



Free vibrations of a solid cylinder

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1. Introduction

The value of Young's modulus may be measured from the free vibrations of a bar, providing the relations between the frequency of the vibrations and the elastic constants of the material are known. The following summarises the theory and gives approximate formulae; in general the analysis follows the classic work of Love [1]. It is assumed that the vibrations are uncoupled, which is true for cases where the length is very much larger than the diameter of the bar or vice versa. The case of coupled vibrations is treated in UKNF/Target/N2/2010.

2. The Propagation of Waves in Elastic Solid Media

2.1 Waves of Dilation and of Distortion

In Cartesian coordinates, x, y, z , where the displacements along these axes are u, v, w , respectively, the equations of motion are given by [2],

$$\begin{aligned}(\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 u &= \rho \frac{\partial^2 u}{\partial t^2} \\(\lambda + \mu)\frac{\partial \Delta}{\partial y} + \mu \nabla^2 v &= \rho \frac{\partial^2 v}{\partial t^2} \\(\lambda + \mu)\frac{\partial \Delta}{\partial z} + \mu \nabla^2 w &= \rho \frac{\partial^2 w}{\partial t^2}\end{aligned}\tag{2.1}$$

Differentiating the three equations in (2.1) by x, y and z respectively and adding them together gives a compact form of (2.1) when the body forces are constant throughout the volume of the body,

$$(\lambda + 2\mu)\nabla^2 \Delta = \rho \frac{\partial^2 \Delta}{\partial t^2}\tag{2.1a}$$

The terms, $\rho \frac{\partial^2 u_{x,y,z}}{\partial t^2}$, are the body forces due to the mass of the body, where ρ is the density of the material. The volume expansion, or cubical dilation, Δ , is the sum of the individual strains,

$$\Delta = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\tag{2.2}$$

λ and μ are Lamé's constants, which, when given in terms of Young's modulus of elasticity, E , the modulus of elasticity in shear (or modulus of rigidity), G , and Poisson's ratio, ν , become,

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}\tag{2.3}$$

$$\mu = G = \frac{E}{2(1 + \nu)}\tag{2.4}$$

If the volume expansion is zero, then,

$$\begin{aligned}
\mu \nabla^2 u - \rho \frac{\partial^2 u}{\partial t^2} &= 0 \\
\mu \nabla^2 v - \rho \frac{\partial^2 v}{\partial t^2} &= 0 \\
\mu \nabla^2 w - \rho \frac{\partial^2 w}{\partial t^2} &= 0
\end{aligned} \tag{2.5}$$

These are equations of waves called *equivoluminal waves*, or *waves of distortion*; and their velocity is given by,

$$c_2 = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2\rho(1+\nu)}} \tag{2.6}$$

Differentiating the three equations (2.1) with respect to x , y , z , respectively, and adding the results, gives,

$$(\lambda + 2\mu) \nabla^2 \Delta = \rho \frac{\partial^2 \Delta}{\partial t^2} \tag{2.7}$$

as the equivalent to (2.1).

For the case where the deformation produced by the waves is not accompanied by rotation, whose components of rotational strain are the sums of simple shear components and are given by [1],

$$2\varpi_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0; \quad 2\varpi_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0; \quad 2\varpi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \tag{2.8}$$

then,

$$\begin{aligned}
(\lambda + 2\mu) \nabla^2 u - \rho \frac{\partial^2 u}{\partial t^2} &= 0 \\
(\lambda + 2\mu) \nabla^2 v - \rho \frac{\partial^2 v}{\partial t^2} &= 0 \\
(\lambda + 2\mu) \nabla^2 w - \rho \frac{\partial^2 w}{\partial t^2} &= 0
\end{aligned} \tag{2.9}$$

These equations represent waves which are called *irrotational waves*, or *waves of dilation*; their velocity is.

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{M}{\rho}} = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} \tag{2.10}$$

Where M is the longitudinal or axial modulus (see 2.5 for a definition of M).

2.2 Plane Waves

If a disturbance is produced at a point in an elastic medium, waves radiate from this point in all directions. At a great distance from the centre of the disturbance the waves can be considered as plane waves and it can be assumed that all particles in the medium are moving parallel to the direction of wave propagation (*longitudinal waves* or *waves of dilation*) or perpendicular to this direction (*transverse waves* or *waves of distortion*). Longitudinal waves have a higher velocity than transverse waves.

2.3 Rayleigh Surface Waves [3]

The propagation of disturbances in an elastic medium can be represented as a superposition of irrotational and equivoluminal waves. However when there are free boundaries, or interfaces between two media, other velocities of propagation are possible. Surface waves can appear in a thin layer, similar to the ripples caused by a stone thrown into a pond. The velocity of these waves is given by [1,2],

$$c_3 = \alpha c_2 = \alpha \sqrt{\frac{\mu}{\rho}} \quad (2.11)$$

where α is 0.91-0.96 depending on the value of ν .

2.4 Waves –Transverse and Parallel to their Direction of Propagation- Coupling-Harmonics

In general, waves can propagate along all three axes and the vibration can be either transverse or parallel to that axis. In addition these waves can be coupled to each other through the elastic medium to give frequencies which are different to the frequencies of their components. Moreover, the vibrations can have harmonics of the fundamental frequencies. As a result the vibrations of a body are in general quite complex. In the following sections, the vibrations of free solid bars will be studied for a limited number of cases.

