j)\lambda_l^4\\
& +(190080\beta - 4354560 + 2030400\beta^2 + 518400\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^3\\
& +(57600\beta^3 + 561600\beta^2 + 2972160 + 2689920\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^3\\
& +(129600\beta^3 - 1728000 + 2298240\beta + 1097280\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^2\\
& +(151200\beta^2 + 411840\beta + 299520)(\lambda_j^4 + \lambda_i^4)\lambda_l^2\\
& +(6946560 + 1477440\beta^2 + 388800\beta^3 + 1347840\beta)\lambda_i^2\lambda_j^2\lambda_l^2\\
& +2(924480\beta + 1140480 + 237600\beta^2 + 25920\beta^3)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l\\
& +2(152640\beta + 23040 + 90720\beta^2)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l\\
& +(2880\beta^3 + 43200\beta^2 + 345600 + 420480\beta)\lambda_i^3\lambda_j^3\\
& +(30240\beta^2 + 129600\beta + 138240)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2),
\end{aligned}
$$

and, hence,

$$
\begin{aligned}
\xi_0^{(5)}(\lambda_l;\lambda_j;\lambda_i;\beta) = {} & (2073600\beta^3 + 74649600\beta^2 + 447897600 + 489369600\beta)\lambda_l\\
& +(258508800\beta + 279244800 + 78796800\beta^2 + 6220800\beta^3)(\lambda_j + \lambda_i)\\
\geq {} & 0 \text{ for all } \beta \geq -1.0915.
\end{aligned}
$$

Define $\xi_{04}(\lambda_j;\lambda_i;\beta) = \xi_0^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{04} is non-negative for all

$\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned}
\xi_{04}(\lambda_j;\lambda_i;\beta) &= (10368000\beta^3 + 137376000\beta^2 + 687744000 + 579916800\beta)\lambda_j^2 \\
&\quad +(267494400\beta + 8294400 + 131673600\beta^2 + 15552000\beta^3)\lambda_i\lambda_j \\
&\quad +(76723200\beta + 184550400 + 21254400\beta^2 + 3110400\beta^3)\lambda_i^2, \\
\xi_{04}^{(1)}(\lambda_j;\lambda_i;\beta) &= (20736000\beta^3 + 274752000\beta^2 + 1375488000 + 1159833600\beta)\lambda_j \\
&\quad +(267494400\beta + 8294400 + 131673600\beta^2 + 15552000\beta^3)\lambda_i, \\
\xi_{04}^{(2)}(\lambda_j;\lambda_i;\beta) &= 20736000\beta^3 + 274752000\beta^2 + 1375488000 + 1159833600\beta \\
&\geq 0 \text{ for all } \beta \geq -1.9649.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{04}(\lambda_i;\lambda_i;\beta) &= 4838400\lambda_i^2(6\beta^3 + 60\beta^2 + 182 + 191\beta) \geq 0 \text{ for all } \beta \geq -1.7321, \\
\xi_{04}^{(1)}(\lambda_i;\lambda_i;\beta) &= 2419200\lambda_i(15\beta^3 + 168\beta^2 + 572 + 590\beta) \geq 0 \text{ for all } \beta \geq -1.5802.
\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j;\lambda_i;\beta) = \xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{03} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}
\xi_{03}(\lambda_j;\lambda_i;\beta) &= (6912000\beta^3 + 76464000\beta^2 + 416655360 + 303678720\beta)\lambda_j^3 \\
&\quad +(15552000\beta^3 - 157455360 + 139380480\beta + 104457600\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(77863680\beta + 158423040 + 33436800\beta^2 + 6220800\beta^3)\lambda_i^2\lambda_j \\
&\quad +(345600\beta^3 + 3369600\beta^2 + 17832960 + 16139520\beta)\lambda_i^3, \\
\xi_{03}^{(1)}(\lambda_j;\lambda_i;\beta) &= (20736000\beta^3 + 229392000\beta^2 + 1249966080 + 911036160\beta)\lambda_j^2 \\
&\quad +(31104000\beta^3 - 314910720 + 278760960\beta + 208915200\beta^2)\lambda_i\lambda_j \\
&\quad +(77863680\beta + 158423040 + 33436800\beta^2 + 6220800\beta^3)\lambda_i^2, \\
\xi_{03}^{(2)}(\lambda_j;\lambda_i;\beta) &= (41472000\beta^3 + 458784000\beta^2 + 2499932160 + 1822072320\beta)\lambda_j \\
&\quad +(31104000\beta^3 - 314910720 + 278760960\beta + 208915200\beta^2)\lambda_i, \\
\xi_{03}^{(3)}(\lambda_j;\lambda_i;\beta) &= 41472000\beta^3 + 458784000\beta^2 + 2499932160 + 1822072320\beta \\
&\geq 0 \text{ for all } \beta \geq -3.2529.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{03}(\lambda_i;\lambda_i;\beta) &= 14515200\lambda_i^3(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{03}^{(1)}(\lambda_i;\lambda_i;\beta) &= 2419200\lambda_i^2(24\beta^3 + 195\beta^2 + 452 + 524\beta) \geq 0 \text{ for all } \beta \geq -1.8116, \\
\xi_{03}^{(2)}(\lambda_i;\lambda_i;\beta) &= 967680\lambda_i(75\beta^3 + 690\beta^2 + 2258 + 2171\beta) \geq 0 \text{ for all } \beta \geq -2.3123.
\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j;\lambda_i;\beta) = \xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$
\begin{aligned}
\xi_{02}(\lambda_j;\lambda_i;\beta) &= (3024000\beta^3 + 30542400\beta^2 + 175910400 + 118800000\beta)\lambda_j^4 \\
&\quad +(9072000\beta^3 - 118517760 + 53314560\beta + 53948160\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(42197760\beta + 80040960 + 25764480\beta^2 + 5443200\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(604800\beta^3 + 14376960 + 20736000\beta + 5564160\beta^2)\lambda_i^3\lambda_j \\
&\quad +(302400\beta^2 + 823680\beta + 599040)\lambda_i^4, \\
\xi_{02}^{(1)}(\lambda_j;\lambda_i;\beta) &= (12096000\beta^3 + 122169600\beta^2 + 703641600 + 475200000\beta)\lambda_j^3 \\
&\quad +(27216000\beta^3 - 355553280 + 159943680\beta + 161844480\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(84395520\beta + 160081920 + 51528960\beta^2 + 10886400\beta^3)\lambda_i^2\lambda_j \\
&\quad +(604800\beta^3 + 14376960 + 20736000\beta + 5564160\beta^2)\lambda_i^3, \\
\xi_{02}^{(2)}(\lambda_j;\lambda_i;\beta) &= (36288000\beta^3 + 366508800\beta^2 + 2110924800 + 1425600000\beta)\lambda_j^2 \\
&\quad +(54432000\beta^3 - 711106560 + 319887360\beta + 323688960\beta^2)\lambda_i\lambda_j \\
&\quad +(84395520\beta + 160081920 + 51528960\beta^2 + 10886400\beta^3)\lambda_i^2, \\
\xi_{02}^{(3)}(\lambda_j;\lambda_i;\beta) &= (72576000\beta^3 + 733017600\beta^2 + 4221849600 + 2851200000\beta)\lambda_j \\
&\quad +(54432000\beta^3 - 711106560 + 319887360\beta + 323688960\beta^2)\lambda_i, \\
\xi_{02}^{(4)}(\lambda_j;\lambda_i;\beta) &= 72576000\beta^3 + 733017600\beta^2 + 4221849600 + 2851200000\beta \\
&\geq 0 \text{ for all } \beta \geq -3.7755.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{02}(\lambda_i;\lambda_i;\beta) &= 3628800\lambda_i^4(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\
\xi_{02}^{(1)}(\lambda_i;\lambda_i;\beta) &= 7257600\lambda_i^3(\beta+3)(\beta+2)(7\beta+12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}, \\
\xi_{02}^{(2)}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i^2(210\beta^3 + 1533\beta^2 + 3224 + 3782\beta) \geq 0 \text{ for all } \beta \geq -3.0192, \\
\xi_{02}^{(3)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i(525\beta^3 + 4368\beta^2 + 14512 + 13108\beta) \geq 0 \text{ for all } \beta \geq -3.2688.
\end{aligned}
$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{01} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$
\begin{aligned}
\xi_{01}(\lambda_j;\lambda_i;\beta) &= (967680\beta^3 + 9434880\beta^2 + 55641600 + 36529920\beta)\lambda_j^4 \\
&\quad +(3628800\beta^3 - 49996800 + 17740800\beta + 20563200\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(17902080\beta + 33868800 + 13063680\beta^2 + 2903040\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(483840\beta^3 + 4354560\beta^2 + 7741440 + 14515200\beta)\lambda_i^3\lambda_j \\
&\quad +(483840\beta^2 + 1128960\beta + 645120)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3870720\beta^3 + 37739520\beta^2 + 222566400 + 146119680\beta)\lambda_j^3 \\
&\quad +(10886400\beta^3 - 149990400 + 53222400\beta + 61689600\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(35804160\beta + 67737600 + 26127360\beta^2 + 5806080\beta^3)\lambda_i^2\lambda_j \\
&\quad +(483840\beta^3 + 4354560\beta^2 + 7741440 + 14515200\beta)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= (11612160\beta^3 + 113218560\beta^2 + 667699200 + 438359040\beta)\lambda_j^2 \\
&\quad +(21772800\beta^3 - 299980800 + 106444800\beta + 123379200\beta^2)\lambda_i\lambda_j \\
&\quad +(35804160\beta + 67737600 + 26127360\beta^2 + 5806080\beta^3)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j;\lambda_i;\beta) &= (23224320\beta^3 + 226437120\beta^2 + 1335398400 + 876718080\beta)\lambda_j \\
&\quad +(21772800\beta^3 - 299980800 + 106444800\beta + 123379200\beta^2)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j;\lambda_i;\beta) &= 23224320\beta^3 + 226437120\beta^2 + 1335398400 + 876718080\beta \\
&\geq 0 \text{ for all } \beta \geq -3.7954.
\end{aligned}
$$

