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Beauty in Physics: the Legacy of Paul Dirac*

N. A. McCubbin**

In 2002 physicists around the world celebrated the centenary of the birth of Paul Dirac, OM, FRS, Nobel Laureate, who was one of the greatest physicists of the 20th century. He made towering contributions to the formulation of quantum mechanics and he was one of the principal creators of quantum field theory. In 1928 he combined relativity and quantum mechanics in the Dirac equation, which provides a natural description for the spin of the electron and which led to the prediction, by Dirac himself, of the existence of anti-matter. In this article I try to explain, in the simplest terms, these major contributions to physics and to give some flavour of the man himself.

1. Introduction

Throughout his life Paul Dirac loved walking. It was his habit, as a young post-graduate in Cambridge, to take long walks in the Cambridge countryside at the weekend. On one of these walks in the autumn of 1925, probably Sunday 20th September [1], he was thinking about a paper by a young German physicist, Werner Heisenberg, which suggested a new approach to quantum theory. The mathematics in the paper had a peculiar feature that Heisenberg found disturbing. Dirac thought suddenly of a similar feature in classical mechanics, but he couldn't quite remember the details, and, frustratingly, libraries in Cambridge were closed on Sundays. It is reasonable to suppose that Dirac was at the library's doors as they opened on Monday morning; he checked his hunch and found, to his delight, that the pieces fitted: the feature that the paper's author had found so disturbing in fact mapped perfectly on to the Poisson Brackets of classical mechanics. The young post-graduate had discovered the bridge between classical mechanics and a general theory of quantum mechanics.

This story of Dirac's discovery is well known, delightful, and all the better for being true, since we have it from Dirac himself [2]. (I will say more about it later.) It occurred at the beginning of a 'golden period' in physics to which Dirac himself would contribute so much. Just eight years later, in 1933, he was a Nobel Prize winner, Fellow of the Royal Society, and Lucasian Professor of Mathematics at Cambridge. He lived until 1984 and did outstanding work after 1933. But his place in the pantheon of physics rests on the work of those eight years.

In the following sections I describe and discuss Dirac's major contributions to the development of quantum theory, and thus to modern physics. My primary aim is to make the main ideas of Dirac's principal contributions as accessible as possible, so that, for example, the interested physics undergraduate should be able to follow the mathematics very easily, and, I hope, learn something of these great years in the history of physics. There is, therefore, no attempt whatsoever to give a complete or in-depth survey of Dirac's work, which would, in any case, far outstrip my competence.

Section 2 is devoted to the birth of quantum mechanics in 1925. I describe Heisenberg's odd-looking multiplication rule for quantum quantities and Dirac's discovery that this rule had an analogy in the Poisson Brackets of classical mechanics. Dirac's discovery leads directly to one of the defining principles of modern quantum theory.

** Particle Physics Department, Rutherford Appleton Laboratory, CCLRC, Chilton, Didcot, Oxfordshire, OX11 0QX, United Kingdom. E-mail: N.A.McCubbin@rl.ac.uk

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Section 3 is a brief summary of the remarkable blossoming of quantum theory that took place in 1926 and 1927, which sets the scene for sections 4 and 5 that describe what are, by common consent, Dirac's greatest contributions: the relativistic equation for the electron (which everyone, except Dirac, has always referred to as the Dirac Equation), and the prediction, from the equation, of anti-matter.

Section 6 is a very light-touch discussion of Dirac's contribution to quantum field theory, which underpins so much of the modern theory of particle physics. As is well known, Dirac himself was deeply sceptical of the development of quantum field theory in the second half of the last century, even though he was the originator of several of the key ideas.

I have attempted to give enough mathematical detail to give at least some flavour of Dirac's work and style, but I have cut mathematical corners aplenty in the interests of simplicity and accessibility to as wide an audience as possible.

2. 1925: Quantum Mechanics

The year 1925 saw the climax of several years of struggle to turn the early quantum ideas of Planck (1901, black-body radiation), Einstein (1905, photo-electric effect; 1916, emission and absorption of radiation), and Bohr (1913, spectrum of hydrogen) into a fully-fledged theory.

Bohr's 1913 picture of atomic structure has great heuristic value, even today: the hydrogen atom comprises a positively charged proton round which orbits the much lighter negatively charged electron. Only certain discrete orbits (and hence energies) are allowed, and radiation is emitted or absorbed by transitions between these discrete orbits.