2.5 The Longitudinal or Axial Modulus

The axial modulus M [4,5], is defined similarly to Young's modulus, being the ratio of the longitudinal stress to the longitudinal strain, but the strain in the transverse directions is maintained at zero by lateral stresses. It may be shown that the relationship of the axial modulus to other moduli is,

$$M = E \frac{(1-\nu)}{(1+\nu)(1-2\nu)} = \lambda + 2\mu \quad (2.12)$$

It can be seen that $(\lambda+2\mu)$ occurs frequently in the above equations (2.10) and in the following sections. Clearly the axial modulus is an important parameter – although rarely used. Also it can be seen from (2.10) that it determines the velocity of irrotational or longitudinal waves (and the radial velocity in long free bars). M determines the velocity of ultrasonic stress pulses in solids [5] and is sometimes called the P-wave (primary wave) modulus.

2.6 Observations on Poisson's Ratio

Soft metals like gold and lead have $\nu \approx 0.44$ and for rubber $\nu \approx 0.48$. Most other materials have values in the range 0.35-0.25. Considering the relationship between the elastic constants it is found that Poisson's ratio must be between 0 and 0.5. At $\nu = 0.5$ there is a singularity in the equations involving $(1-2\nu)$ since the term becomes zero and values of $\nu \geq 0.5$ are unrealistic.

2.7 Particle Velocity

If a bar, for example, is struck axially on its end with a compressive stress σ , it can be shown [1,2] that the longitudinal velocity of the particles of the bar is,

$$v = \frac{c\sigma}{E} = \frac{\sigma}{\sqrt{\rho E}} \quad (2.13)$$

This velocity is proportional to the applied stress, whereas the velocity of the wave propagation, c , is independent of the stress.

2.8 Coupled Vibrations

Coupled vibrations will be treated in more detail in UKNF/Target/N4/2010 but it is worthwhile pointing out the following salient points. A piece of elastic material will not in general have a single mode of vibration but several modes which are coupled to each other through the three dimensional elastic forces. The equations of motion (2.1) and (3.1-3.3) represent these coupled vibrations.

3. Vibrations of a Free Solid Circular Cylinder

Consider a solid cylinder, length $2L$ and radius a . The general equations of small motion of the body in polar coordinates, r , θ and z , where the z axis is coincident with that of the cylinder and $z = 0$ is at the centre of the cylinder, are given by [1],

$$(\lambda + 2\mu)\frac{\partial \Delta}{\partial r} - \frac{2\mu}{r}\frac{\partial \varpi_z}{\partial \theta} + 2\mu\frac{\partial \varpi_\theta}{\partial z} - \rho\frac{\partial^2 u_r}{\partial t^2} = 0 \quad (3.1)$$

$$(\lambda + 2\mu)\frac{1}{r}\frac{\partial \Delta}{\partial \theta} - 2\mu\frac{\partial \varpi_r}{\partial z} + 2\mu\frac{\partial \varpi_z}{\partial r} - \rho\frac{\partial^2 u_\theta}{\partial t^2} = 0 \quad (3.2)$$

$$(\lambda + 2\mu)\frac{\partial \Delta}{\partial z} - \frac{2\mu}{r}\frac{\partial(r\varpi_\theta)}{\partial r} + 2\mu\frac{\partial \varpi_r}{\partial \theta} - \rho\frac{\partial^2 u_z}{\partial t^2} = 0 \quad (3.3)$$

where Δ is the cubical dilation given by,

$$\Delta = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \quad (3.4)$$

The terms, $\rho\frac{\partial^2 u_{r,\theta,z}}{\partial t^2}$, are the body forces due to the mass of the cylinder. The displacements of a point in the cylinder from its position, r , θ , z when unstressed, are given by u_r , u_θ , u_z and ϖ_r , ϖ_θ , ϖ_z , which are the components of rotational strain given by,

$$2\varpi_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}, \quad 2\varpi_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \quad 2\varpi_z = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right) \quad (3.5)$$

For a cylinder unrestrained over all its surfaces, the stresses at the curved surface, $r = a$ and the stresses at the ends of the cylinder, $z = \pm L$, vanish. These stress components are expressed by,

$$\overline{rr} = \lambda\Delta + 2\mu\frac{\partial u_r}{\partial r}, \quad \overline{r\theta} = \mu\left\{\frac{1}{r} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right\}, \quad \overline{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \quad (3.6)$$

Generally it is not possible to solve the equations (3.1-3.3) with all these boundary conditions but if the bar is long compared to its diameter ($L \gg a$) approximate solutions to equations of longitudinal and radial motions are possible (see Love [1] chapter XII for a discussion of these points).

The vibrations along the three axes will take the form,

$$u_r = Ue^{i(\gamma z + pt)}, \quad u_\theta = Ve^{i(\gamma z + pt)}, \quad u_z = We^{i(\gamma z + pt)} \quad (3.7)$$

where U , V and W are functions of r and θ . The exponential term represents a sinusoidal oscillation in time and distance along the z axis with a velocity of propagation γ/p along the z axis. Similar waves can be formed travelling along the other axes and expressed by equations like (3.7).