Now

$$
\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 7983360\lambda_i^4(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 725760\lambda_i^3(\beta+3)(\beta+2)(29\beta+34) \geq 0 \text{ for all } \beta \geq -\frac{34}{29}, \\
\xi_{01}^{(2)}(\lambda_i;\lambda_i;\beta) &= 1451520\lambda_i^2(\beta+3)(27\beta^2+100\beta+100) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i(93\beta^3 + 723\beta^2 + 2140 + 2032\beta) \geq 0 \text{ for all } \beta \geq -3.1755.
\end{aligned}
$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j^2$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= (241920\beta^3+2358720\beta^2+13910400+9132480\beta)\lambda_j^4 \\
&\quad +(1088640\beta^3-14999040+5322240\beta+6168960\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(6713280\beta+12700800+4898880\beta^2+1088640\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(241920\beta^3+2177280\beta^2+3870720+7257600\beta)\lambda_i^3\lambda_j \\
&\quad +(362880\beta^2+846720\beta+483840)\lambda_i^4, \\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= (967680\beta^3+9434880\beta^2+55641600+36529920\beta)\lambda_j^3 \\
&\quad +(3265920\beta^3-44997120+15966720\beta+18506880\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(13426560\beta+25401600+9797760\beta^2+2177280\beta^3)\lambda_i^2\lambda_j \\
&\quad +(241920\beta^3+2177280\beta^2+3870720+7257600\beta)\lambda_i^3, \\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= (2903040\beta^3+28304640\beta^2+166924800+109589760\beta)\lambda_j^2 \\
&\quad +(6531840\beta^3-89994240+31933440\beta+37013760\beta^2)\lambda_i\lambda_j \\
&\quad +(13426560\beta+25401600+9797760\beta^2+2177280\beta^3)\lambda_i^2, \\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= (5806080\beta^3+56609280\beta^2+333849600+219179520\beta)\lambda_j \\
&\quad +(6531840\beta^3-89994240+31933440\beta+37013760\beta^2)\lambda_i, \\
\xi_{00}^{(4)}(\lambda_j;\lambda_i;\beta) &= 5806080\beta^3+56609280\beta^2+333849600+219179520\beta \\
&\geq 0 \text{ for all } \beta \geq -3.7954.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 2661120\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 6652800\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^2(\beta+3)(\beta+2)(32\beta+47) > 0 \text{ for all } \beta > -\frac{47}{32}, \\
\xi_{00}^{(3)}(\lambda_i;\lambda_i;\beta) &= 725760\lambda_i(\beta+3)(17\beta^2+78\beta+112) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (311040\beta^2+2039040\beta+1866240+8640\beta^3)\lambda_l^5\\
&\quad +(5385600\beta+5817600+1641600\beta^2+129600\beta^3)(\lambda_j+\lambda_i)\lambda_l^4\\
&\quad +(6393600\beta+15379200+1771200\beta^2+259200\beta^3)(\lambda_j^2+\lambda_i^2)\lambda_l^3\\
&\quad +(777600\beta^3-22579200+748800\beta+4406400\beta^2)\lambda_i\lambda_j\lambda_l^3\\
&\quad +(285120\beta-6531840+3045600\beta^2+777600\beta^3)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^2\\
&\quad +(4034880\beta+4458240+842400\beta^2+86400\beta^3)(\lambda_j^3+\lambda_i^3)\lambda_l^2\\
&\quad +(129600\beta^3-1728000+2298240\beta+1097280\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l\\
&\quad +(151200\beta^2+411840\beta+299520)\lambda_j^4+\lambda_i^4)\lambda_l\\
&\quad +(6946560+1477440\beta^2+388800\beta^3+1347840\beta)\lambda_i^2\lambda_j^2\lambda_l\\
&\quad +(152640\beta+23040+90720\beta^2)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j)\\
&\quad +(924480\beta+1140480+237600\beta^2+25920\beta^3)(\lambda_i^3\lambda_j^2+\lambda_i^2\lambda_j^3),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_1^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (37324800\beta^2+244684800\beta+223948800+1036800\beta^3)\lambda_l\\
&\quad +(129254400\beta+139622400+39398400\beta^2+3110400\beta^3)(\lambda_i+\lambda_j)\\
&\geq 0 \text{ for all } \beta\geq -1.0915.
\end{aligned}$$

Define $\xi_{13}(\lambda_j;\lambda_i;\beta)=\xi_1^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{13} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}
\xi_{13}(\lambda_j;\lambda_i;\beta) &= (68688000\beta^2+289958400\beta+343872000+5184000\beta^3)\lambda_j^2\\
&\quad +(133747200\beta+4147200+65836800\beta^2+7776000\beta^3)\lambda_i\lambda_j\\
&\quad +(38361600\beta+92275200+10627200\beta^2+1555200\beta^3)\lambda_i^2,\\
\xi_{13}^{(1)}(\lambda_j;\lambda_i;\beta) &= (137376000\beta^2+579916800\beta+687744000+10368000\beta^3)\lambda_j\\
&\quad +(133747200\beta+4147200+65836800\beta^2+7776000\beta^3)\lambda_i,\\
\xi_{13}^{(2)}(\lambda_j;\lambda_i;\beta) &= 137376000\beta^2+579916800\beta+687744000+10368000\beta^3\\
&\geq 0 \text{ for all } \beta\geq -1.9649.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{13}(\lambda_i;\lambda_i;\beta) &= 2419200\lambda_i^2(60\beta^2+191\beta+182+6\beta^3)\geq 0 \text{ for all } \beta\geq -1.7321,\\
\xi_{13}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1209600\lambda_i(168\beta^2+590\beta+572+15\beta^3)\geq 0 \text{ for all } \beta\geq -1.5802.
\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j;\lambda_i;\beta)=\xi_1^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{12} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned}
\xi_{12}(\lambda_j;\lambda_i;\beta) &= (38232000\beta^2+151839360\beta+208327680+3456000\beta^3)\lambda_j^3\\
&\quad +(7776000\beta^3-78727680+69690240\beta+52228800\beta^2)\lambda_i\lambda_j^2\\
&\quad +(38931840\beta+79211520+16718400\beta^2+3110400\beta^3)\lambda_i^2\lambda_j\\
&\quad +(8069760\beta+8916480+1684800\beta^2+172800\beta^3)\lambda_i^3,\\
\xi_{12}^{(1)}(\lambda_j;\lambda_i;\beta) &= (114696000\beta^2+455518080\beta+624983040+10368000\beta^3)\lambda_j^2\\
&\quad +(15552000\beta^3-157455360+139380480\beta+104457600\beta^2)\lambda_i\lambda_j\\
&\quad +(38931840\beta+79211520+16718400\beta^2+3110400\beta^3)\lambda_i^2,\\
\xi_{12}^{(2)}(\lambda_j;\lambda_i;\beta) &= (229392000\beta^2+911036160\beta+1249966080+20736000\beta^3)\lambda_j\\
&\quad +(15552000\beta^3-157455360+139380480\beta+104457600\beta^2)\lambda_i,\\
\xi_{12}^{(3)}(\lambda_j;\lambda_i;\beta) &= 229392000\beta^2+911036160\beta+1249966080+20736000\beta^3\\
&\geq 0 \text{ for all } \beta \geq -3.2529.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{12}(\lambda_i;\lambda_i;\beta) &= 7257600\lambda_i^3(2\beta+5)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2,\\
\xi_{12}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1209600\lambda_i^2(195\beta^2+524\beta+452+24\beta^3) \geq 0 \text{ for all } \beta \geq -1.8116,\\
\xi_{12}^{(2)}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i(690\beta^2+2171\beta+2258+75\beta^3) \geq 0 \text{ for all } \beta \geq -2.3123.
\end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j;\lambda_i;\beta) = \xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j;\lambda_i;\beta) &= (15271200\beta^2+59400000\beta+87955200+1512000\beta^3)\lambda_j^4\\
&\quad +(4536000\beta^3-59258880+26657280\beta+26974080\beta^2)\lambda_i\lambda_j^3\\
&\quad +(21098880\beta+40020480+12882240\beta^2+2721600\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(10368000\beta+7188480+2782080\beta^2+302400\beta^3)\lambda_i^3\lambda_j\\
&\quad +(151200\beta^2+411840\beta+299520)\lambda_i^4,\\
\xi_{11}^{(1)}(\lambda_j;\lambda_i;\beta) &= (61084800\beta^2+237600000\beta+351820800+6048000\beta^3)\lambda_j^3\\
&\quad +(13608000\beta^3-177776640+79971840\beta+80922240\beta^2)\lambda_i\lambda_j^2\\
&\quad +(42197760\beta+80040960+25764480\beta^2+5443200\beta^3)\lambda_i^2\lambda_j\\
&\quad +(10368000\beta+7188480+2782080\beta^2+302400\beta^3)\lambda_i^3,\\
\xi_{11}^{(2)}(\lambda_j;\lambda_i;\beta) &= (183254400\beta^2+712800000\beta+1055462400+18144000\beta^3)\lambda_j^2\\
&\quad +(27216000\beta^3-355553280+159943680\beta+161844480\beta^2)\lambda_i\lambda_j\\
&\quad +(42197760\beta+80040960+25764480\beta^2+5443200\beta^3)\lambda_i^2,\\
\xi_{11}^{(3)}(\lambda_j;\lambda_i;\beta) &= (366508800\beta^2+1425600000\beta+2110924800+36288000\beta^3)\lambda_j\\
&\quad +(27216000\beta^3-355553280+159943680\beta+161844480\beta^2)\lambda_i,\\
\xi_{11}^{(4)}(\lambda_j;\lambda_i;\beta) &= 366508800\beta^2+1425600000\beta+2110924800+36288000\beta^3\\
&\geq 0 \text{ for all } \beta \geq -3.7755.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i;\lambda_i;\beta) &= 1814400\lambda_i^4(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5},\\
\xi_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= 3628800\lambda_i^3(\beta+3)(\beta+2)(7\beta+12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7},\\
\xi_{11}^{(2)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i^2(1533\beta^2+3782\beta+3224+210\beta^3) \geq 0 \text{ for all } \beta \geq -3.0192,\\
\xi_{11}^{(3)}(\lambda_i;\lambda_i;\beta) &= 120960\lambda_i(4368\beta^2+13108\beta+14512+525\beta^3) \geq 0 \text{ for all } \beta \geq -3.2688.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j;\lambda_i;\beta) = \xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= (4717440\beta^2+18264960\beta+27820800+483840\beta^3)\lambda_j^4\\
&\quad +(1814400\beta^3-24998400+8870400\beta+10281600\beta^2)\lambda_i\lambda_j^3\\
&\quad +(8951040\beta+16934400+6531840\beta^2+1451520\beta^3)\lambda_i^2\lambda_j^2\\
&\quad +(7257600\beta+3870720+2177280\beta^2+241920\beta^3)\lambda_i^3\lambda_j\\
&\quad +(564480\beta+322560+241920\beta^2)\lambda_i^4,\\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= (18869760\beta^2+73059840\beta+111283200+1935360\beta^3)\lambda_j^3\\
&\quad +(5443200\beta^3-74995200+26611200\beta+30844800\beta^2)\lambda_i\lambda_j^2\\
&\quad +(17902080\beta+33868800+13063680\beta^2+2903040\beta^3)\lambda_i^2\lambda_j\\
&\quad +(7257600\beta+3870720+2177280\beta^2+241920\beta^3)\lambda_i^3,\\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= (56609280\beta^2+219179520\beta+333849600+5806080\beta^3)\lambda_j^2\\
&\quad +(10886400\beta^3-149990400+53222400\beta+61689600\beta^2)\lambda_i\lambda_j\\
&\quad +(17902080\beta+33868800+13063680\beta^2+2903040\beta^3)\lambda_i^2,\\
\xi_{10}^{(3)}(\lambda_j;\lambda_i;\beta) &= (113218560\beta^2+438359040\beta+667699200+11612160\beta^3)\lambda_j\\
&\quad +(10886400\beta^3-149990400+53222400\beta+61689600\beta^2)\lambda_i,\\
\xi_{10}^{(4)}(\lambda_j;\lambda_i;\beta) &= 113218560\beta^2+438359040\beta+667699200+11612160\beta^3\\
&\geq 0 \text{ for all } \beta \geq -3.7954.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 3991680\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1,\\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 362880\lambda_i^3(\beta+3)(\beta+2)(29\beta+34) > 0 \text{ for all } \beta > -\frac{34}{29},\\
\xi_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= 725760\lambda_i^2(\beta+3)(27\beta^2+100\beta+100) > 0 \text{ for all } \beta > -3,\\
\xi_{10}^{(3)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i(723\beta^2+2032\beta+2140+93\beta^3) > 0 \text{ for all } \beta > -3.1755.
\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= (682560\beta^2+4337280\beta+4008960+17280\beta^3)\lambda_l^4 \\
&\quad +(9227520\beta+9400320+2989440\beta^2+207360\beta^3)(\lambda_i+\lambda_j)\lambda_l^3 \\
&\quad +(2488320\beta^2+9279360\beta+18766080+311040\beta^3)(\lambda_j^2+\lambda_i^2)\lambda_l^2 \\
&\quad +(-30412800+933120\beta^3+6376320\beta^2+2868480\beta)\lambda_i\lambda_j\lambda_l^2 \\
&\quad +(4371840\beta+4354560+734400\beta^2+69120\beta^3)(\lambda_j^3+\lambda_i^3)\lambda_l \\
&\quad +(207360+622080\beta^3+2980800\beta^2+3888000\beta)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l \\
&\quad +(1848960\beta+2280960+475200\beta^2+51840\beta^3)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j) \\
&\quad +(10264320+699840\beta^2+155520\beta^3+2177280\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(60480\beta^2+259200\beta+276480)(\lambda_j^4+\lambda_i^4),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_2^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (16381440\beta^2+104094720\beta+96215040+414720\beta^3)\lambda_l \\
&\quad +(55365120\beta+56401920+17936640\beta^2+1244160\beta^3)(\lambda_j+\lambda_i) \\
&\geq 0 \text{ for all } \beta \geq -1.1141.
\end{aligned}$$