By 1925 much effort and ingenuity had been expended in elaborating Bohr's picture and calculations, and it was well established that:

(a) classical physics could not explain atomic structure, for the simple reason that classical physics predicted that electrons orbiting a nucleus was not a stable configuration: any accelerating (by virtue of the orbital motion) electron would rapidly radiate off its energy and collapse into the nucleus;

(b) there was something 'discrete' (or 'quantized') about the atomic world: only certain values of energy, angular momenta,….. were allowed. In particular there must be energy levels which atomic electrons can occupy without radiating, in order to explain the stability of atomic structure;

(c) atomic spectroscopy (the frequencies, intensities and polarisations of light from atoms), which provided much of the information about the atomic world, could not be calculated in detail, except in the simplest cases (e.g. the hydrogen atom);

(d) the frequency, ν_{21}, of a spectral line was assumed to correspond to a *transition* between discrete energy levels E_2 and E_1, such that : $h\nu_{21} = E_2 - E_1$, where h is Planck's constant. This formula was found to work very well in those simple cases for which the energy levels (E) could be calculated. It also implied that the observed frequencies should often be simple combinations of each other. For example, in a three-level system with $E_3 > E_2 > E_1$, and in which transitions took place between all three levels, there is the simple prediction that $\nu_{31} = \nu_{32} + \nu_{21}$.

2.1. *Heisenberg's paper*

By the summer of 1925 Heisenberg, although still only 24 years old, had been fully engaged in the battle to tease a proper quantum theory out of spectroscopy for some time. He had worked at Göttingen and Copenhagen with some of the other leading young physicists, published several papers on quantum theory, and discussed the problems with Bohr and Einstein.

He was mindful of one of the lessons of Einstein's relativity theory: pay careful attention to what is measurable, and how you measure it. A classical approach to the atom would start, as Bohr had done, with the electron's orbit. But electron orbits are not the directly measured quantities. On the other hand spectral lines are observable; wavelengths (and hence frequencies) can be measured.

It was, and is, a valuable and powerful technique to seek solutions to problems in classical physics in terms of a Fourier series. For example, a position *X(t)* might be expanded as:

$$X(t) = \sum_{\alpha=-\infty}^{\infty} x_\alpha \exp(i\alpha\omega t); \omega = 2\pi\nu, \quad (1)$$

where ω is the angular frequency (I consider, for simplicity, a system with just a single characteristic frequency). The x_α are the coefficients of the expansion, and α takes positive and negative integer values so that all integer multiples of ω participate in the series.[1]

Consider now a two-level quantum system, see figure 1. There is only one measurable angular frequency: $\hbar\omega_{21} = E_2 - E_1$, where $\hbar \equiv h/2\pi$. It is convenient to define also $\omega_{11}=\omega_{22}=0$, and $\omega_{12} = -\omega_{21}$. If we insist on using measurable quantities only, then the Fourier series for *X(t)* has to be:

$$X(t) = x_{11}e^{i\omega_{11}t} + x_{12}e^{i\omega_{12}t} + x_{21}e^{i\omega_{21}t} + x_{22}e^{i\omega_{22}t}, \quad (2)$$

where the x_{ij} are expansion coefficients similar to the x_α in equation (1).

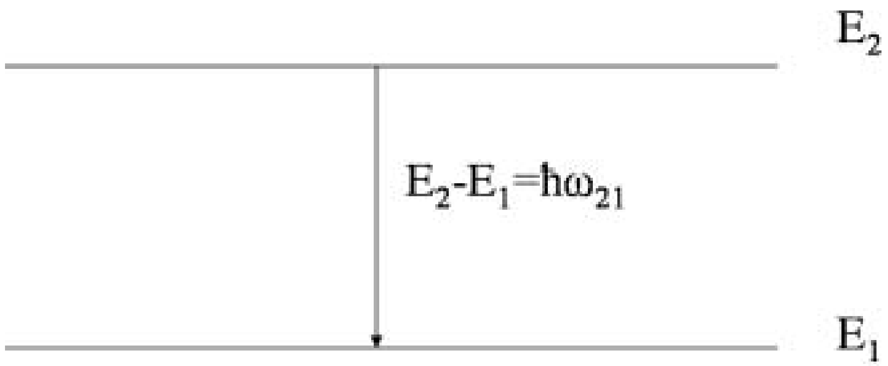

Figure 1. Two-level system

Another quantity *Y(t)* would be written:

$$Y(t) = y_{11}e^{i\omega_{11}t} + y_{12}e^{i\omega_{12}t} + y_{21}e^{i\omega_{21}t} + y_{22}e^{i\omega_{22}t}. \quad (3)$$