In the following sections the free vibrations in the longitudinal, transverse (both flexural and simple radial, where the axis is stationary) and torsional directions will be covered with emphasis on the radial which is of the main interest in the current investigations.

4. Torsional Vibrations (from Love [1])

For a pure torsional vibration, U and W vanish and V is independent of θ . Then Δ and ϖ_θ also vanish, $2\varpi_r = -\frac{\partial u_\theta}{\partial z}$, $2\varpi_z = \frac{1}{r} \frac{\partial(ru_\theta)}{\partial r}$ and (3.2) becomes,

$$\mu \frac{\partial^2 u_\theta}{\partial z^2} + \mu \frac{1}{r} \left\{ \left[2 \frac{\partial u_\theta}{\partial r} + r \frac{\partial^2 u_\theta}{\partial r^2} \right] - \frac{1}{r} \left[u_\theta + r \frac{\partial u_\theta}{\partial r} \right] \right\} - \rho \frac{\partial^2 u_\theta}{\partial t^2} = 0 \quad (4.1)$$

Putting in u_θ from (3.7) gives,

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} - \frac{1}{r^2} V + \kappa^2 V = 0, \text{ where } \kappa^2 = \frac{p^2 \rho}{\mu} - \gamma^2 \quad (4.2)$$

The above equation in V is a Bessel equation and the solution is of the form, $V = BJ_1(\kappa r)$ where B is a constant and J_1 denotes Bessel's function of order unity. The conditions at the surface of a free bar, $r = a$, are that the stresses vanish, so this gives the condition that κa is the root of the equation $\frac{\partial}{\partial a} \left\{ \frac{J_1(\kappa a)}{a} \right\} = 0$. One solution is $\kappa = 0$, giving $V = Br$ and hence the equation of the torsional vibration is $u_\theta = Bre^{i(\gamma z + pt)}$, where $\gamma^2 = p^2 \rho / \mu$. These are waves of torsion and are propagated along the bar at a velocity $c_2 = \sqrt{\mu / \rho}$.

4.1 Torsional Vibrations (from Weaver [2])

The torsional frequency of vibration of a bar can be arrived at in a somewhat simpler way [2]. Let the torque on the cross section at z be T . Then the torques on a small slice of the bar, length dz at z are, $T + \frac{\partial T}{\partial z} dz$ at $z + dz$ and $-T$ at z . In addition the body force due to the rotating mass is $-\rho I_p dz \frac{\partial^2 \theta}{\partial t^2}$, where I_p is the polar moment of inertia of the cross section.

Adding these torques together gives, $\frac{\partial T}{\partial z} - \rho I_p dz \frac{\partial^2 \theta}{\partial t^2} = 0$. Since, from elementary torsion theory, $T = \mu I_p \frac{\partial \theta}{\partial z}$, so, $\frac{\partial T}{\partial z} = \mu I_p \frac{\partial^2 \theta}{\partial z^2}$, and this gives the familiar wave equation,

$\frac{\partial^2 \theta}{\partial z^2} = \frac{1}{c_2^2} \frac{\partial^2 \theta}{\partial t^2}$, where the velocity is $c_2 = \sqrt{\frac{\mu}{\rho}}$. The solution for free vibrations of the bar is of the form,

$$u_\theta = C_n \cos\left(\frac{n\pi}{2L} c_2 t\right) \cos\left(\frac{n\pi}{2L} z\right), \text{ where } C_n \text{ is a constant.}$$

5. Longitudinal Vibrations (from Love [1])

Consider the vibration equations given by (3.1), (3.2) and (3.3); the torsional vibration vanishes, so $V = 0$ and U and W are independent of θ . Equation (3.2) vanishes and equations (3.1) and (3.3) can be expressed as,

$$\frac{\partial^2 \Delta}{\partial r^2} + \frac{1}{r} \frac{\partial \Delta}{\partial r} + h^2 \Delta = 0 \quad (5.1)$$

$$\frac{\partial^2 \varpi_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial \varpi_\theta}{\partial r} - \frac{\varpi_\theta}{r} + \kappa \varpi_\theta = 0 \quad (5.2)$$

where,

$$h^2 = \frac{p^2 \rho}{(\lambda + 2\mu)} - \gamma^2 \quad \kappa^2 = \frac{p^2 \rho}{\mu} - \gamma^2 \quad (5.3)$$

$$\Delta = \left(\frac{\partial U}{\partial r} + \frac{U}{r} + i\gamma W \right) e^{i(\gamma z + pt)} \quad 2\varpi_\theta = \left(i\gamma U - \frac{\partial W}{\partial r} \right) e^{i(\gamma z + pt)} \quad (5.4)$$