Define $\xi_{22}(\lambda_j;\lambda_i;\beta)=\xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}
\xi_{22}(\lambda_j;\lambda_i;\beta) &= (31104000\beta^2+125971200\beta+142041600+2073600\beta^3)\lambda_j^2 \\
&\quad +(61102080\beta-4423680+30689280\beta^2+3110400\beta^3)\lambda_i\lambda_j \\
&\quad +(4976640\beta^2+18558720\beta+37532160+622080\beta^3)\lambda_i^2, \\
\xi_{22}^{(1)}(\lambda_j;\lambda_i;\beta) &= (62208000\beta^2+251942400\beta+284083200+4147200\beta^3)\lambda_j \\
&\quad +(61102080\beta-4423680+30689280\beta^2+3110400\beta^3)\lambda_i, \\
\xi_{22}^{(2)}(\lambda_j;\lambda_i;\beta) &= 62208000\beta^2+251942400\beta+284083200+4147200\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -1.9255.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{22}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i^2(138\beta^2+425\beta+362+12\beta^3) \geq 0 \text{ for all } \beta \geq -1.4432, \\
\xi_{22}^{(1)}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i(192\beta^2+647\beta+578+15\beta^3) \geq 0 \text{ for all } \beta \geq -1.4382, \\
\xi_{22}^{(2)}(\lambda_i;\lambda_i;\beta) &= 62208000\beta^2+251942400\beta+284083200+4147200\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -1.9255.
\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j;\lambda_i;\beta)=\xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{21} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}
\xi_{21}(\lambda_j;\lambda_i;\beta) &= (17409600\beta^2 + 67962240\beta + 86123520 + 1382400\beta^3)\lambda_j^3 \\
&\quad +(37307520\beta - 32417280 + 24701760\beta^2 + 3110400\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(7957440\beta^2 + 22446720\beta + 37739520 + 1244160\beta^3)\lambda_i^2\lambda_j \\
&\quad +(4371840\beta + 4354560 + 734400\beta^2 + 69120\beta^3)\lambda_i^3, \\
\xi_{21}^{(1)}(\lambda_j;\lambda_i;\beta) &= (52228800\beta^2 + 203886720\beta + 258370560 + 4147200\beta^3)\lambda_j^2 \\
&\quad +(74615040\beta - 64834560 + 49403520\beta^2 + 6220800\beta^3)\lambda_i\lambda_j \\
&\quad +(7957440\beta^2 + 22446720\beta + 37739520 + 1244160\beta^3)\lambda_i^2, \\
\xi_{21}^{(2)}(\lambda_j;\lambda_i;\beta) &= (104457600\beta^2 + 407773440\beta + 516741120 + 8294400\beta^3)\lambda_j \\
&\quad +(74615040\beta - 64834560 + 49403520\beta^2 + 6220800\beta^3)\lambda_i, \\
\xi_{21}^{(3)}(\lambda_j;\lambda_i;\beta) &= 104457600\beta^2 + 407773440\beta + 516741120 + 8294400\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -6.5166.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{21}(\lambda_i;\lambda_i;\beta) &= 1451520\lambda_i^3(\beta+3)(4\beta^2+23\beta+22) \geq 0 \text{ for all } \beta \geq -1.2120, \\
\xi_{21}^{(1)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i^2(453\beta^2 + 1244\beta + 956 + 48\beta^3) \geq 0 \text{ for all } \beta \geq -1.2965, \\
\xi_{21}^{(2)}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i(318\beta^2 + 997\beta + 934 + 30\beta^3) \geq 0 \text{ for all } \beta \geq -1.7649.
\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j;\lambda_i;\beta) = \xi_2(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{20} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}
\xi_{20}(\lambda_j;\lambda_i;\beta) &= (6955200\beta^2 + 27475200\beta + 36806400 + 604800\beta^3)\lambda_j^4 \\
&\quad +(-18524160 + 1814400\beta^3 + 12821760\beta^2 + 17832960\beta)\lambda_i\lambda_j^3 \\
&\quad +(6168960\beta^2 + 15344640\beta + 29237760 + 1088640\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(6220800\beta + 6635520 + 1209600\beta^2 + 120960\beta^3)\lambda_i^3\lambda_j \\
&\quad +(60480\beta^2 + 259200\beta + 276480)\lambda_i^4, \\
\xi_{20}^{(1)}(\lambda_j;\lambda_i;\beta) &= (27820800\beta^2 + 109900800\beta + 147225600 + 2419200\beta^3)\lambda_j^3 \\
&\quad +(-55572480 + 5443200\beta^3 + 38465280\beta^2 + 53498880\beta)\lambda_i\lambda_j^2 \\
&\quad +(12337920\beta^2 + 30689280\beta + 58475520 + 2177280\beta^3)\lambda_i^2\lambda_j \\
&\quad +(6220800\beta + 6635520 + 1209600\beta^2 + 120960\beta^3)\lambda_i^3, \\
\xi_{20}^{(2)}(\lambda_j;\lambda_i;\beta) &= (83462400\beta^2 + 329702400\beta + 441676800 + 7257600\beta^3)\lambda_j^2 \\
&\quad +(-111144960 + 10886400\beta^3 + 76930560\beta^2 + 106997760\beta)\lambda_i\lambda_j \\
&\quad +(12337920\beta^2 + 30689280\beta + 58475520 + 2177280\beta^3)\lambda_i^2, \\
\xi_{20}^{(3)}(\lambda_j;\lambda_i;\beta) &= (166924800\beta^2 + 659404800\beta + 883353600 + 14515200\beta^3)\lambda_j \\
&\quad +(-111144960 + 10886400\beta^3 + 76930560\beta^2 + 106997760\beta)\lambda_i, \\
\xi_{20}^{(4)}(\lambda_j;\lambda_i;\beta) &= 166924800\beta^2 + 659404800\beta + 883353600 + 14515200\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -3.4191.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{20}(\lambda_i;\lambda_i;\beta) &= 1814400\lambda_i^4(2\beta+5)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -2,\\
\xi_{20}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1451520\lambda_i^3(\beta+3)(7\beta^2+34\beta+36) > 0 \text{ for all } \beta > -1.5596,\\
\xi_{20}^{(2)}(\lambda_i;\lambda_i;\beta) &= 1451520\lambda_i^2(119\beta^2+322\beta+268+14\beta^3) > 0 \text{ for all } \beta > -1.6020,\\
\xi_{20}^{(3)}(\lambda_i;\lambda_i;\beta) &= 725760\lambda_i(336\beta^2+1056\beta+1064+35\beta^3) > 0 \text{ for all } \beta > -2.1389.
\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &= (984960\beta^2+5633280\beta+5391360+17280\beta^3)\lambda_l^3\\
&\quad +(9210240\beta+7395840+3602880\beta^2+155520\beta^3)(\lambda_j+\lambda_i)\lambda_l^2\\
&\quad +(2151360\beta^2+8657280\beta+10160640+155520\beta^3)(\lambda_j^2+\lambda_i^2)\lambda_l\\
&\quad +(6359040\beta-23500800+5909760\beta^2+466560\beta^3)\lambda_i\lambda_j\lambda_l\\
&\quad +(17280\beta^3+259200\beta^2+2073600+2522880\beta)(\lambda_j^3+\lambda_i^3)\\
&\quad +(5546880\beta+6842880+1425600\beta^2+155520\beta^3)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j),\\
\xi_3^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (2954880\beta^2+16899840\beta+16174080+51840\beta^3)\lambda_l^2\\
&\quad +((18420480\beta+14791680+7205760\beta^2+311040\beta^3)\lambda_j\\
&\quad +2(9210240\beta+7395840+3602880\beta^2+155520\beta^3)\lambda_i)\lambda_l\\
&\quad +(6359040\beta-23500800+5909760\beta^2+466560\beta^3)\lambda_i\lambda_j\\
&\quad +(2151360\beta^2+8657280\beta+10160640+155520\beta^3)\lambda_j^2\\
&\quad +(2151360\beta^2+8657280\beta+10160640+155520\beta^3)\lambda_i^2,\\
\xi_3^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (5909760\beta^2+33799680\beta+32348160+103680\beta^3)\lambda_l\\
&\quad +(18420480\beta+14791680+7205760\beta^2+311040\beta^3)(\lambda_i+\lambda_j)\\
&\geq 0 \text{ for all } \beta \geq -1.2060.
\end{aligned}$$

Define $\xi_{31}(\lambda_j;\lambda_i;\beta) = \xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{31} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j;\lambda_i;\beta) &= (12312000\beta^2+43977600\beta+41126400+518400\beta^3)\lambda_j^2\\
&\quad +(24779520\beta-8709120+13115520\beta^2+777600\beta^3)\lambda_i\lambda_j\\
&\quad +(2151360\beta^2+8657280\beta+10160640+155520\beta^3)\lambda_i^2,\\
\xi_{31}^{(1)}(\lambda_j;\lambda_i;\beta) &= (24624000\beta^2+87955200\beta+82252800+1036800\beta^3)\lambda_j\\
&\quad +(24779520\beta-8709120+13115520\beta^2+777600\beta^3)\lambda_i,\\
\xi_{31}^{(2)}(\lambda_j;\lambda_i;\beta) &= 24624000\beta^2+87955200\beta+82252800+1036800\beta^3 \geq 0 \text{ for all } \beta \geq -1.6188.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{31}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i^2(57\beta^2+160\beta+88+3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350,\\
\xi_{31}^{(1)}(\lambda_i;\lambda_i;\beta) &= 120960\lambda_i(312\beta^2+932\beta+608+15\beta^3) \geq 0 \text{ for all } \beta \geq -0.9275.
\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j;\lambda_i;\beta) = \xi_3(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}
\xi_{30}(\lambda_j;\lambda_i;\beta) &= (6998400\beta^2 + 26023680\beta + 25021440 + 345600\beta^3)\lambda_j^3 \\
&\quad +(21116160\beta - 9262080 + 10938240\beta^2 + 777600\beta^3)\lambda_i\lambda_j^2 \\
&\quad +(3576960\beta^2 + 14204160\beta + 17003520 + 311040\beta^3)\lambda_i^2\lambda_j \\
&\quad +(17280\beta^3 + 259200\beta^2 + 2073600 + 2522880\beta)\lambda_i^3, \\
\xi_{30}^{(1)}(\lambda_j;\lambda_i;\beta) &= (20995200\beta^2 + 78071040\beta + 75064320 + 1036800\beta^3)\lambda_j^2 \\
&\quad +(42232320\beta - 18524160 + 21876480\beta^2 + 1555200\beta^3)\lambda_i\lambda_j \\
&\quad +(3576960\beta^2 + 14204160\beta + 17003520 + 311040\beta^3)\lambda_i^2, \\
\xi_{30}^{(2)}(\lambda_j;\lambda_i;\beta) &= (41990400\beta^2 + 156142080\beta + 150128640 + 2073600\beta^3)\lambda_j \\
&\quad +(42232320\beta - 18524160 + 21876480\beta^2 + 1555200\beta^3)\lambda_i, \\
\xi_{30}^{(3)}(\lambda_j;\lambda_i;\beta) &= 41990400\beta^2 + 156142080\beta + 150128640 + 2073600\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -1.5818.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{30}(\lambda_i;\lambda_i;\beta) &= 1451520\lambda_i^3(\beta+3)(\beta^2+12\beta+8) > 0 \text{ for all } \beta > -0.7085, \\
\xi_{30}^{(1)}(\lambda_i;\lambda_i;\beta) &= 967680\lambda_i^2(48\beta^2+139\beta+76+3\beta^3) > 0 \text{ for all } \beta > -0.7158, \\
\xi_{30}^{(2)}(\lambda_i;\lambda_i;\beta) &= 241920\lambda_i(264\beta^2+820\beta+544+15\beta^3) > 0 \text{ for all } \beta > -0.9237.
\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_4(\lambda_l;\lambda_j;\lambda_i;\beta) &= (725760\beta^2 + 3110400\beta + 3317760)\lambda_l^2 \\
&\quad +(3663360\beta + 552960 + 2177280\beta^2)(\lambda_j+\lambda_i)\lambda_l \\
&\quad +(3663360\beta + 552960 + 2177280\beta^2)\lambda_i\lambda_j \\
&\quad +(725760\beta^2 + 3110400\beta + 3317760)(\lambda_j^2+\lambda_i^2), \\
\xi_4^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (1451520\beta^2 + 6220800\beta + 6635520)\lambda_l \\
&\quad +(3663360\beta + 552960 + 2177280\beta^2)\lambda_j \\
&\quad +(3663360\beta + 552960 + 2177280\beta^2)\lambda_i, \\
\xi_4^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 1451520\beta^2 + 6220800\beta + 6635520 \geq 0 \text{ for all } \beta \geq -2.
\end{aligned}$$