And what about *X(t)Y(t)*? Presumably this is perfectly straight-forward:

$$X(t)Y(t) = [x_{11}e^{i\omega_{11}t} + x_{12}e^{i\omega_{12}t} + x_{21}e^{i\omega_{21}t} + x_{22}e^{i\omega_{22}t}]\times[y_{11}e^{i\omega_{11}t} + y_{12}e^{i\omega_{12}t} + y_{21}e^{i\omega_{21}t} + y_{22}e^{i\omega_{22}t}]. \quad (4)$$

Mathematically this multiplication is of course perfectly natural. However, when multiplied out, the expression for *X(t)Y(t)* contains terms like $x_{21} \times y_{21} \times \exp(i2\omega_{21}t)$. Such a term is *not*

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[1] A position, *X*, must of course be a real quantity, so there is a convention (e.g. 'take the real part') or constraint (e.g. $x_\alpha = x_{-\alpha}$) required when writing a real quantity in terms of complex exponentials $\exp(i\alpha\omega t)$. However, it is algebraically extremely convenient to work with the complex exponential even in classical physics, and mandatory to do so in quantum theory. For a discussion of the significance of '*i*' in quantum mechanics, see [3].

allowed if we continue to insist that *only observed frequencies* appear in the Fourier series, since $2\omega_{21}$ does not correspond to any transition in our simple 2-level system.

Taking a cue from his theoretical studies of 'dispersion' (scattering of light by atoms) with Kramers, Heisenberg suggested in his ground-breaking 1925 paper [4] a new rule for multiplication in the quantum world. Instead of (4), Heisenberg's rule gives:

$$\begin{aligned} X(t)Y(t) &= x_{11}y_{11}e^{i2\omega_{11}t} + x_{12}y_{21}e^{i(\omega_{12}+\omega_{21})t} + x_{11}y_{12}e^{i(\omega_{11}+\omega_{12})t} + x_{12}y_{22}e^{i(\omega_{12}+\omega_{22})t} \\ &+ x_{21}y_{11}e^{i(\omega_{21}+\omega_{11})t} + x_{22}y_{21}e^{i(\omega_{22}+\omega_{21})t} + x_{21}y_{12}e^{i(\omega_{21}+\omega_{12})t} + x_{22}y_{22}e^{i2\omega_{22}t}. \end{aligned} \quad (5)$$

Stated a little clumsily, the rule is to multiply each '*ij*' term in the *X* series by all the '*jk*' terms in the *Y* series, e.g. the x_{21} term gets multiplied by the y_{11} and y_{12} terms, but not by the y_{21} or y_{22} term.

At first glance equation (5) may also seem to be littered with frequencies which don't correspond to actual transitions, but, remembering that $\omega_{11}=\omega_{22}=0$, and $\omega_{21}=-\omega_{12}$, the only non-trivial exponential terms are in fact $\exp(\pm i\omega_{21}t)$, i.e. measurable frequencies only.

Generalising to a multi-level system, and considering:

$$X(t) = ...+ x_{nj}e^{i\omega_{nj}t} + ...$$

$$Y(t) = ...+ y_{jm}e^{i\omega_{jm}t} + ...,$$

then Heisenberg's rule for $Z(t) = X(t)Y(t)$ is that

$$Z(t) = ...+ z_{nm}e^{i\omega_{nm}t} + ...$$

$$\text{where} \quad z_{nm}e^{i\omega_{nm}t} = \sum_j x_{nj}y_{jm}e^{i(\omega_{nj}+\omega_{jm})t}. \quad (6)$$

We note that $\omega_{nj} + \omega_{jm} = (E_n - E_j + E_j - E_m)/\hbar = \omega_{nm}$ for all *j*, and that Heisenberg's rule therefore guarantees that the same frequencies, and only the same frequencies, appear in the product as appear in the multiplicands.