Taking Δ and ϖ_θ as functions of r , they can be expressed as proportional to the Bessel functions $J_0(hr)$ and $J_1(\kappa r)$. To satisfy the equations (5.4) U and W take the forms,

$$U = A \frac{\partial}{\partial r} J_0(hr) + C \gamma J_1(\kappa r) \quad (5.5)$$

$$W = A i \gamma J_0(hr) + \frac{iC}{r} \frac{\partial}{\partial r} \{r J_1(\kappa r)\} \quad (5.6)$$

where A and C are constants. At the surface of the bar, $r = a$, the stress vanishes, so this gives for A and C ,

$$A \left[2\mu \frac{\partial^2 J_0(ha)}{\partial a^2} - \frac{p^2 \rho \lambda}{\lambda + 2\gamma} J_0(\kappa a) \right] + 2\mu C \gamma \frac{\partial J_1(\kappa a)}{\partial a} = 0 \quad (5.7)$$

$$2A \gamma \frac{\partial J_0(ha)}{\partial a} + C \left(2\gamma^2 - \frac{p^2 \rho}{\mu} \right) J_1(\kappa a) = 0 \quad (5.8)$$

Eliminating A/C , gives,

$$2\mu \gamma \frac{\partial J_1(\kappa a)}{\partial a} - 2\gamma \frac{\partial J_0(ha)}{\partial a} - \left(2\gamma^2 - \frac{p^2 \rho}{\mu} \right) J_1(\kappa a) \left[2\mu \frac{\partial^2 J_0(ha)}{\partial a^2} - \frac{p^2 \rho \lambda}{\lambda + 2\gamma} J_0(\kappa a) \right] = 0 \quad (5.9)$$

The angular frequency of the longitudinal vibrations, p , can be found from (5.9). For $a \ll L$ the Bessel functions may be approximated by [3],

$$J_0(ha) = 1 - \left(\frac{ha}{2} \right)^2 + \frac{1}{4} \left(\frac{ha}{2} \right)^4 \quad J_1(\kappa a) = \frac{1}{2} \left(\kappa a - \left(\frac{\kappa a}{2} \right)^3 \right) \quad (5.10)$$

When $\kappa a = 0$ there is no longitudinal vibration. If terms in κa are omitted and terms up to a^2 retained, then the approximate result for p in terms of γ is $p = \gamma \sqrt{E/\rho} \left(1 - \frac{1}{4} \nu^2 \gamma^2 a^2\right)$. The oscillations, both radial and longitudinal, are propagated along the z axis with a velocity,

$$c = \sqrt{\frac{E}{\rho} \left[1 - \left(\frac{a\pi\nu}{2L}\right)^2\right]} \quad (5.11)$$

which for small a/L can be approximated by,

$$c = \sqrt{\frac{E}{\rho}} \quad (5.12)$$

The frequencies are given by,

$$f_n = \frac{(2n+1)c}{4L} \approx \frac{(2n+1)}{4L} \sqrt{\frac{E}{\rho} \left[1 - \left(\frac{a\pi\nu}{2L}\right)^2\right]} \quad (5.13)$$

where $n = 0, 1, 2, 3, \dots$

An interesting case occurs when the bar is restrained radially so that u_r is zero. Effectively this is the definition of the *Axial* or *Longitudinal Modulus*. Putting $U = 0$, $V = 0$, and W is not a function of r or θ , then (3.3) becomes,

$$(\lambda + 2\mu) \frac{\partial^2 u_z}{\partial z^2} = \rho \frac{\partial^2 u_z}{\partial t^2} \quad (5.14)$$

This represents a wave (*irrotational*) propagating along the z axis with a velocity,

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{M}{\rho}} = \sqrt{\frac{E}{\rho} \frac{(1-\nu)}{(1+\nu)(1-2\nu)}} \quad (5.15)$$

Note the similarity of c and c_1 with the change of E to M .

5.1 Longitudinal Vibrations (from Timoshenko [2])

This simple analysis is taken from Timoshenko and Goodier [2]. Assume that the bar is subject to a simple longitudinal stress only and the other components of stress are negligible. Then the equation of axial stress is,

$$\sigma_z = E \frac{\partial u_z}{\partial z} \quad (5.16)$$

$$\frac{\partial \sigma_z}{\partial z} - \rho \frac{\partial^2 u_z}{\partial t^2} = 0 \quad (5.17)$$

$$\frac{\partial^2 u_z}{\partial z^2} = \frac{\rho}{E} \frac{\partial^2 u_z}{\partial t^2} \quad (5.18)$$

This represents a wave propagating along the z axis with a velocity,

$$c = \sqrt{\frac{E}{\rho}} \quad (5.19)$$

which is the same as (5.12). The solution to (5.18) can be expressed as,

$$u_r = f(z + ct) + f_1(z - ct) \quad (5.20)$$

which represents forward and backward travelling waves, with the “bar velocity”, c .

The original solution to (5.1) and (5.2) incorporates the radial influence on the longitudinal motion. When $a \ll L$ the answer becomes the same as (5.19) even though there is no reference to any other vibrations than the longitudinal direction in its calculation. However, a totally different answer is found when U vanishes in addition to V and W is not a function of r , see (5.14) above.

