AN ELEMENTARY ANALYSIS OF COUPLED-BUNCH INSTABILITIES*

K. M. Hock#, A. Wolski, Physics Dept, Liverpool University, Liverpool, L69 7ZE, UK and Cockcroft Institute, Daresbury, Warrington, WA4 4AD, UK.

Abstract

We reconsider the equations of motion of wake field coupled bunches in the light of recent developments in Delay Differential Equations. In the case of uniform resistive wall in a storage ring, we demonstrate an alternative way to characterize the growth modes. For each multibunch Fourier mode, an infinite number of time domain modes can arise from an exact solution of the equation of motion. The growth rate as it is commonly defined corresponds to only one of them. The amplitude of each Fourier mode can therefore evolve with time in a way that is not a simple exponential. This is a result that has been observed in simulations of wake field coupled bunches.

INTRODUCTION

Charged particles in a storage ring generate wake fields as a result of impedance of the beam pipe. This gives rise to coupling between bunches, and is an important source of instability. In this paper, we study the effect of this coupling on the transverse displacement of the bunches. The wake fields can potentially cause this displacement to grow exponentially, resulting in decoherence and beam loss. For simplicity, we only consider the case where the impedance arises from a uniform, cylindrical beam pipe with resistive wall, though our approach is equally valid in the case where additional sources of wake field (for example, higher-order modes in RF cavities) are present.

Beam instabilities driven by resistive-wall wake fields have been widely studied. The standard formalism is to transform the bunch displacements into Fourier modes. The equations of motion are then decoupled, and the amplitude of each mode is assumed to follow a simple exponential. This has allowed an analytic expression for the growth rate to be obtained [1], and an analytic formula for the bunch trajectory to be derived [2]. Our study is motivated by the need to achieve much more challenging levels of beam stability in future accelerators, such as the ILC damping rings, and to develop a detailed understanding of the dynamics that develops in the transient regime during injection and extraction.

Simulation studies of bunch motion in the presence of coupling from wake fields show mode amplitudes that appear to deviate from simple exponentials [3]. In our analysis, we find that the standard formalism neglects the existence of other solutions to the equations of motion. This means that although the growth rate formula is valid, the trajectory formula is incomplete. We derive the correct formula for the trajectory, and also show that this is valid only in the limit of small wake field.

FORMALISM

Equations of Motion

The equation of motion for each bunch is given by:

$$\ddot{x}_m(t) + \omega_\beta^2 x_m(t) = -\frac{N r_0 c}{\gamma T_0}\{W_1(-c\tau)x_{m+1}(t-\tau) + W_1(-2c\tau)x_{m+2}(t-2\tau) + \ldots\} \tag{1}$$

for $m = 0, 1, \ldots, M-1$. The notation is as follows:

$x_m(t)$	transverse displacement of the m^{th} bunch
ω_β	betatron frequency
$W_1(z)$	wake function
τ	time from one bunch to the next
N	number of particles in each bunch
r_0	classical electron radius
c	speed of light
T_0	period of revolution
γ	energy of each particle in units of its rest mass

The mode is defined by the Fourier transform

$$\tilde{x}_\mu(t) = \sum_{m=0}^{M-1} x_m(t) e^{-i\frac{2\pi m\mu}{M}} \tag{2}$$

The new parameter μ represents the mode number. In order to transform to modes, the equation for each bunch in eq. (1) is multiplied by $\exp(-i2\pi m\mu/M)$. The equations are then summed and rearranged to give a set of decoupled equations that can be written in the form:

$$\ddot{\tilde{x}}_\mu(t) + \omega_\beta^2 \tilde{x}_\mu(t) = \sum_{n=1} b_n e^{i\frac{2\pi n\mu}{M}} \tilde{x}_\mu(t - n\tau) \tag{3}$$

for $\mu = 0, 1, \ldots, M-1$. Substituting into eq. (3) the elementary solution $e^{-i\Omega t}$, we obtain, in the case of resistive wall wake field, the characteristic equation:

$$-\Omega^2 + \omega_\beta^2 = b_0\left(e^{i\frac{2\pi\mu}{N}} e^{i\Omega\tau} + \frac{1}{\sqrt{2}} e^{i\frac{2\pi 2\mu}{N}} e^{i2\Omega\tau} + \frac{1}{\sqrt{3}} e^{i\frac{2\pi 3\mu}{N}} e^{i3\Omega\tau} + \ldots\right) \tag{4}$$

where

$$b_0 = \frac{N r_0 c}{\gamma T_0}\frac{2}{\pi b^3}\sqrt{\frac{1}{\sigma\tau}}C \tag{5}$$

In eq. (5), C is the ring circumference, b is the pipe radius and σ is the conductivity.