Define $\xi_{41}(\lambda_j;\lambda_i;\beta) = \xi_4^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}
\xi_{41}(\lambda_j;\lambda_i;\beta) &= (3628800\beta^2 + 9884160\beta + 7188480)\lambda_j \\
&\quad +(3663360\beta + 552960 + 2177280\beta^2)\lambda_i, \\
\xi_{41}^{(1)}(\lambda_j;\lambda_i;\beta) &= 3628800\beta^2 + 9884160\beta + 7188480 \geq 0 \text{ for all } \beta.
\end{aligned}$$

Now

$$\xi_{41}(\lambda_i;\lambda_i;\beta) \quad = \quad 1935360\lambda_i(3\beta+4)(\beta+1) \geq 0 \text{ for all } \beta \geq -1.$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j;\lambda_i;\beta) = \xi_4(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}
\xi_{40}(\lambda_j;\lambda_i;\beta) &= (3628800\beta^2+9884160\beta+7188480)\lambda_j^2 \\
&\quad +(7326720\beta+1105920+4354560\beta^2)\lambda_i\lambda_j \\
&\quad +(725760\beta^2+3110400\beta+3317760)\lambda_i^2, \\
\xi_{40}^{(1)}(\lambda_j;\lambda_i;\beta) &= (7257600\beta^2+19768320\beta+14376960)\lambda_j \\
&\quad +(7326720\beta+1105920+4354560\beta^2)\lambda_i, \\
\xi_{40}^{(2)}(\lambda_j;\lambda_i;\beta) &= 7257600\beta^2+19768320\beta+14376960 \geq 0 \text{ for all } \beta.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{40}(\lambda_i;\lambda_i;\beta) &= 2903040\lambda_i^2(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{40}^{(1)}(\lambda_i;\lambda_i;\beta) &= 3870720\lambda_i(3\beta+4)(\beta+1) > 0 \text{ for all } \beta > -1.
\end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.18 Let

$$\begin{aligned}
\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 20160(322560+564480\beta+241920\beta^2)t(0,0,1,4) \\
&\quad +(60480\beta^3+3870720+3628800\beta+725760\beta^2)t(0,0,2,3) \\
&\quad +(1935360\beta^2+181440\beta^3+645120+4757760\beta)t(0,1,1,3) \\
&\quad +(2903040\beta^2+544320\beta^3+13789440+6894720\beta)t(0,1,2,2) \\
&\quad +(-3628800\beta+6531840\beta^2-31933440+1632960\beta^3)t(1,1,1,2),
\end{aligned}$$

where

$$t(a,b,c,d) = \sum_{\substack{(n_1,n_2,n_3,n_4) \\ \in \mathrm{perm}(a,b,c,d)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_k^{n_4}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (2257920\beta + 967680\beta^2 + 1290240)\lambda_k^3\hat{s}(0,0,1) \\
&\quad +(1935360 + 14273280\beta + 544320\beta^3 + 5806080\beta^2)\lambda_k^2\hat{s}(0,1,1) \\
&\quad +(11612160 + 2177280\beta^2 + 181440\beta^3 + 10886400\beta)\lambda_k^2\hat{s}(0,0,2) \\
&\quad +(7741440 + 7257600\beta + 1451520\beta^2 + 120960\beta^3)\lambda_k\hat{s}(0,0,3) \\
&\quad +(-7257600\beta + 13063680\beta^2 - 63866880 + 3265920\beta^3)\lambda_k\hat{s}(1,1,1) \\
&\quad +(13789440\beta + 1088640\beta^3 + 5806080\beta^2 + 27578880)\lambda_k\hat{s}(0,1,2) \\
&\quad +(1632960\beta^3 - 3628800\beta - 31933440 + 6531840\beta^2)\hat{s}(1,1,2) \\
&\quad +(564480\beta + 322560 + 241920\beta^2)\hat{s}(0,0,4) \\
&\quad +(1935360\beta^2 + 181440\beta^3 + 4757760\beta + 645120)\hat{s}(0,1,3) \\
&\quad +(544320\beta^3 + 6894720\beta + 13789440 + 2903040\beta^2)\hat{s}(0,2,2), \\
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 3(2257920\beta + 967680\beta^2 + 1290240)\lambda_k^2\hat{s}(0,0,1) \\
&\quad +2(1935360 + 14273280\beta + 544320\beta^3 + 5806080\beta^2)\lambda_k\hat{s}(0,1,1) \\
&\quad +2(11612160 + 2177280\beta^2 + 181440\beta^3 + 10886400\beta)\lambda_k\hat{s}(0,0,2) \\
&\quad +(7741440 + 7257600\beta + 1451520\beta^2 + 120960\beta^3)\hat{s}(0,0,3) \\
&\quad +(-7257600\beta + 13063680\beta^2 - 63866880 + 3265920\beta^3)\hat{s}(1,1,1) \\
&\quad +(13789440\beta + 1088640\beta^3 + 5806080\beta^2 + 27578880)\hat{s}(0,1,2), \\
\xi^{(3)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(2257920\beta + 967680\beta^2 + 1290240)\lambda_k\hat{s}(0,0,1) \\
&\quad +2(1935360 + 14273280\beta + 544320\beta^3 + 5806080\beta^2)\hat{s}(0,1,1) \\
&\quad +2(11612160 + 2177280\beta^2 + 181440\beta^3 + 10886400\beta)\hat{s}(0,0,2), \\
\xi^{(4)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 6(2257920\beta + 967680\beta^2 + 1290240)\hat{s}(0,0,1) \geq 0 \text{ for all } \beta \geq -1.
\end{aligned}$$

Define

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta), \\
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta), \\
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta), \\
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (8386560\beta+1935360\beta^2+8386560+120960\beta^3)\lambda_l^5\\
&+(15724800+17539200\beta+907200\beta^3+7257600\beta^2)(\lambda_j+\lambda_i)\lambda_l^4\\
&+(35320320+1209600\beta^3+7257600\beta^2+21047040\beta)(\lambda_j^2+\lambda_i^2)\lambda_l^3\\
&+(16934400\beta^2+3628800\beta^3-62576640+2257920\beta)\lambda_i\lambda_j\lambda_l^3\\
&+(10160640\beta-4354560+2721600\beta^3+12337920\beta^2)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l^2\\
&+(8386560+12015360\beta+3386880\beta^2+302400\beta^3)(\lambda_j^3+\lambda_i^3)\lambda_l^2\\
&+(483840\beta^2+645120+1128960\beta)(\lambda_j^4+\lambda_i^4)\lambda_l\\
&+(9515520\beta+362880\beta^3+1290240+3870720\beta^2)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j)\lambda_l\\
&+(5806080\beta^2+13789440\beta+1088640\beta^3+27578880)\lambda_i^2\lambda_j^2\lambda_l\\
&+(725760\beta^2+60480\beta^3+3870720+3628800\beta)(\lambda_i^3\lambda_j^2+\lambda_i^2\lambda_j^3)\\
&+(564480\beta+322560+241920\beta^2)(\lambda_i\lambda_j^4+\lambda_i^4\lambda_j),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_0^{(4)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (1006387200\beta+232243200\beta^2+1006387200+14515200\beta^3)\lambda_l\\
&+(377395200+420940800\beta+21772800\beta^3+174182400\beta^2)(\lambda_i+\lambda_j)\\
&\geq 0 \text{ for all } \beta\geq -1.4294.
\end{aligned}$$