6.3 Longitudinal Waves in Free Bars (Sievers [8])

Sievers [8] has treated the vibrations of a bar where they are excited by a virtually instantaneous pulse of a proton beam causing a step rise in the temperature of the bar. He obtained a Fourier series for the longitudinal strain on a free bar, representing two square waves travelling in opposite directions (also see [2] for a similar calculation).

$$\sigma_z(z, t) = A \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left\{ \cos \frac{(2n+1)}{4L} 2\pi(z+ct) + \cos \frac{(2n+1)}{4L} 2\pi(z-ct) \right\} \quad (5.21)$$

where $n = 0, 1, 2, 3, \dots$, z is the distance along the axis of bar with the origin at the centre of the bar and A is a constant determined by the magnitude of the applied stress at $t = 0$. The stress is initially uniform along the length of the bar and then shrinks in length about the centre of the bar until the stress vanishes after a time L/c , where c is the bar velocity given above. The repeat frequency of the wave pattern along the rod is,

$$f_z = \frac{c}{4L} = \frac{1}{4L} \sqrt{\frac{E}{\rho}} \quad (5.22)$$

Sievers [8] also considers the radial stresses, but concludes that it is not possible to calculate them when the bar is free.

6. Radial Vibrations (Airey [9])

Airey [9] has treated radial vibrations of solid cylinders; his analysis, with additions, follows. Taking the equations of motion (3.1), (3.2) and (3.3), for the case where there is no motion in θ ; u_r and u_z are not functions of θ or z (this gives waves propagating radially), then (3.1) reduces to,

$$(\lambda + 2\mu) \frac{\partial}{\partial r} \left(\frac{\partial u_r}{r} + \frac{u_r}{r} \right) - \rho \frac{\partial^2 u_r}{\partial t^2} = 0 \quad (6.1)$$

and (3.2) and (3.3) vanish. It is assumed that u_r can be expressed as a function of t as,

$$u_r(t) = \alpha e^{i\omega_r t} \quad (6.2)$$

and as a function of r by the solution of the equation,

$$(\lambda + 2\mu) \frac{\partial}{\partial r} \left(\frac{\partial u_r}{r} + \frac{u_r}{r} \right) = -\rho \omega_r^2 u_r \quad (6.3)$$

where ω_r is the radial angular frequency of the oscillation. Putting,

$$\kappa^2 (\lambda + 2\mu) = \rho \omega_r^2 \quad (6.4)$$

gives,

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \left[\kappa^2 - \frac{1}{r^2} \right] \cdot u_r = 0 \quad (6.5)$$

A solution to this Bessel equation is [7],

$$u = (AJ_0(\kappa r) + BY_0(\kappa r))e^{i\omega t} \quad (6.6)$$

where $J_0(\kappa r)$ is a Bessel function of the first kind and order zero and $Y_0(\kappa r)$ is a Bessel function of the second kind and order zero. At the axis, $r = 0$, $Y_0(0)$ becomes minus infinity and is unrealistic so B must be zero. In addition, since the cylinder is free from stress at its surface, this gives the condition at $r = a$,

$$(\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda\frac{u}{r} = 0 \quad (6.7)$$

Substituting (6.6) into (6.7) gives,

$$\kappa a J_1'(\kappa a) + \frac{\lambda}{\lambda + 2\mu} J_1(\kappa a) = 0 \quad (6.8)$$

and letting

$$\kappa a = \xi \quad (6.9)$$

gives,

$$\xi J_1'(\xi) + \frac{\lambda}{\lambda + 2\mu} J_1(\xi) = 0 \quad (6.10)$$

Since $xJ_n'(x) = nJ_n(x) - xJ_{n-1}(x)$, [7], equation (6.10) may be expressed as,

$$\xi J_0(\xi) - \frac{2\mu}{\lambda + 2\mu} J_1(\xi) = 0 \quad (6.11)$$

where $J_0(\xi)$ is a Bessel function of the first kind and order zero. Expressing Lamé's constants in (6.11) in terms of ν ,

$$\frac{2\mu}{\lambda + 2\mu} = \frac{(1 - 2\nu)}{(1 - \nu)} \quad (6.12)$$

Equation (6.11) becomes,

$$\xi J_0(\xi) - \frac{(1 - 2\nu)}{(1 - \nu)} J_1(\xi) = 0 \quad (6.13)$$

The first 5 roots of (6.13), for various ν , are listed below in Table 1.

ν	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
0	1.841184	5.520078	8.536316	11.706005	14.863589
0.1	1.923306	5.352994	8.549502	11.715562	14.871069
0.2	2.017172	5.379773	8.565949	11.727495	14.880473
0.3	2.125749	5.413895	8.587023	11.74281	14.892517
0.35	2.186766	5.434695	8.599944	11.752217	14.899919
0.4	2.25309	5.458748	8.614968	11.763171	14.908546
0.45	2.325478	5.486853	8.632644	11.776086	14.918726
0.5	2.404826	5.520078	8.653728	11.791534	14.930918

Table 1. The first 5 roots (excluding $\xi = 0$) of $\xi J_0(\xi) - \frac{(1 - 2\nu)}{(1 - \nu)} J_1(\xi) = 0$ at $r = a$.