*Work supported by Science and Technology Facilities Council, UK
#k.meng_hock@dl.ac.uk

In the standard formalism, the approximation $\Omega \approx \omega_\beta$ is then made, to obtain:

$$\Omega \approx \omega_\beta - \frac{b_0}{2\omega_\beta} \times\left(e^{i\frac{2\pi\mu}{N}} e^{i\omega_\beta \tau} + \frac{1}{\sqrt{2}} e^{i\frac{2\pi 2\mu}{N}} e^{i2\omega_\beta \tau} + \frac{1}{\sqrt{3}} e^{i\frac{2\pi 3\mu}{N}} e^{i3\omega_\beta \tau} + ... \right) \tag{6}$$

from which the growth rate formula can be derived [1].

Delay Differential Equations

The equation of motion (1) is in fact a Delay Differential Equation; equations of this type are the subject of active research [4]. Here, we use some of the ideas to study eq. (3) more closely. Eq. (4) is in fact a transcendental equation with an infinite number of roots. By rearranging this into:

$$f(\Omega_\mu) = \Omega_\mu^2 - \omega_\beta^2 + \sum_{n=1}^{N} b_n e^{i\frac{2\pi n\mu}{M}} e^{i\Omega_\mu \tau} \tag{7}$$

such that the roots are just the zeros of this function, we can then get a snapshot of the distribution of the roots from a contour of the absolute value of this function. Figure 1 shows this contour plot for the case when the infinite series is truncated at N = 100. The time variable has been transformed using $t \rightarrow t' = t/\tau$.

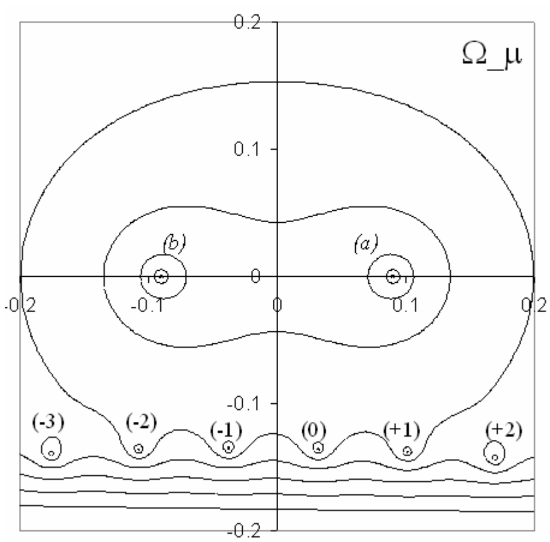

Figure 1. Contour plot from the characteristic equation for mode 500, showing the distribution of roots – located at the centres of the concentric contours.

The general solution is therefore given by:

$$\tilde{x}_\mu = Ae^{-i\Omega_\mu^{(a)} t} + Be^{-i\Omega_\mu^{(b)} t} + \sum_{n=-\infty}^{\infty} C_n e^{-i\Omega_\mu^{(n)} t} \tag{8}$$

The root in eq. (5) corresponds to root (a) in fig. 1, which appears in the first term on the right hand side of this equation. In principle, the unknown constants A, B and C_n can be obtained by fitting with the initial conditions. The reason why there is an infinite number of constants is because the initial condition is in fact an initial history. If we consider the case when the series is truncated at N = 100, then integrating eq. (1) or (3) requires not only the initial values at t = 0, but also all the values in the time interval $[-N\tau, 0)$. The form of eq. (8) also means that the mode amplitude is in general not a simple exponential.