In his 1925 paper, Heisenberg emphasised this feature of his multiplication rule: '..in fact this type of combination is an almost necessary consequence of the frequency combination rules' [4].

But there is another striking feature. Heisenberg again [4]:

'A significant difficulty arises, however, if we consider two quantities *X(t)*, *Y(t)*, and ask after their product *X(t)Y(t)*. …. Whereas in classical theory *X(t)Y(t)* is always equal to *Y(t) X(t)*, this is not necessarily the case in quantum theory'.[2]

And indeed this 'significant difficulty' is immediately apparent in our simple two-level example. For example, according to the Heisenberg multiplication rule, the product *X(t)Y(t)* contains the term $x_{11}y_{12}\exp(i(\omega_{11}+\omega_{12})t)$ (equation (5)), but there is no term in $x_{11}y_{12}$ in the Heisenberg expression for *Y(t)X(t)*. Modern readers will no doubt recognise that Heisenberg's multiplication rule, equation (6), is exactly that of matrix multiplication, but, at the time of writing his 1925 paper, Heisenberg, in common with almost all physicists of the period, had never heard of a matrix!

Having stated his multiplication rule, and noted the 'significant difficulty', Heisenberg proceeded in his 1925 paper to apply it to simple cases. Significantly and quite deliberately, Heisenberg took pains to consider cases involving products like X^2 in which the '$XY \neq YX$' difficulty did not appear, or at least was not manifest. The results looked encouraging.

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[2] As the second sentence is one of the most pregnant in all science, perhaps one should note it in the original German: 'Während klassisch *X(t)Y(t)* stets gleich *Y(t)X(t)* wird, brauch dies in der Quantentheorie im allgemeinen nicht der Fall zu sein.'

Crucially, he found that the principle of energy conservation survived his weird multiplication rule. He thought he was on to something, and he was right.

2.2. *Dirac's contribution*

In September 1925 Paul Dirac had been a post-graduate research student in Cambridge for about two years. Born in Bristol in 1902, he had been a precocious schoolboy, entering Bristol University at the age of 16, and gaining first-class honours in electrical engineering in 1921. Since his mathematical interest and gifts were already clear at school, it is at first sight surprising that he did not study mathematics at university. His father, who was by all accounts a severe and rather forbidding character from the French-speaking part of Switzerland, may have insisted that Paul should study a subject with good employment prospects, as he had insisted with Paul's elder brother. Perhaps Paul himself was attracted by the fact that the engineering department was in the same building as his secondary school, so the transition from school to university would be in a largely familiar environment; he was, after all, only sixteen. Whatever the reason, Paul excelled in the course, particularly its theoretical aspects, and he never regretted his engineering studies.

He then tried to get a job a job as an engineer: and failed. However disappointing this may have been personally, it is one of the best things that could have happened for the development of physics. He won a scholarship (by examination) to continue his studies at Cambridge, but the scholarship was not enough to support him there. So he stayed in Bristol where the University mathematics department invited him to take the undergraduate mathematics course, free of charge. He completed the three-year course in two years, and in 1923, encouraged and recommended by the Bristol mathematicians, he obtained a grant from the Department of Scientific and Industrial Research, which, when combined with his earlier scholarship, was enough to study at Cambridge. Cambridge would be his professional home for the next forty-six years.

Dirac's supervisor as a research student at Cambridge was Ralph Fowler, who had trained as a mathematician, but who had turned to theoretical physics, and developed a strong interest in the doings of experimentalists. (Perhaps not coincidentally, he was also Rutherford's son-in-law.) He had mastered the quantum theory such as it was, and Dirac attended his lectures on quantum physics. Fowler probably had a better understanding of the latest developments from the continental physicists than anyone else in Britain. Certainly he had spent the early months of 1925 working at the Niels Bohr Institute in Copenhagen; Bohr had visited and spoken in Cambridge in May 1925; two months later Heisenberg also lectured in Cambridge; certainly Fowler asked Heisenberg for early sight of his latest work, and in August 1925 Fowler asked his young research student, Paul Dirac, to take a look at the proof-sheets (prior to publication) of Heisenberg's paper.