The frequencies of the radial oscillations are given by (6.4) and using the values of ξ_n (Table 1) in (6.3) provides the solution,

$$f_n = \frac{\xi_n(\nu)}{2\pi a} \sqrt{\frac{(\lambda + 2\mu)}{\rho}} = \frac{\xi_n(\nu)}{2\pi a} c_1 \quad (6.14)$$

or, in terms of Young's modulus and Poisson's ratio,

$$f_n = \frac{\xi_n(\nu)}{2\pi a} \sqrt{\frac{E}{\rho} \frac{(1-\nu)}{(1+\nu)(1-2\nu)}} \quad (6.15)$$

Equation (6.6) represents a series of radial vibrations (see Figure 1 for the variation of $J_1(\xi_n r/a)$ with r/a) with frequencies f_n given by (6.14). The fundamental frequency, f_1 , has a radial amplitude of vibration increasing from zero on the axis of the cylinder to a maximum near $r = a$. (Note that $\nu = 0.5$ makes the frequency infinite.) The harmonics scale very approximately as whole numbers: 2, 3, 4, etc. The harmonics have nodes at radii where $u(r) = 0$. Hence, from (6.6), these nodal cylinders are given by the roots of,

$$AJ_1(\kappa r)e^{i\omega_n t} = 0 \quad (6.16)$$

which are shown in Table 2 for $\nu = 0.3$.

$(\kappa r)_1$	$(\kappa r)_2$	$(\kappa r)_3$	$(\kappa r)_4$	$(\kappa r)_5$
3.831706	7.015587	10.173468	13.323692	16.470630

Table 2. The first 5 roots of $J_1(\kappa r) = 0$ for $\nu = 0.3$.

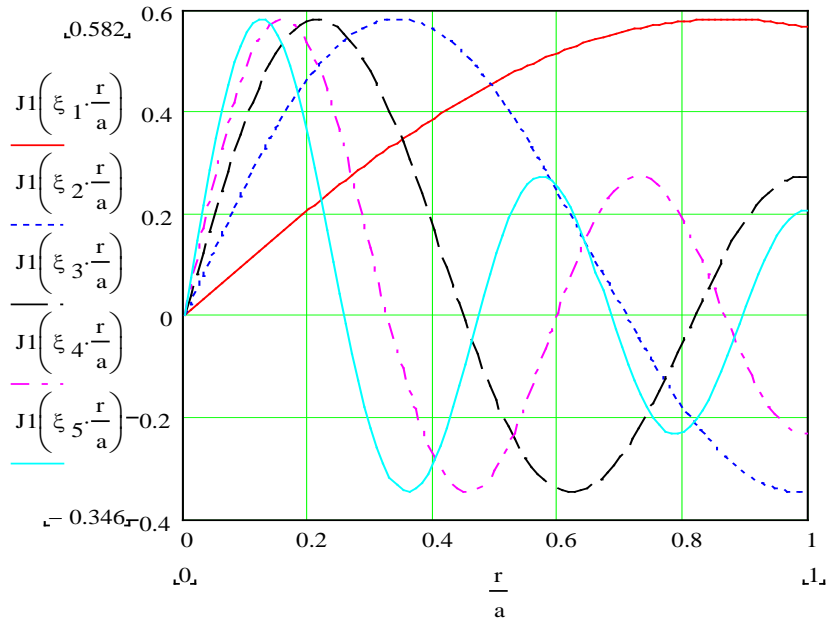


Figure 1. The radial oscillations $J_1(\xi_n r/a)$ as a function of radius, r/a , for the resonant frequencies given by ξ_n in Table 1, for $\nu = 0.3$.

The first harmonic has nodes at $r = 0$ and r_1 . The frequency of the oscillation is given by (6.14) as, $f_2 = \frac{\xi_2(\nu)}{2\pi a} \sqrt{\frac{(\lambda + 2\mu)}{\rho}}$ where, from (6.9), $\xi_2(\nu) = \kappa_2(\nu)a$. The value of κ in $(\kappa r)_1$ is of course κ_2 , thus $r_1 = (\kappa r)_1 / \kappa_2$. Hence, expressing the radius of the first nodal cylinder in terms of ξ_n and $(\kappa r)_n$, gives,

$$r_1 = \frac{(\kappa r)_1}{\kappa_2} = \frac{(\kappa r)_1}{\xi_2} a, \quad \text{and for } \nu = 0.3, \quad r_1 = \frac{3.8317}{5.4139} a = 0.7101a$$

With 2 nodes, in addition to the one at $r = 0$, the frequency is given by ξ_3 (or κ_3) and the radii of the nodes, r_2 , are given by,

$$r_2 = \frac{(\kappa r)_{1,2}}{\kappa_3} = \frac{(\kappa r)_{1,2}}{\xi_3} a, \quad \text{for } \nu = 0.3, \quad r_2 = 0.4462a : 0.8170a$$

And with 3 nodal cylinders, the radii, r_3 , are,

$$r_3 = \frac{(\kappa r)_{1,2,3}}{\kappa_4} = \frac{(\kappa r)_{1,2,3}}{\xi_4} a, \quad \text{and for } \nu = 0.3, \quad r_3 = 0.3263a : 0.5974a : 0.8664a$$

6.1 Radial Vibrations (Abe, Tanaka and Uo)

Abe, Tanaka and Uo [10-13] have also calculated the frequency of radial vibrations of a solid cylinder where $L \gg a$. Their approach is rather different from that of Airey (9). They start from an electrical analogy of oscillations (see for example [14] for an analogy between mechanical and electrical vibrations) and calculate the longitudinal and radial resonant frequencies and then combine the two vibrations with a three dimensional coupling constant which they identify with Poisson's ratio by analogy to the results given by Love (1). They obtain the relationship for the uncoupled vibration as,

$$F_n = \frac{\varsigma_n}{2\pi a} \sqrt{\frac{E}{\rho} \frac{1}{(1+\nu)^2(1-2\nu)}} \quad (6.17)$$

where ς_n are the roots of the equation,

$$\varsigma_n J_0(\varsigma_n) - (1-\sigma) J_1(\varsigma_n) = 0 \quad (6.18)$$

Equations (6.17) and (6.18) are similar to (6.13) and (6.15), respectively. For $\nu \ll 1$, (6.13) and (6.15) approximate to (6.18) and (6.17). However, the differences in the frequencies are ~1% with $\nu = 0.3$. The roots of (6.18) are shown below in Table 3 for $\nu = 0.3$.