Bunch Trajectory

In practice, it is difficult to know the initial history of the bunch displacements. In order to make use of these roots to give a useful solution, we make the following interpretation of the roots. Roots (a) and (b) are very close to $+\omega_\beta$ and $-\omega_\beta$ respectively. In the limit of zero wake field, the right hand side of eq. (3) vanishes. Eq. (3) becomes the equation for a simple harmonic oscillator, which has only two solutions. These would be just the first two terms on the right hand side of eq. (8). The infinite sum in eq. (8) may therefore be attributed to the presence of the wake field, which has led to the dependence on initial history. The infinite sum is likely to behave like a Fourier sum that can be fitted to arbitrary functions on the interval $[-N\tau, 0)$ [4]. If the wake field is small, it may be possible to neglect this sum and keep only the first two terms. Then for each mode, we can obtain the constants A and B by specifying just the initial values at t = 0:

$$\tilde{x}_\mu(t) = A_\mu e^{-i\Omega_\mu^{(a)} t} + B_\mu e^{-i\Omega_\mu^{(b)} t} \tag{9}$$

where

$$A_\mu = \frac{i\Omega_\mu^{(b)} \tilde{x}_\mu(0) + \dot{\tilde{x}}_\mu(0)}{i\Omega_\mu^{(b)} - i\Omega_\mu^{(a)}} \tag{10}$$

and

$$B_\mu = \frac{i\Omega_\mu^{(a)} \tilde{x}_\mu(0) + \dot{\tilde{x}}_\mu(0)}{i\Omega_\mu^{(a)} - i\Omega_\mu^{(b)}} \tag{11}$$

Inverting the transform, we get:

$$x_m(t) = \frac{1}{M} \sum_{\mu=0}^{M-1} \left[A_\mu e^{-i\Omega_\mu^{(a)} t} + B_\mu e^{-i\Omega_\mu^{(b)} t} \right] e^{i\frac{2\pi m\mu}{M}} \tag{12}$$

for the trajectories in real space.

As it turns out, there is already an accurate formula for root (a), given by eq. (6). A corresponding formula can be derived in a similar way for root (b) by making the approximation that $\Omega \approx -\omega_\beta$. This gives:

$$\Omega \approx -\omega_\beta + \frac{b_0}{2\omega_\beta} \times\left(e^{i\frac{2\pi\mu}{N}} e^{-i\omega_\beta \tau} + \frac{1}{\sqrt{2}} e^{i\frac{2\pi 2\mu}{N}} e^{-i2\omega_\beta \tau} + \frac{1}{\sqrt{3}} e^{i\frac{2\pi 3\mu}{N}} e^{-i3\omega_\beta \tau} + ... \right) \tag{13}$$

RESULTS AND DISCUSSION

The above reasoning leading to the trajectory formula in eq. (12) is heuristic and needs to be validated. We do so by comparing the analytical result with simulation. We carry out the calculation using the following parameters for the OCS6 damping ring in the ILC [5]:

Circumference of ring	6695.057 m
Particle energy	5.0 GeV
Horizontal tune	52.397
Number of particles per bunch	2×10^{10}
Number of bunches	3649
Beam pipe (aluminium) conductivity	3.2×10^{17} s^{-1}
Beam pipe radius	10 mm

The integration is carried out on eq. (3) using the method described in ref. [3]. The objective is to determine whether eq. (9) remains accurate when the initial history is neglected, i.e. when it is set to zero over the interval $[-N\tau,0)$. Before the actual comparison is made, however, we must consider the error inherent in the numerical integration itself. To make an estimate of the error, we construct a history using eq. (9) on $[-N\tau,0)$. If the numerical integration has no error, the result of the integration for $t > 0$ will agree exactly with that of eq. (9). Any difference between the numerical integration and eq. (9) is therefore the numerical error. Figure 2 illustrates the integration result for mode 500.