Define $\xi_{03}(\lambda_j;\lambda_i;\beta)=\xi_0^{(3)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{03} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}
\xi_{03}(\lambda_j;\lambda_i;\beta) &= (1050416640\beta+333849600\beta^2+1092510720+36288000\beta^3)\lambda_j^2\\
&+(1935360+434488320\beta+43545600\beta^3+275788800\beta^2)\lambda_i\lambda_j\\
&+(211921920+7257600\beta^3+43545600\beta^2+126282240\beta)\lambda_i^2,\\
\xi_{03}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2100833280\beta+667699200\beta^2+2185021440+72576000\beta^3)\lambda_j\\
&+(1935360+434488320\beta+43545600\beta^3+275788800\beta^2)\lambda_i,\\
\xi_{03}^{(2)}(\lambda_j;\lambda_i;\beta) &= 2100833280\beta+667699200\beta^2+2185021440+72576000\beta^3\\
&\geq 0 \text{ for all } \beta\geq -2.3123.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{03}(\lambda_i;\lambda_i;\beta) &= 43545600\lambda_i^2(2\beta+5)(\beta+3)(\beta+2)\geq 0 \text{ for all } \beta\geq -2,\\
\xi_{03}^{(1)}(\lambda_i;\lambda_i;\beta) &= 4838400\lambda_i(524\beta+195\beta^2+452+24\beta^3)\geq 0 \text{ for all } \beta\geq -1.8116.
\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{02}(\lambda_j;\lambda_i;\beta)=\xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{02} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned}
\xi_{02}(\lambda_j;\lambda_i;\beta) &= (528514560\beta + 176117760\beta^2 + 585123840 + 21168000\beta^3)\lambda_j^3 \\
&\quad +(213373440\beta^2 + 38102400\beta^3 - 195471360 + 244339200\beta)\lambda_i\lambda_j^2 \\
&\quad +(203212800 + 12700800\beta^3 + 68221440\beta^2 + 146603520\beta)\lambda_i^2\lambda_j \\
&\quad +(16773120 + 24030720\beta + 6773760\beta^2 + 604800\beta^3)\lambda_i^3, \\
\xi_{02}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1585543680\beta + 528353280\beta^2 + 1755371520 + 63504000\beta^3)\lambda_j^2 \\
&\quad +(426746880\beta^2 + 76204800\beta^3 - 390942720 + 488678400\beta)\lambda_i\lambda_j \\
&\quad +(203212800 + 12700800\beta^3 + 68221440\beta^2 + 146603520\beta)\lambda_i^2, \\
\xi_{02}^{(2)}(\lambda_j;\lambda_i;\beta) &= (3171087360\beta + 1056706560\beta^2 + 3510743040 + 127008000\beta^3)\lambda_j \\
&\quad +(426746880\beta^2 + 76204800\beta^3 - 390942720 + 488678400\beta)\lambda_i, \\
\xi_{02}^{(3)}(\lambda_j;\lambda_i;\beta) &= 3171087360\beta + 1056706560\beta^2 + 3510743040 + 127008000\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -3.2688.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{02}(\lambda_i;\lambda_i;\beta) &= 14515200\lambda_i^3(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(1)}(\lambda_i;\lambda_i;\beta) &= 21772800\lambda_i^2(\beta+3)(\beta+2)(7\beta+12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}, \\
\xi_{02}^{(2)}(\lambda_i;\lambda_i;\beta) &= 967680\lambda_i(3782\beta + 1533\beta^2 + 3224 + 210\beta^3) \geq 0 \text{ for all } \beta \geq -3.0192.
\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{01} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j;\lambda_i;\beta) &= (200390400\beta + 67737600\beta^2 + 228211200 + 8467200\beta^3)\lambda_j^4 \\
&\quad +(-132249600 + 106767360\beta + 20321280\beta^3 + 108380160\beta^2)\lambda_i\lambda_j^3 \\
&\quad +(124830720 + 10160640\beta^3 + 52254720\beta^2 + 97251840\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(18063360 + 33546240\beta + 10644480\beta^2 + 967680\beta^3)\lambda_i^3\lambda_j \\
&\quad +(483840\beta^2 + 645120 + 1128960\beta)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= (801561600\beta + 270950400\beta^2 + 912844800 + 33868800\beta^3)\lambda_j^3 \\
&\quad +(-396748800 + 320302080\beta + 60963840\beta^3 + 325140480\beta^2)\lambda_i\lambda_j^2 \\
&\quad +(249661440 + 20321280\beta^3 + 104509440\beta^2 + 194503680\beta)\lambda_i^2\lambda_j \\
&\quad +(18063360 + 33546240\beta + 10644480\beta^2 + 967680\beta^3)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= (2404684800\beta + 812851200\beta^2 + 2738534400 + 101606400\beta^3)\lambda_j^2 \\
&\quad +(-793497600 + 640604160\beta + 121927680\beta^3 + 650280960\beta^2)\lambda_i\lambda_j \\
&\quad +(249661440 + 20321280\beta^3 + 104509440\beta^2 + 194503680\beta)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j;\lambda_i;\beta) &= (4809369600\beta + 1625702400\beta^2 + 5477068800 + 203212800\beta^3)\lambda_j \\
&\quad +(-793497600 + 640604160\beta + 121927680\beta^3 + 650280960\beta^2)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j;\lambda_i;\beta) &= 4809369600\beta + 1625702400\beta^2 + 5477068800 + 203212800\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -3.3100.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 39916800\lambda_i^4(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 14515200\lambda_i^3(8\beta+9)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{9}{8}, \\
\xi_{01}^{(2)}(\lambda_i;\lambda_i;\beta) &= 34836480\lambda_i^2(\beta+3)(7\beta^2+24\beta+21) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i;\lambda_i;\beta) &= 7741440\lambda_i(704\beta + 294\beta^2 + 605 + 42\beta^3) \geq 0 \text{ for all } \beta \geq -3.0629.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)/\lambda_j$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= (60117120\beta + 20321280\beta^2 + 68463360 + 2540160\beta^3)\lambda_j^4 \\
&\quad +(40642560\beta^2 + 7620480\beta^3 - 49593600 + 40037760\beta)\lambda_i\lambda_j^3 \\
&\quad +(62415360 + 5080320\beta^3 + 26127360\beta^2 + 48625920\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(13547520 + 25159680\beta + 7983360\beta^2 + 725760\beta^3)\lambda_i^3\lambda_j \\
&\quad +(725760\beta^2 + 967680 + 1693440\beta)\lambda_i^4, \\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= (240468480\beta + 81285120\beta^2 + 273853440 + 10160640\beta^3)\lambda_j^3 \\
&\quad +(121927680\beta^2 + 22861440\beta^3 - 148780800 + 120113280\beta)\lambda_i\lambda_j^2 \\
&\quad +(124830720 + 10160640\beta^3 + 52254720\beta^2 + 97251840\beta)\lambda_i^2\lambda_j \\
&\quad +(13547520 + 25159680\beta + 7983360\beta^2 + 725760\beta^3)\lambda_i^3, \\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= (721405440\beta + 243855360\beta^2 + 821560320 + 30481920\beta^3)\lambda_j^2 \\
&\quad +(243855360\beta^2 + 45722880\beta^3 - 297561600 + 240226560\beta)\lambda_i\lambda_j \\
&\quad +(124830720 + 10160640\beta^3 + 52254720\beta^2 + 97251840\beta)\lambda_i^2, \\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= (1442810880\beta + 487710720\beta^2 + 1643120640 + 60963840\beta^3)\lambda_j \\
&\quad +(243855360\beta^2 + 45722880\beta^3 - 297561600 + 240226560\beta)\lambda_i, \\
\xi_{00}^{(4)}(\lambda_j;\lambda_i;\beta) &= 1442810880\beta + 487710720\beta^2 + 1643120640 + 60963840\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -3.3100.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 15966720\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 43908480\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 725760\lambda_i^2(\beta+3)(\beta+2)(119\beta+149) > 0 \text{ for all } \beta > -\frac{149}{119}, \\
\xi_{00}^{(3)}(\lambda_i;\lambda_i;\beta) &= 2177280\lambda_i(\beta+3)(49\beta^2+189\beta+206) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (20966400 + 302400\beta^3 + 20966400\beta + 4838400\beta^2)\lambda_l^4 \\
&\quad +(31449600 + 1814400\beta^3 + 14515200\beta^2 + 35078400\beta)(\lambda_i+\lambda_j)\lambda_l^3 \\
&\quad +(31570560\beta + 52980480 + 1814400\beta^3 + 10886400\beta^2)(\lambda_j^2+\lambda_i^2)\lambda_l^2 \\
&\quad +(25401600\beta^2 + 3386880\beta + 5443200\beta^3 - 93864960)\lambda_i\lambda_j\lambda_l^2 \\
&\quad +(10160640\beta - 4354560 + 2721600\beta^3 + 12337920\beta^2)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l \\
&\quad +(8386560 + 12015360\beta + 3386880\beta^2 + 302400\beta^3)(\lambda_j^3+\lambda_i^3)\lambda_l \\
&\quad +(544320\beta^3 + 6894720\beta + 13789440 + 2903040\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad +(564480\beta + 322560 + 241920\beta^2)(\lambda_j^4+\lambda_i^4) \\
&\quad +(1935360\beta^2 + 181440\beta^3 + 4757760\beta + 645120)(\lambda_i\lambda_j^3+\lambda_i^3\lambda_j),
\end{aligned}$$

and, hence,

$$\begin{aligned}\xi_1^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (503193600+7257600\beta^3+503193600\beta+116121600\beta^2)\lambda_l\\ &\quad +(188697600+10886400\beta^3+87091200\beta^2+210470400\beta)(\lambda_i+\lambda_j)\\ &\geq 0 \text{ for all } \beta\geq -1.4294.\end{aligned}$$

Define $\xi_{12}(\lambda_j;\lambda_i;\beta)=\xi_1^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{12} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned}\xi_{12}(\lambda_j;\lambda_i;\beta) &= (546255360+18144000\beta^3+525208320\beta+166924800\beta^2)\lambda_j^2\\ &\quad +(967680+21772800\beta^3+137894400\beta^2+217244160\beta)\lambda_i\lambda_j\\ &\quad +(63141120\beta+105960960+3628800\beta^3+21772800\beta^2)\lambda_i^2,\\ \xi_{12}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1092510720+36288000\beta^3+1050416640\beta+333849600\beta^2)\lambda_j\\ &\quad +(967680+21772800\beta^3+137894400\beta^2+217244160\beta)\lambda_i,\\ \xi_{12}^{(2)}(\lambda_j;\lambda_i;\beta) &= 1092510720+36288000\beta^3+1050416640\beta+333849600\beta^2\\ &\geq 0 \text{ for all } \beta\geq -2.3123.\end{aligned}$$

Now

$$\begin{aligned}\xi_{12}(\lambda_i;\lambda_i;\beta) &= 21772800\lambda_i^2(2\beta+5)(\beta+3)(\beta+2)\geq 0 \text{ for all } \beta\geq -2,\\ \xi_{12}^{(1)}(\lambda_i;\lambda_i;\beta) &= 2419200\lambda_i(452+24\beta^3+524\beta+195\beta^2)\geq 0 \text{ for all } \beta\geq -1.8116.\end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j;\lambda_i;\beta)=\xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{11} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}\xi_{11}(\lambda_j;\lambda_i;\beta) &= (292561920+10584000\beta^3+264257280\beta+88058880\beta^2)\lambda_j^3\\ &\quad +(106686720\beta^2+122169600\beta+19051200\beta^3-97735680)\lambda_i\lambda_j^2\\ &\quad +(73301760\beta+101606400+6350400\beta^3+34110720\beta^2)\lambda_i^2\lambda_j\\ &\quad +(8386560+12015360\beta+3386880\beta^2+302400\beta^3)\lambda_i^3,\\ \xi_{11}^{(1)}(\lambda_j;\lambda_i;\beta) &= (877685760+31752000\beta^3+792771840\beta+264176640\beta^2)\lambda_j^2\\ &\quad +(213373440\beta^2+38102400\beta^3-195471360+244339200\beta)\lambda_i\lambda_j\\ &\quad +(73301760\beta+101606400+6350400\beta^3+34110720\beta^2)\lambda_i^2,\\ \xi_{11}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1755371520+63504000\beta^3+1585543680\beta+528353280\beta^2)\lambda_j\\ &\quad +(213373440\beta^2+38102400\beta^3-195471360+244339200\beta)\lambda_i,\\ \xi_{11}^{(3)}(\lambda_j;\lambda_i;\beta) &= 1755371520+63504000\beta^3+1585543680\beta+528353280\beta^2\\ &\geq 0 \text{ for all } \beta\geq -3.2688.\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i;\lambda_i;\beta) &= 7257600\lambda_i^3(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5},\\
\xi_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= 10886400\lambda_i^2(\beta+3)(\beta+2)(7\beta+12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7},\\
\xi_{11}^{(2)}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i(3224+210\beta^3+3782\beta+1533\beta^2) \geq 0 \text{ for all } \beta \geq -3.0192.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j;\lambda_i;\beta) = \xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= (114105600+4233600\beta^3+100195200\beta+33868800\beta^2)\lambda_j^4\\
&\quad+(-66124800+10160640\beta^3+54190080\beta^2+53383680\beta)\lambda_i\lambda_j^3\\
&\quad+(48625920\beta+62415360+5080320\beta^3+26127360\beta^2)\lambda_i^2\lambda_j^2\\
&\quad+(9031680+16773120\beta+5322240\beta^2+483840\beta^3)\lambda_i^3\lambda_j\\
&\quad+(564480\beta+322560+241920\beta^2)\lambda_i^4,\\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= (456422400+16934400\beta^3+400780800\beta+135475200\beta^2)\lambda_j^3\\
&\quad+(-198374400+30481920\beta^3+162570240\beta^2+160151040\beta)\lambda_i\lambda_j^2\\
&\quad+(97251840\beta+124830720+10160640\beta^3+52254720\beta^2)\lambda_i^2\lambda_j\\
&\quad+(9031680+16773120\beta+5322240\beta^2+483840\beta^3)\lambda_i^3,\\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1369267200+50803200\beta^3+1202342400\beta+406425600\beta^2)\lambda_j^2\\
&\quad+(-396748800+60963840\beta^3+325140480\beta^2+320302080\beta)\lambda_i\lambda_j\\
&\quad+(97251840\beta+124830720+10160640\beta^3+52254720\beta^2)\lambda_i^2,\\
\xi_{10}^{(3)}(\lambda_j;\lambda_i;\beta) &= (2738534400+101606400\beta^3+2404684800\beta+812851200\beta^2)\lambda_j\\
&\quad+(-396748800+60963840\beta^3+325140480\beta^2+320302080\beta)\lambda_i,\\
\xi_{10}^{(4)}(\lambda_j;\lambda_i;\beta) &= 2738534400+101606400\beta^3+2404684800\beta+812851200\beta^2\\
&\geq 0 \text{ for all } \beta \geq -3.3100.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 19958400\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1,\\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 7257600\lambda_i^3(8\beta+9)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -\frac{9}{8},\\
\xi_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= 17418240\lambda_i^2(\beta+3)(7\beta^2+24\beta+21) > 0 \text{ for all } \beta > -3,\\
\xi_{10}^{(3)}(\lambda_i;\lambda_i;\beta) &= 3870720\lambda_i(605+42\beta^3+704\beta+294\beta^2) > 0 \text{ for all } \beta > -3.0629.
\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_2(\lambda_l;\lambda_j;\lambda_i;\beta) &= (34836480+8709120\beta^2+35804160\beta+483840\beta^3)\lambda_l^3\\
&\quad +(2177280\beta^3+20321280\beta^2+49109760\beta+35320320)(\lambda_j+\lambda_i)\lambda_l^2\\
&\quad +(10160640\beta^2+35562240\beta+50803200+1451520\beta^3)(\lambda_j^2+\lambda_i^2)\lambda_l\\
&\quad +(-59996160+4354560\beta^3+24675840\beta^2+21288960\beta)\lambda_i\lambda_j\lambda_l\\
&\quad +(7741440+7257600\beta+1451520\beta^2+120960\beta^3)(\lambda_j^3+\lambda_i^3)\\
&\quad +(5806080\beta^2+13789440\beta+1088640\beta^3+27578880)(\lambda_i^2\lambda_j+\lambda_i\lambda_j^2),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_2^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 209018880+52254720\beta^2+214824960\beta+2903040\beta^3\\
&\geq 0 \text{ for all } \beta\geq -1.4322.
\end{aligned}$$