ς_1	ς_2	ς_3	ς_4	ς_5
2.048850	5.389364	8.571859	11.731787	14.883847

Table 3. The first 5 roots of (6.18) for $\nu = 0.3$.

The ratio of the fundamental frequencies according to Airey [9] to that of Abe, Tanaka and Uo [10-13] is,

$$\Pi_1 = f_1 / F_1$$

$$\Pi_1 = \xi_1 / \varsigma_1 \sqrt{1-\nu^2}, \quad \text{which, for } \nu = 0.3, \text{ gives } \Pi_1 = 0.9897$$

The 3rd harmonic ratio is, $\Pi_4 = \frac{\xi_4}{\varsigma_4} \sqrt{1-\nu^2}$, which for $\nu = 0.3$, gives $\Pi_4 = 0.9548$. Clearly

the agreement gets worse for higher harmonics and for larger values of ν . With $\nu = 0.4$, $\Pi_1 = 0.9790$, and $\Pi_4 = 0.9183$.

7. Transverse Vibrations of Strings [14]

In the case of a violin string for example or a tuning fork, the whole of the string or prong of the fork can be vibrated transversely about its longitudinal axis. In the case of a string, the string is attached to supports at its ends and it is assumed that there are no bending forces, hence a tension, X , must be applied to provide the restoring force when the string is displaced transversely. The velocity of the wave propagated along the string is given by [14],

$$c_s = \sqrt{\frac{X}{m}} \quad (7.1)$$

where m is the mass per unit length of the string. The fundamental, simplest frequency of the transverse vibration is,

$$f_s = \frac{c}{4L} \quad (7.2)$$

where $2L$ is the length of the string. In terms of the density, ρ , of the string and its radius, a , (7.2) becomes,

$$f_s = \frac{1}{4aL} \sqrt{\frac{X}{\pi\rho}} \quad (7.3)$$

Other harmonic frequencies can be excited, given by nf_s , where n is 1, 2, 3....

8. Transverse Vibrations of Bars [14]

The transverse vibration of bars is different from those of strings; it is assumed that strings have no bending force, whereas bars do. Hence the bar does not require the application of an axial stress. The equation of lateral motion, u , is [14],

$$\frac{\partial^4 u}{\partial z^4} + \frac{1}{(ck)^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad (8.1)$$

where k is the radius of gyration of the cross-section about the axis. For a solid cylinder,

$$k = \frac{a}{\sqrt{2}} \quad (8.2)$$

Substituting $e^{i\omega t} u(z)$ into (8.1) gives,

$$\frac{\partial^4 u}{\partial z^4} + \frac{\omega^2}{(ck)^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad (8.3)$$

where ω is the angular frequency of the transverse vibrations of the bar. The solution may be put in the form,

$$u(z) = A \cos(bz) + B \sin(bz) + \cosh(bz) + D \sinh(bz) \quad (8.4)$$

where,

$$b = \sqrt{\frac{\omega}{ck}} \quad (8.5)$$

If an end of the cylinder is free, $\frac{\partial^2 u}{\partial z^2} = 0$ and $\frac{\partial^3 u}{\partial z^3} = 0$; if an end is clamped, u and $\frac{\partial u}{\partial z} = 0$; and if an end is simply supported, $\frac{\partial^2 u}{\partial z^2} = 0$ and $u = 0$. For the case where one end ($z = 0$) of the bar is clamped and the other end ($z = 2L$) is free, the conditions at the clamped end are,

$$\begin{aligned} A + C &= 0 \\ B + D &= 0 \end{aligned}$$

while at the free end, the conditions are,

$$-A \cos \zeta - B \sin \zeta + C \cosh \zeta + D \sinh \zeta = 0$$

$$A \sin \zeta - B \cos \zeta + C \sinh \zeta + D \cosh \zeta = 0$$

where $2bL = \zeta$.