At first reading Dirac was not overly impressed.[3] On second reading he focussed on precisely the 'significant difficulty' that Heisenberg had noted, but not pursued. If the difference between *XY* and *YX* is not zero, what *is* it equal to?

Dirac considered typical terms in the Heisenberg expression for *XY*-*YX*:

$$\begin{gathered}......+x_{100,97}y_{97,95}e^{i\omega_{100,95}t}+..... \\ ....-y_{100,98}x_{98,95}e^{i\omega_{100,95}t}-...... \;.\end{gathered} \tag{7}$$

[3] As already noted, Heisenberg's paper was written in German, which up to the middle of the last century had at least equal status to English in academic physics. Dirac had already studied with the aid of a dictionary the recently published 4th edition of Sommerfeld's Atombau und Spektrallinien.

I shall now assume that $\omega_{n,n-1} = \Omega$ where Ω is the (classical) angular frequency of the system,[4] so that $\omega_{100,95} = 5\Omega$. We now add and subtract the term $x_{98,95} y_{97,95} \exp[i(\omega_{98,95} + \omega_{97,95})t]$. Such a term does not conform to the Heisenberg product rule since the frequency combination $\omega_{98,95} + \omega_{97,95}$ is not guaranteed in general to be one of the observed transitional frequencies. But in this case $\omega_{n,n-1} = \Omega$, so $\omega_{98,95} + \omega_{97,95} = 5\Omega$, and we can write equation (7) as:

$$\begin{aligned} &......+(x_{100,97} - x_{98,95}) y_{97,95} e^{i5\Omega t} + ... \\ &.....................-(y_{100,98} - y_{97,95}) x_{98,95} e^{i5\Omega t} - ...... \; . \end{aligned} \tag{8}$$

(Of course coefficients like $x_{98,95}$ and $y_{97,95}$ are simple numbers which obey $x_{98,95} y_{97,95} = y_{97,95} x_{98,95}$.)

Generalising equation (8), we write:

$$\begin{aligned} &......+(x_{n,n-\alpha} - x_{n-\beta,n-\beta-\alpha}) y_{n-\alpha,n-\alpha-\beta} e^{i(\alpha+\beta)\Omega\, t} + ... \\ &.....................-(y_{n,n-\beta} - y_{n-\alpha,n-\alpha-\beta}) x_{n-\beta,n-\alpha-\beta} e^{i(\alpha+\beta)\Omega\, t} - ...... \; . \end{aligned} \tag{9}$$

We now consider large n and small $|\alpha|$ and $|\beta|$, and assume that the x and y coefficients can be treated as smoothly varying functions of a continuous variable $J(\equiv n\hbar)$, so that:

$$x_{n,n-\alpha} - x_{n-\beta,n-\beta-\alpha} \approx \hbar\beta \frac{dx_{n,n-\alpha}}{dJ} \text{ and } y_{n,n-\beta} - y_{n-\alpha,n-\alpha-\beta} \approx \hbar\alpha \frac{dy_{n,n-\beta}}{dJ}.$$

In setting $J = n\hbar$ we are of course echoing Bohr's quantization condition for angular momentum, as discussed further below.

The two terms of equation (9) can now be written:

$$\hbar\left[\frac{dx_{n,n-\alpha}}{dJ} e^{i\alpha\Omega t}\right]\beta y_{n-\alpha,n-\alpha-\beta} e^{i\beta\Omega t} - \hbar\left[\frac{dy_{n,n-\beta}}{dJ} e^{i\beta\Omega t}\right]\alpha x_{n-\beta,n-\alpha-\beta} e^{i\alpha\Omega t}.$$

If we further define $\theta \equiv \Omega t$, then

$$\alpha e^{i\alpha\Omega t} = -i\frac{de^{i\alpha\Omega t}}{d\theta} \text{ and } \beta e^{i\beta\Omega t} = -i\frac{de^{i\beta\Omega t}}{d\theta},$$

and equation (9) becomes:

$$-i\hbar\left\{\left[\frac{dx_{n,n-\alpha}}{dJ} e^{i\alpha\Omega t}\right] y_{n-\alpha,n-\alpha-\beta} \frac{de^{i\beta\Omega t}}{d\theta} - \left[\frac{dy_{n,n-\beta}}{dJ} e^{i\beta\Omega t}\right] x_{n-\beta,n-\alpha-\beta} \frac{de^{i\alpha\Omega t}}{d\theta}\right\}.$$