Define $\xi_{22}(\lambda_j;\lambda_i;\beta)=\xi_2^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{22} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}
\xi_{22}(\lambda_j;\lambda_i;\beta) &= (279659520+92897280\beta^2+313044480\beta+7257600\beta^3)\lambda_j\\
&\quad +(4354560\beta^3+40642560\beta^2+98219520\beta+70640640)\lambda_i,\\
\xi_{22}^{(1)}(\lambda_j;\lambda_i;\beta) &= 279659520+92897280\beta^2+313044480\beta+7257600\beta^3\\
&\geq 0 \text{ for all } \beta\geq -1.4382.
\end{aligned}$$

Now

$$\xi_{22}(\lambda_i;\lambda_i;\beta) = 967680\lambda_i(362+138\beta^2+425\beta+12\beta^3)\geq 0 \text{ for all } \beta\geq -1.4432.$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j\geq\lambda_i>0$ and $\beta\geq-\frac{2}{5}$.

Define $\xi_{21}(\lambda_j;\lambda_i;\beta)=\xi_2^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{21} is non-negative for all $\lambda_j\geq\lambda_i>0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}
\xi_{21}(\lambda_j;\lambda_i;\beta) &= (225953280+76930560\beta^2+241194240\beta+7257600\beta^3)\lambda_j^2\\
&\quad +(8709120\beta^3+65318400\beta^2+119508480\beta+10644480)\lambda_i\lambda_j\\
&\quad +(10160640\beta^2+35562240\beta+50803200+1451520\beta^3)\lambda_i^2,\\
\xi_{21}^{(1)}(\lambda_j;\lambda_i;\beta) &= (451906560+153861120\beta^2+482388480\beta+14515200\beta^3)\lambda_j\\
&\quad +(8709120\beta^3+65318400\beta^2+119508480\beta+10644480)\lambda_i,\\
\xi_{21}^{(2)}(\lambda_j;\lambda_i;\beta) &= 451906560+153861120\beta^2+482388480\beta+14515200\beta^3\\
&\geq 0 \text{ for all } \beta\geq -1.7649.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{21}(\lambda_i;\lambda_i;\beta) &= 4354560\lambda_i^2(\beta+3)(4\beta^2+23\beta+22)\geq 0 \text{ for all } \beta\geq -1.2120,\\
\xi_{21}^{(1)}(\lambda_i;\lambda_i;\beta) &= 483840\lambda_i(956+453\beta^2+1244\beta+48\beta^3)\geq 0 \text{ for all } \beta\geq -1.2965.
\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j;\lambda_i;\beta) = \xi_2(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{20} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}
\xi_{20}(\lambda_j;\lambda_i;\beta) &= (128701440 + 40642560\beta^2 + 127733760\beta + 4233600\beta^3)\lambda_j^3 \\
&\quad +(2903040 + 7620480\beta^3 + 50803200\beta^2 + 84188160\beta)\lambda_i\lambda_j^2 \\
&\quad +(15966720\beta^2 + 49351680\beta + 78382080 + 2540160\beta^3)\lambda_i^2\lambda_j \\
&\quad +(7741440 + 7257600\beta + 1451520\beta^2 + 120960\beta^3)\lambda_i^3, \\
\xi_{20}^{(1)}(\lambda_j;\lambda_i;\beta) &= (386104320 + 121927680\beta^2 + 383201280\beta + 12700800\beta^3)\lambda_j^2 \\
&\quad +(5806080 + 15240960\beta^3 + 101606400\beta^2 + 168376320\beta)\lambda_i\lambda_j \\
&\quad +(15966720\beta^2 + 49351680\beta + 78382080 + 2540160\beta^3)\lambda_i^2, \\
\xi_{20}^{(2)}(\lambda_j;\lambda_i;\beta) &= (772208640 + 243855360\beta^2 + 766402560\beta + 25401600\beta^3)\lambda_j \\
&\quad +(5806080 + 15240960\beta^3 + 101606400\beta^2 + 168376320\beta)\lambda_i, \\
\xi_{20}^{(3)}(\lambda_j;\lambda_i;\beta) &= 772208640 + 243855360\beta^2 + 766402560\beta + 25401600\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -2.1389.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{20}(\lambda_i;\lambda_i;\beta) &= 7257600\lambda_i^3(2\beta+5)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -2, \\
\xi_{20}^{(1)}(\lambda_i;\lambda_i;\beta) &= 4354560\lambda_i^2(\beta+3)(7\beta^2+34\beta+36) > 0 \text{ for all } \beta > -1.5596, \\
\xi_{20}^{(2)}(\lambda_i;\lambda_i;\beta) &= 2903040\lambda_i(268+119\beta^2+322\beta+14\beta^3) > 0 \text{ for all } \beta > -1.6020.
\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_3(\lambda_l;\lambda_j;\lambda_i;\beta) &= (362880\beta^3 + 10160640\beta^2 + 30965760 + 35320320\beta)\lambda_l^2 \\
&\quad +(1088640\beta^3 + 42094080\beta + 11612160 + 17418240\beta^2)(\lambda_j+\lambda_i)\lambda_l \\
&\quad +(4354560\beta^2 + 21772800\beta + 362880\beta^3 + 23224320)\lambda_j^2 \\
&\quad +(1088640\beta^3 + 28546560\beta + 3870720 + 11612160\beta^2)\lambda_i\lambda_j \\
&\quad +(4354560\beta^2 + 21772800\beta + 362880\beta^3 + 23224320)\lambda_i^2,
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_3^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= 725760\beta^3 + 20321280\beta^2 + 61931520 + 70640640\beta \\
&\geq 0 \text{ for all } \beta \geq -1.4510.
\end{aligned}$$

Define $\xi_{31}(\lambda_j;\lambda_i;\beta) = \xi_3^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{31} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j;\lambda_i;\beta) &= (1814400\beta^3 + 37739520\beta^2 + 73543680 + 112734720\beta)\lambda_j \\
&\quad +(1088640\beta^3 + 42094080\beta + 11612160 + 17418240\beta^2)\lambda_i, \\
\xi_{31}^{(1)}(\lambda_j;\lambda_i;\beta) &= 1814400\beta^3 + 37739520\beta^2 + 73543680 + 112734720\beta \\
&\geq 0 \text{ for all } \beta \geq -0.9275.
\end{aligned}$$

Now

$$\xi_{31}(\lambda_i;\lambda_i;\beta) = 967680\lambda_i(3\beta^3 + 57\beta^2 + 88 + 160\beta) \geq 0 \text{ for all } \beta \geq -0.7350.$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j;\lambda_i;\beta) = \xi_3(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}
\xi_{30}(\lambda_j;\lambda_i;\beta) &= (1814400\beta^3 + 31933440\beta^2 + 65802240 + 99187200\beta)\lambda_j^2 \\
&\quad +(2177280\beta^3 + 70640640\beta + 15482880 + 29030400\beta^2)\lambda_i\lambda_j \\
&\quad +(4354560\beta^2 + 21772800\beta + 362880\beta^3 + 23224320)\lambda_i^2, \\
\xi_{30}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3628800\beta^3 + 63866880\beta^2 + 131604480 + 198374400\beta)\lambda_j \\
&\quad +(2177280\beta^3 + 70640640\beta + 15482880 + 29030400\beta^2)\lambda_i, \\
\xi_{30}^{(2)}(\lambda_j;\lambda_i;\beta) &= 3628800\beta^3 + 63866880\beta^2 + 131604480 + 198374400\beta \\
&\geq 0 \text{ for all } \beta \geq -0.9237.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{30}(\lambda_i;\lambda_i;\beta) &= 4354560\lambda_i^2(\beta+3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -0.7085, \\
\xi_{30}^{(1)}(\lambda_i;\lambda_i;\beta) &= 1935360\lambda_i(3\beta^3 + 48\beta^2 + 76 + 139\beta) > 0 \text{ for all } \beta > -0.7158.
\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.19 Let

$$\begin{aligned}\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (725760\beta^2+967680+1693440\beta)t(0,0,0,4)\\ &+(10321920+483840\beta^3+19031040\beta+6289920\beta^2)t(0,0,1,3)\\ &+(38223360\beta+23950080\beta^2+25159680+4354560\beta^3)t(0,1,1,2)\\ &+(50803200+10160640\beta^2+35562240\beta+1451520\beta^3)t(0,0,2,2)\\ &+(-29030400\beta+52254720\beta^2-255467520+13063680\beta^3)t(1,1,1,1),\end{aligned}$$

where

$$t(a,b,c,d)=\sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in\text{perm}(a,b,c,d)}}\lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_l^{n_4}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k\geq\lambda_l\geq\lambda_j\geq\lambda_i$, and $\beta\geq-\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a,b,c)=\sum_{\substack{(n_1,n_2,n_3)\\ \in\text{perm}(a,b,c)}}\lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$\begin{aligned}\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (6773760\beta+3870720+2903040\beta^2)\lambda_k^3\\ &+((1451520\beta^3+30965760+18869760\beta^2+57093120\beta)\lambda_k^2\hat{s}(0,0,2)\\ &+(101606400+71124480\beta+20321280\beta^2+2903040\beta^3)\lambda_l^2\lambda_k\hat{s}(0,0,2)\\ &+(8709120\beta^3+47900160\beta^2+76446720\beta+50319360)\lambda_k\hat{s}(0,1,1)\\ &+(-29030400\beta+52254720\beta^2+13063680\beta^3-255467520)\hat{s}(1,1,1)\\ &+(10321920+19031040\beta+6289920\beta^2+483840\beta^3)\hat{s}(0,0,3)\\ &+(4354560\beta^3+23950080\beta^2+25159680+38223360\beta)\hat{s}(0,1,2),\\ \xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 3(6773760\beta+3870720+2903040\beta^2)\lambda_k^2\\ &+2((1451520\beta^3+30965760+18869760\beta^2+57093120\beta)\lambda_k\hat{s}(0,0,2)\\ &+(101606400+71124480\beta+20321280\beta^2+2903040\beta^3)\lambda_l^2\hat{s}(0,0,2)\\ &+(8709120\beta^3+47900160\beta^2+76446720\beta+50319360)\hat{s}(0,1,1)\\ &\geq 0 \text{ for all } \beta\geq-0.6924.\end{aligned}$$