These 4 equations give,

$$A(\cos \zeta + \cosh \zeta) + B(\sin \zeta + \sinh \zeta) = 0$$

$$A(\sin \zeta - \sinh \zeta) - B(\cos \zeta + \cosh \zeta) = 0$$

and eliminating A (or B) from these last two equations gives,

$$\cos \zeta \cosh \zeta + 1 = 0 \quad (8.6)$$

The equivalent equation to (8.6) for the bar clamped (or free) at both ends is,

$$\cos \zeta \cosh \zeta - 1 = 0 \quad (8.7)$$

For the bar clamped at one end and simply supported at the other, the conditions at the clamped end, $z = 0$, are,

$$\begin{aligned} A + C &= 0 \\ B + D &= 0 \end{aligned} \quad (8.8)$$

and at the simply supported end, $z = 2L$,

$$\begin{aligned} A \cos \zeta + B \sin \zeta + C \cosh \zeta + D \sinh \zeta &= 0 \\ -A \cos \zeta - B \sin \zeta + C \cosh \zeta + D \sinh \zeta &= 0 \end{aligned} \quad (8.9)$$

Substituting for C and D , from (8.8) into (8.9) gives.

$$\begin{aligned} A(\cos \zeta - \cosh \zeta) + B(\sin \zeta - \sinh \zeta) &= 0 \\ -A(\cos \zeta + \cosh \zeta) - B(\sin \zeta + \sinh \zeta) &= 0 \end{aligned} \quad (8.10)$$

And eliminating A (or B) from the last two equations (8.10) gives,

$$\cos \zeta \sinh \zeta = 0 \quad (8.11)$$

Table 4 shows the values of ζ for four options of support for the bar.

Support	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
Clamped/Free	1.875	4.694	7.855	10.996	14.137	----
Clamped/Clamped		4.730	7.853	10.996	14.137	----
Free/Free		4.730	7.853	10.996	14.137	----
Clamped/Simply Supported		3.9266	7.0686	10.2102	13.3518	16.4934

Table 4. Values of ζ for different types of end support of the bar.

The transverse bar frequencies are given by (8.5),

$$f_{trn} = \frac{ck}{2\pi} \left(\frac{\zeta_n}{2L} \right)^2 \quad (8.12)$$

Substituting (8.2) for a bar of circular section, gives,

$$f_{trn} = \frac{ca}{2\sqrt{2}\pi} \left(\frac{\zeta_n}{2L} \right)^2, \text{ where } c = \sqrt{E/\rho}. \text{ This gives,}$$

$$f_{trn} = \frac{a}{2\pi} \sqrt{\frac{E}{2\rho}} \left(\frac{\zeta_n}{2L} \right)^2 \quad (8.13)$$

Figure 2 shows the transverse displacement as a function of z for a bar clamped at $z = 0$ and simply supported at $z = 2L$ and Figure 3 shows the transverse displacement as a function of z for a bar clamped at $z = 0$ and at $z = 2L$.

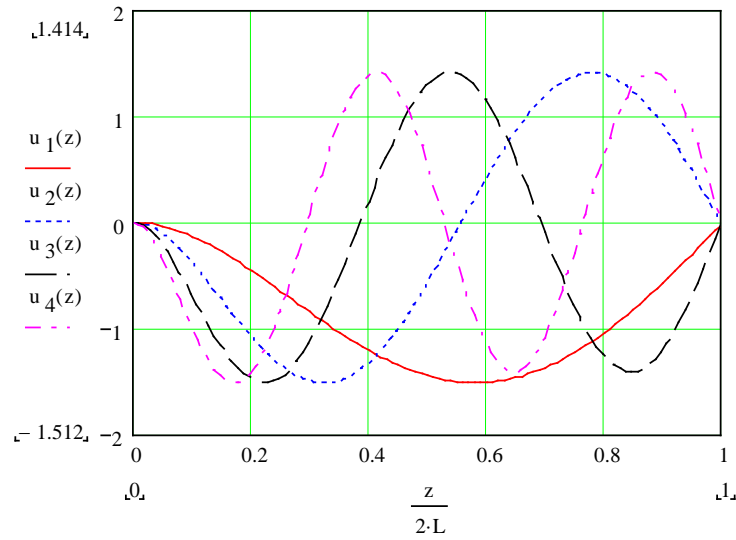


Figure 2. Transverse vibration of a bar clamped at one end, $z = 0$, and simply supported at the other, $z = 2L$.

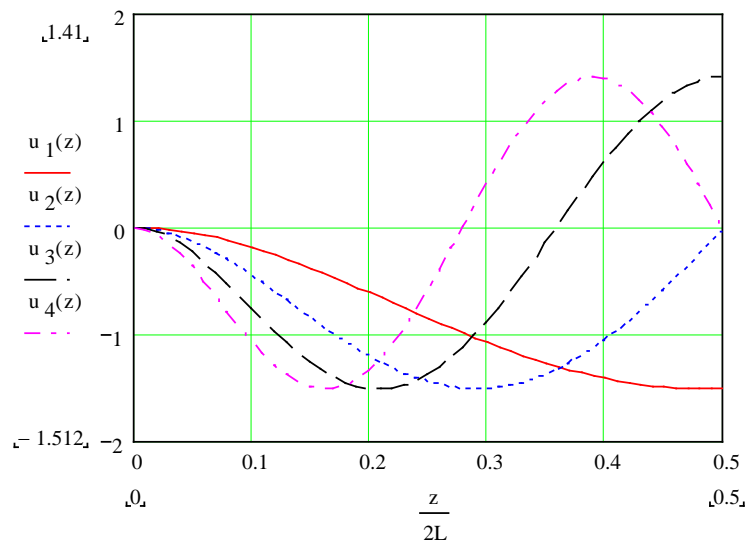


Figure 3. Transverse vibrations of a bar clamped at both ends. The vibrations are only shown for the first half of the bar; the second half is a mirror image of the first half.

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