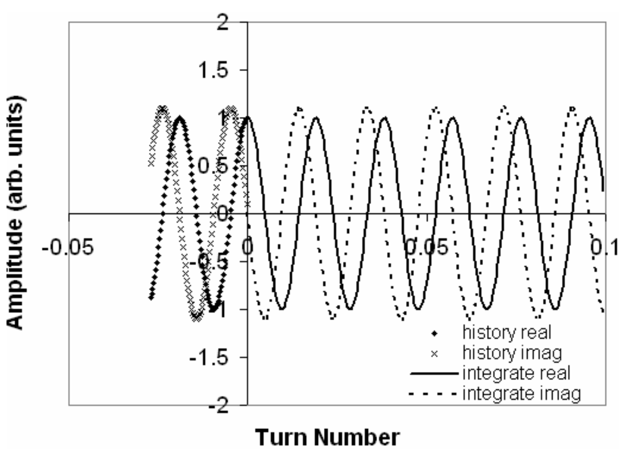

Figure 2. Single-mode simulation of mode 500 using initial history created from the analytic solution.

We then set the initial history to zero and repeat the integration. To account for the numerical error, in the case of mode 500 for example, we compare the zero history result with that in fig. 2 (which gives the result of the numerical integration with non-zero initial history) instead of eq. (9) directly. The difference between the numerical integration in the two cases we refer to as the "history-induced error". We find that the history-induced error is fairly small and approaches a constant value at large turn numbers. In the case of mode 500, the relative error in eq. (9) as a result of neglecting the initial history is about 10^{-5} at turn number 500. A similar calculation for an individual bunch rather than an individual mode can be performed using eq. (12). In the case of bunch number 100, we find a relative error of about 0.001 at turn number 100.

Since the coupling between bunches arise from the wake field, the strength of the wake field would have a direct impact on the size of the history-induced error. We insert on the right hand side of eq. (3) a multiplying factor, which we call the wake field strength, and repeat the error calculation for different values of this factor. The result for mode 500 at turn number 500 is shown in fig. 3. The history-induced error is nearly proportional to the wake field strength. This provides an estimate on the range of validity of eq. (12).

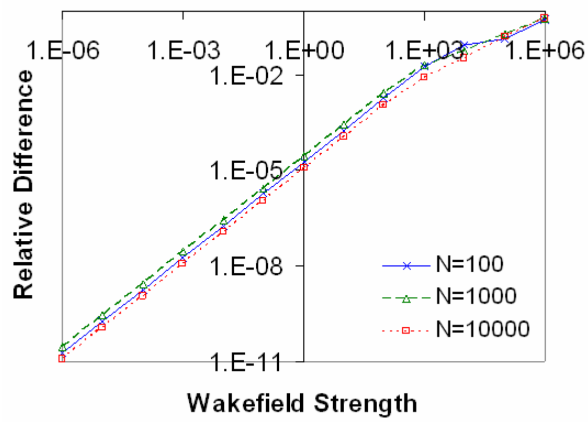

Figure 3. History-induced error for mode 500 as a function of wake field strength.

Finally, it should be mentioned that in this analysis, the maximum growth rate remains unaffected. The growth rate in the standard formalism is given by the imaginary part of eq. (5). Although eq. (13) introduces a second growth rate for each mode, this equation is simply a reflection ($\mu \rightarrow -\mu$) of eq. (5), and the maximum growth rate remains the same. What this analysis achieves, however, is an explanation of some otherwise puzzling features in the simulations of coupled-bunch motion. This is important if the simulation methods are to be applied (as intended) to cases representing more realistic conditions, including details of the optical functions and the fill patterns.

REFERENCES

[1] A. W. Chao, *Physics of Collective Beam Instabilities* (John Wiley & Sons, New York, 1993).

[2] K. A. Thompson and R. D. Ruth, Phys. Rev. D, **43**, 3049 (1991).

[3] A. Wolski, LBNL-59526, LCC-0149, CBP Tech Note-322 (July 2004).

[4] F. M. Asi, and A. G. Ulsoy, Journal of Dynamic Systems, Measurement, and Control, **125**, 215 (2003).

[5] A. Wolski, J. Gao, and S. Guiducci (eds.), *Configuration Studies and Recommendations for the ILC Damping Rings*, LBNL-59449 (2006). http://www.linearcollider.org/cms/?pid=100027