Define

$$\begin{aligned}\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta).\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (24192000\beta^2 + 73382400 + 77011200\beta + 2419200\beta^3)\lambda_l^4 \\
&\quad +(9676800\beta^3 + 60480000\beta^2 + 70963200 + 114508800\beta)(\lambda_i+\lambda_j)\lambda_l^3 \\
&\quad +(7257600\beta^3 + 44271360\beta^2 + 126766080 + 109347840\beta)(\lambda_j^2+\lambda_i^2)\lambda_l^2 \\
&\quad +(21772800\beta^3 + 100154880\beta^2 - 205148160 + 47416320\beta)\lambda_i\lambda_j\lambda_l^2 \\
&\quad +2(10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)(\lambda_j^3+\lambda_i^3)\lambda_l \\
&\quad +2(4354560\beta^3 + 23950080\beta^2 + 25159680 + 38223360\beta)(\lambda_i\lambda_j^2+\lambda_i^2\lambda_j)\lambda_l \\
&\quad +(725760\beta^2 + 1693440\beta + 967680)\lambda_j^4 \\
&\quad +(10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_i\lambda_j^3 \\
&\quad +(10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_i^3\lambda_j \\
&\quad +(10160640\beta^2 + 50803200 + 35562240\beta + 1451520\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad +(725760\beta^2 + 1693440\beta + 967680)\lambda_i^4,
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_0^{(3)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (580608000\beta^2 + 1761177600 + 1848268800\beta + 58060800\beta^3)\lambda_l \\
&\quad +(58060800\beta^3 + 362880000\beta^2 + 425779200 + 687052800\beta)(\lambda_i+\lambda_j) \\
&\geq 0 \text{ for all } \beta \geq -1.7321.
\end{aligned}$$

Define $\xi_{02}(\lambda_j;\lambda_i;\beta) = \xi_0^{(2)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned}
\xi_{02}(\lambda_j;\lambda_i;\beta) &= (741726720\beta^2 + 1559900160 + 1829882880\beta + 101606400\beta^3)\lambda_j^2 \\
&\quad +(101606400\beta^3 + 563189760\beta^2 + 15482880 + 781885440\beta)\lambda_i\lambda_j \\
&\quad +(14515200\beta^3 + 88542720\beta^2 + 253532160 + 218695680\beta)\lambda_i^2, \\
\xi_{02}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1483453440\beta^2 + 3119800320 + 3659765760\beta + 203212800\beta^3)\lambda_j \\
&\quad +(101606400\beta^3 + 563189760\beta^2 + 15482880 + 781885440\beta)\lambda_i, \\
\xi_{02}^{(2)}(\lambda_j;\lambda_i;\beta) &= 1483453440\beta^2 + 3119800320 + 3659765760\beta + 203212800\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -3.0192.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{02}(\lambda_i;\lambda_i;\beta) &= 43545600\lambda_i^2(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\
\xi_{02}^{(1)}(\lambda_i;\lambda_i;\beta) &= 43545600\lambda_i(\beta+3)(\beta+2)(7\beta+12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}.
\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j;\lambda_i;\beta) &= (379330560\beta^2 + 780595200 + 908328960\beta + 54190080\beta^3)\lambda_j^3 \\
&\quad +(81285120\beta^3 + 429649920\beta^2 - 147087360 + 514805760\beta)\lambda_i\lambda_j^2 \\
&\quad +(23224320\beta^3 + 136442880\beta^2 + 303851520 + 295142400\beta)\lambda_i^2\lambda_j \\
&\quad +(20643840 + 38062080\beta + 12579840\beta^2 + 967680\beta^3)\lambda_i^3, \\
\xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1137991680\beta^2 + 2341785600 + 2724986880\beta + 162570240\beta^3)\lambda_j^2 \\
&\quad +(162570240\beta^3 + 859299840\beta^2 - 294174720 + 1029611520\beta)\lambda_i\lambda_j \\
&\quad +(23224320\beta^3 + 136442880\beta^2 + 303851520 + 295142400\beta)\lambda_i^2, \\
\xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= (2275983360\beta^2 + 4683571200 + 5449973760\beta + 325140480\beta^3)\lambda_j \\
&\quad +(162570240\beta^3 + 859299840\beta^2 - 294174720 + 1029611520\beta)\lambda_i, \\
\xi_{01}^{(3)}(\lambda_j;\lambda_i;\beta) &= 2275983360\beta^2 + 4683571200 + 5449973760\beta + 325140480\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -3.0629.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 159667200\lambda_i^3(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 43545600\lambda_i^2(8\beta+9)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{9}{8}, \\
\xi_{01}^{(2)}(\lambda_i;\lambda_i;\beta) &= 69672960\lambda_i(\beta+3)(7\beta^2+24\beta+21) \geq 0 \text{ for all } \beta \geq -3.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{00} is non-negative for all

$\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= (142248960\beta^2 + 292723200 + 340623360\beta + 20321280\beta^3)\lambda_j^4 \\
&\quad +(40642560\beta^3 + 214824960\beta^2 - 73543680 + 257402880\beta)\lambda_i\lambda_j^3 \\
&\quad +(17418240\beta^3 + 102332160\beta^2 + 227888640 + 221356800\beta)\lambda_i^2\lambda_j^2 \\
&\quad +(30965760 + 57093120\beta + 18869760\beta^2 + 1451520\beta^3)\lambda_i^3\lambda_j \\
&\quad +(725760\beta^2 + 1693440\beta + 967680)\lambda_i^4, \\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= (568995840\beta^2 + 1170892800 + 1362493440\beta + 81285120\beta^3)\lambda_j^3 \\
&\quad +(121927680\beta^3 + 644474880\beta^2 - 220631040 + 772208640\beta)\lambda_i\lambda_j^2 \\
&\quad +(34836480\beta^3 + 204664320\beta^2 + 455777280 + 442713600\beta)\lambda_i^2\lambda_j \\
&\quad +(30965760 + 57093120\beta + 18869760\beta^2 + 1451520\beta^3)\lambda_i^3, \\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= (1706987520\beta^2 + 3512678400 + 4087480320\beta + 243855360\beta^3)\lambda_j^2 \\
&\quad +(243855360\beta^3 + 1288949760\beta^2 - 441262080 + 1544417280\beta)\lambda_i\lambda_j \\
&\quad +(34836480\beta^3 + 204664320\beta^2 + 455777280 + 442713600\beta)\lambda_i^2, \\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= (3413975040\beta^2 + 7025356800 + 8174960640\beta + 487710720\beta^3)\lambda_j \\
&\quad +(243855360\beta^3 + 1288949760\beta^2 - 441262080 + 1544417280\beta)\lambda_i, \\
\xi_{00}^{(4)}(\lambda_j;\lambda_i;\beta) &= 3413975040\beta^2 + 7025356800 + 8174960640\beta + 487710720\beta^3 \\
&> 0 \quad \text{for all } \beta > -3.0629.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 79833600\lambda_i^4(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 239500800\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 65318400\lambda_i^2(8\beta+9)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -\frac{9}{8}, \\
\xi_{00}^{(3)}(\lambda_i;\lambda_i;\beta) &= 104509440\lambda_i(\beta+3)(7\beta^2+24\beta+21) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (154022400\beta + 146764800 + 48384000\beta^2 + 4838400\beta^3)\lambda_l^3 \\
&\quad +(106444800 + 171763200\beta + 90720000\beta^2 + 14515200\beta^3)(\lambda_j+\lambda_i)\lambda_l^2 \\
&\quad +(7257600\beta^3 + 44271360\beta^2 + 126766080 + 109347840\beta)(\lambda_j^2+\lambda_i^2)\lambda_l \\
&\quad +(21772800\beta^3 + 100154880\beta^2 - 205148160 + 47416320\beta)\lambda_i\lambda_j\lambda_l \\
&\quad +(10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_j^3 \\
&\quad +(4354560\beta^3 + 23950080\beta^2 + 25159680 + 38223360\beta)\lambda_i\lambda_j^2 \\
&\quad +(10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_i^3 \\
&\quad +(4354560\beta^3 + 23950080\beta^2 + 25159680 + 38223360\beta)\lambda_i^2\lambda_j,
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_1^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (924134400\beta + 880588800 + 290304000\beta^2 + 29030400\beta^3)\lambda_l \\
&\quad +(29030400\beta^3 + 181440000\beta^2 + 212889600 + 343526400\beta)(\lambda_i+\lambda_j) \\
&\geq 0 \text{ for all } \beta \geq -1.7321.
\end{aligned}$$

Define $\xi_{11}(\lambda_j;\lambda_i;\beta) = \xi_1^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j;\lambda_i;\beta) &= (9149941440\beta + 779950080 + 370863360\beta^2 + 50803200\beta^3)\lambda_j^2 \\
&\quad +(50803200\beta^3 + 281594880\beta^2 + 7741440 + 390942720\beta)\lambda_i\lambda_j \\
&\quad +(7257600\beta^3 + 44271360\beta^2 + 126766080 + 109347840\beta)\lambda_i^2, \\
\xi_{11}^{(1)}(\lambda_j;\lambda_i;\beta) &= (741726720\beta^2 + 1559900160 + 1829882880\beta + 101606400\beta^3)\lambda_j \\
&\quad +(50803200\beta^3 + 281594880\beta^2 + 7741440 + 390942720\beta)\lambda_i, \\
\xi_{11}^{(2)}(\lambda_j;\lambda_i;\beta) &= 741726720\beta^2 + 1559900160 + 1829882880\beta + 101606400\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -3.0192.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i;\lambda_i;\beta) &= 21772800\lambda_i^2(5\beta+7)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\
\xi_{11}^{(1)}(\lambda_i;\lambda_i;\beta) &= 21772800\lambda_i(\beta+3)(\beta+2)(7\beta+12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j;\lambda_i;\beta) = \xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= (454164480\beta + 390297600 + 189665280\beta^2 + 27095040\beta^3)\lambda_j^3 \\
&\quad +(40642560\beta^3 + 214824960\beta^2 - 73543680 + 257402880\beta)\lambda_i\lambda_j^2 \\
&\quad +(11612160\beta^3 + 68221440\beta^2 + 151925760 + 147571200\beta)\lambda_i^2\lambda_j \\
&\quad +(10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_i^3, \\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= (1362493440\beta + 1170892800 + 568995840\beta^2 + 81285120\beta^3)\lambda_j^2 \\
&\quad +(81285120\beta^3 + 429649920\beta^2 - 147087360 + 514805760\beta)\lambda_i\lambda_j \\
&\quad +(11612160\beta^3 + 68221440\beta^2 + 151925760 + 147571200\beta)\lambda_i^2, \\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= (2724986880\beta + 2341785600 + 1137991680\beta^2 + 162570240\beta^3)\lambda_j \\
&\quad +(81285120\beta^3 + 429649920\beta^2 - 147087360 + 514805760\beta)\lambda_i, \\
\xi_{10}^{(3)}(\lambda_j;\lambda_i;\beta) &= 2724986880\beta + 2341785600 + 1137991680\beta^2 + 162570240\beta^3 \\
&> 0 \text{ for all } \beta > -3.0629.
\end{aligned}$$

Now

$$
\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 79833600\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1,\\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 21772800\lambda_i^2(8\beta+9)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -\frac{9}{8},\\
\xi_{10}^{(2)}(\lambda_i;\lambda_i;\beta) &= 34836480\lambda_i(\beta+3)(7\beta^2+24\beta+21) > 0 \text{ for all } \beta > -3.
\end{aligned}
$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.20 Let

$$
\begin{aligned}
\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= 1451520(15\beta^2+44\beta+\beta^3+24)t(0,0,0,3)\\
&\quad +4354560(3\beta^3+42+47\beta+20\beta^2)t(0,0,1,2)\\
&\quad +4354560(9\beta^3+45\beta^2-24+46\beta)t(0,1,1,1),
\end{aligned}
$$

where

$$
t(a,b,c,d) = \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in \mathrm{perm}(a,b,c,d)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_l^{n_4}.
$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$
\hat{s}(a,b,c) = \sum_{\substack{(n_1,n_2,n_3)\\ \in \mathrm{perm}(a,b,c)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}.
$$

Differentiating $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_k we obtain

$$
\begin{aligned}
\xi^{(1)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (104509440+4354560\beta^3+65318400\beta^2+191600640\beta)\lambda_k^2\\
&\quad +(409328640\beta+174182400\beta^2+365783040+26127360\beta^3)\lambda_k\hat{s}(0,0,1)\\
&\quad +(39191040\beta^3+195955200\beta^2-104509440+200309760\beta)\hat{s}(0,1,1)\\
&\quad +(13063680\beta^3+204664320\beta+87091200\beta^2+182891520)\hat{s}(0,0,2),
\end{aligned}
$$

$$
\begin{aligned}
\xi^{(2)}(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (209018880+8709120\beta^3+130636800\beta^2+383201280\beta)\lambda_k\\
&\quad +(409328640\beta+174182400\beta^2+365783040+26127360\beta^3)\hat{s}(0,0,1)\\
&\geq 0 \text{ for all } \beta \geq -0.7085.
\end{aligned}
$$

Define

$$\begin{aligned}\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta),\\ \xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta).\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_0(\lambda_l;\lambda_j;\lambda_i;\beta) &= (217728000\beta^2+537062400\beta+29030400\beta^3+435456000)\lambda_l^3\\ &\quad +(65318400\beta^3+609638400\beta+370137600\beta^2+261273600)(\lambda_j+\lambda_i)\lambda_l^2\\ &\quad +(409328640\beta+174182400\beta^2+365783040+26127360\beta^3)(\lambda_j^2+\lambda_i^2)\lambda_l\\ &\quad +2(39191040\beta^3+195955200\beta^2-104509440+200309760\beta)\lambda_i\lambda_j\lambda_l\\ &\quad +(21772800\beta^2+63866880\beta+1451520\beta^3+34836480)\lambda_j^3\\ &\quad +(13063680\beta^3+204664320\beta+87091200\beta^2+182891520)\lambda_i\lambda_j^2\\ &\quad +(21772800\beta^2+63866880\beta+1451520\beta^3+34836480)\lambda_i^3\\ &\quad +(13063680\beta^3+204664320\beta+87091200\beta^2+182891520)\lambda_i^2\lambda_j,\\ \xi_0^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (653184000\beta^2+1611187200\beta+87091200\beta^3+1306368000)\lambda_l^2\\ &\quad +(130636800\beta^3+1219276800\beta+740275200\beta^2+522547200)(\lambda_j+\lambda_i)\lambda_l\\ &\quad +(78382080\beta^3+391910400\beta^2-209018880+400619520\beta)\lambda_i\lambda_j\\ &\quad +(409328640\beta+174182400\beta^2+365783040+26127360\beta^3)(\lambda_i^2+\lambda_j^2),\\ \xi_0^{(2)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (1306368000\beta^2+3222374400\beta+174182400\beta^3+2612736000)\lambda_l\\ &\quad +(130636800\beta^3+1219276800\beta+740275200\beta^2+522547200)(\lambda_i+\lambda_j)\\ &\geq 0 \text{ for all } \beta \geq -\frac{2}{3}.\end{aligned}$$

Define $\xi_{01}(\lambda_j;\lambda_i;\beta) = \xi_0^{(1)}(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}\xi_{01}(\lambda_j;\lambda_i;\beta) &= (1567641600\beta^2+3239792640\beta+243855360\beta^3+2194698240)\lambda_j^2\\ &\quad +(209018880\beta^3+1619896320\beta+1132185600\beta^2+313528320)\lambda_i\lambda_j\\ &\quad +(409328640\beta+174182400\beta^2+365783040+26127360\beta^3)\lambda_i^2,\\ \xi_{01}^{(1)}(\lambda_j;\lambda_i;\beta) &= (3135283200\beta^2+6479585280\beta+487710720\beta^3+4389396480)\lambda_j\\ &\quad +(209018880\beta^3+1619896320\beta+1132185600\beta^2+313528320)\lambda_i,\\ \xi_{01}^{(2)}(\lambda_j;\lambda_i;\beta) &= 3135283200\beta^2+6479585280\beta+487710720\beta^3+4389396480\\ &\geq 0 \text{ for all } \beta \geq -3.\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i;\lambda_i;\beta) &= 479001600\lambda_i^2(\beta+3)(\beta+2)(\beta+1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i;\lambda_i;\beta) &= 87091200\lambda_i(8\beta+9)(\beta+3)(\beta+2) \geq 0 \text{ for all } \beta \geq -\frac{9}{8}.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j;\lambda_i;\beta) = \xi_0(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{00} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}
\xi_{00}(\lambda_j;\lambda_i;\beta) &= (783820800\beta^2 + 1619896320\beta + 121927680\beta^3 + 1097349120)\lambda_j^3 \\
&\quad +(156764160\beta^3 + 849139200\beta^2 + 235146240 + 1214922240\beta)\lambda_i\lambda_j^2 \\
&\quad +(613992960\beta + 261273600\beta^2 + 548674560 + 39191040\beta^3)\lambda_i^2\lambda_j \\
&\quad +(21772800\beta^2 + 63866880\beta + 1451520\beta^3 + 34836480)\lambda_i^3, \\
\xi_{00}^{(1)}(\lambda_j;\lambda_i;\beta) &= (2351462400\beta^2 + 4859688960\beta + 365783040\beta^3 + 3292047360)\lambda_j^2 \\
&\quad +(313528320\beta^3 + 1698278400\beta^2 + 470292480 + 2429844480\beta)\lambda_i\lambda_j \\
&\quad +(613992960\beta + 261273600\beta^2 + 548674560 + 39191040\beta^3)\lambda_i^2, \\
\xi_{00}^{(2)}(\lambda_j;\lambda_i;\beta) &= (4702924800\beta^2 + 9719377920\beta + 731566080\beta^3 + 6584094720)\lambda_j \\
&\quad +(313528320\beta^3 + 1698278400\beta^2 + 470292480 + 2429844480\beta)\lambda_i, \\
\xi_{00}^{(3)}(\lambda_j;\lambda_i;\beta) &= 104509440(\beta+3)(7\beta^2+24\beta+21) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{00}(\lambda_i;\lambda_i;\beta) &= 319334400\lambda_i^3(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(1)}(\lambda_i;\lambda_i;\beta) &= 718502400\lambda_i^2(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{00}^{(2)}(\lambda_i;\lambda_i;\beta) &= 130636800\lambda_i(8\beta+9)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -\frac{9}{8}.
\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l;\lambda_j;\lambda_i;\beta) &= (43545600\beta^3 + 805593600\beta + 326592000\beta^2 + 653184000)\lambda_l^2 \\
&\quad +((65318400\beta^3 + 609638400\beta + 370137600\beta^2 + 261273600)\lambda_j \\
&\quad +(65318400\beta^3 + 609638400\beta + 370137600\beta^2 + 261273600)\lambda_i)\lambda_l \\
&\quad +(13063680\beta^3 + 204664320\beta + 87091200\beta^2 + 182891520)\lambda_i^2 \\
&\quad +(13063680\beta^3 + 204664320\beta + 87091200\beta^2 + 182891520)\lambda_j^2 \\
&\quad +(39191040\beta^3 + 195955200\beta^2 - 104509440 + 200309760\beta)\lambda_i\lambda_j,
\end{aligned}$$

$$\begin{aligned}
\xi_1^{(1)}(\lambda_l;\lambda_j;\lambda_i;\beta) &= (87091200\beta^3+1611187200\beta+653184000\beta^2+1306368000)\lambda_l \\
&\quad +(65318400\beta^3+609638400\beta+370137600\beta^2+261273600)\lambda_j \\
&\quad +(65318400\beta^3+609638400\beta+370137600\beta^2+261273600)\lambda_i \\
&\geq 0 \text{ for all } \beta \geq -\frac{2}{3}.
\end{aligned}$$

Define $\xi_{10}(\lambda_j;\lambda_i;\beta)=\xi_1(\lambda_j;\lambda_j;\lambda_i;\beta)$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}
\xi_{10}(\lambda_j;\lambda_i;\beta) &= (121927680\beta^3+1619896320\beta+783820800\beta^2+1097349120)\lambda_j^2 \\
&\quad +(104509440\beta^3+809948160\beta+566092800\beta^2+156764160)\lambda_i\lambda_j \\
&\quad +(13063680\beta^3+204664320\beta+87091200\beta^2+182891520)\lambda_i^2, \\
\xi_{10}^{(1)}(\lambda_j;\lambda_i;\beta) &= (243855360\beta^3+3239792640\beta+1567641600\beta^2+2194698240)\lambda_j \\
&\quad +(104509440\beta^3+809948160\beta+566092800\beta^2+156764160)\lambda_i, \\
\xi_{10}^{(2)}(\lambda_j;\lambda_i;\beta) &= 34836480(\beta+3)(7\beta^2+24\beta+21) > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{10}(\lambda_i;\lambda_i;\beta) &= 239500800\lambda_i^2(\beta+3)(\beta+2)(\beta+1) > 0 \text{ for all } \beta > -1, \\
\xi_{10}^{(1)}(\lambda_i;\lambda_i;\beta) &= 43545600\lambda_i(8\beta+9)(\beta+3)(\beta+2) > 0 \text{ for all } \beta > -\frac{9}{8}.
\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.21 Let

$$\begin{aligned}
\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta) &= (653184000+326592000\beta^2+43545600\beta^3+805593600\beta)t(0,0,0,2) \\
&\quad +(130636800\beta^3+1219276800\beta+522547200+740275200\beta^2)t(0,0,1,1),
\end{aligned}$$

where

$$t(a,b,c,d) = \sum_{\substack{(n_1,n_2,n_3,n_4)\\ \in \mathrm{perm}(a,b,c,d)}} \lambda_i^{n_1}\lambda_j^{n_2}\lambda_l^{n_3}\lambda_l^{n_4}.$$

The expression $\xi(\lambda_k;\lambda_l;\lambda_j;\lambda_i;\beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$, and $\beta \geq -\frac{2}{5}$.

Proof.

Now

$$\begin{aligned} 653184000 + 326592000\beta^2 + 43545600\beta^3 + 805593600\beta &> 0 \text{ for all } \beta \geq -2, \\ 130636800\beta^3 + 1219276800\beta + 522547200 + 740275200\beta^2 &> 0 \text{ for all } \beta \geq -\frac{2}{3}. \end{aligned}$$

Hence, $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

$\square$

References

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