The '*nm*' term of *X.Y-Y.X* is obtained by summing over α and β with the constraint that $n-\alpha-\beta = m$, i.e. $\alpha+\beta = n-m$:

$$-i\hbar \sum_{\substack{\alpha,\beta \\ \alpha+\beta=n-m}} \left\{\left[\frac{dx_{n,n-\alpha}}{dJ} e^{i\alpha\Omega t}\right] y_{n-\alpha,n-\alpha-\beta} \frac{de^{i\beta\Omega t}}{d\theta} - \left[\frac{dy_{n,n-\beta}}{dJ} e^{i\beta\Omega t}\right] x_{n-\beta,n-\alpha-\beta} \frac{de^{i\alpha\Omega t}}{d\theta}\right\}. \tag{10}$$

Now a term like $x_{n,n-\alpha} \exp(i\alpha\Omega t)$, summed over α, is redolent of the *classical* expression for X(t) in equation (1). Assuming, with Bohr, that there must be a correspondence, in some suitable limit of large quantum numbers, between the quantum and classical descriptions (the

[4] This assumption is actually true for the (important) case of the simple harmonic oscillator, and becomes true for 'large quantum numbers (n)' for a general quantum system by virtue of Bohr's Correspondence Principle. Dirac considered a general system (of several degrees of freedom) and used the Correspondence Principle to justify $\omega_{n,n-1} = \Omega$.

Correspondence Principle), we now identify such sums in equation (10) with the classical descriptions, obtaining:

$$-i\hbar\left\{\frac{\partial X}{\partial J}\frac{\partial Y}{\partial \theta}-\frac{\partial Y}{\partial J}\frac{\partial X}{\partial \theta}\right\}=i\hbar\left\{\frac{\partial X}{\partial \theta}\frac{\partial Y}{\partial J}-\frac{\partial Y}{\partial \theta}\frac{\partial X}{\partial J}\right\}, \qquad (11)$$

where X and Y are the classical descriptions like equation (1).

And it must have been at about this point that Dirac took his Sunday walk.

Where had he seen something like equation (11) before? Years later Dirac recalled: '..I remembered something which I had read up previously in advanced books of dynamics about these strange quantities, Poisson brackets, and from what I could remember, there seemed to be a close similarity…..' [2]. Back at home he searched his notes and books for anything about Poisson brackets, but found nothing.

It is fanciful but not implausible to imagine that he tried out one or two things on paper: the rate of change of any function, $u(\theta, J)$, which is not an explicit function of time *t*, is given by:

$$\frac{du}{dt}=\frac{\partial u}{\partial \theta}\frac{d\theta}{dt}+\frac{\partial u}{\partial J}\frac{dJ}{dt}. \quad (12)$$

That's just mathematics.

If the angle, θ, and angular momentum, *J*, are canonically conjugate variables in the sense of classical mechanics, then Hamilton's equations[5] of classical mechanics apply:

$$\frac{d\theta}{dt}=\frac{\partial H}{\partial J} \text{ and } \frac{dJ}{dt}=-\frac{\partial H}{\partial \theta}, \quad (13)$$

where the *Hamiltonian*, $H(\theta,J)$, is the total energy of the system.

Substituting from equation (13) into equation (12) gives:

$$\frac{du}{dt}=\frac{\partial u}{\partial \theta}\frac{\partial H}{dJ}-\frac{\partial H}{\partial \theta}\frac{\partial u}{\partial J}. \quad (14)$$

The right-hand side of equation (14) is an example of a *Poisson Bracket*. For any two functions $u(\theta, J)$ and $v(\theta, J)$ the Poisson Bracket is defined as:

$$(u,v)_{PB}\equiv\frac{\partial u}{\partial \theta}\frac{\partial v}{dJ}-\frac{\partial v}{\partial \theta}\frac{\partial u}{\partial J}, \quad (15)$$

and so equation (11) is just $i\hbar(X,Y)_{PB}$.

To recapitulate: in the Correspondence Principle limit, in which classical and quantum descriptions should coincide, the difference between the Heisenberg products of two quantum quantities X,Y becomes equal to $i\hbar(X,Y)_{PB}$.

I have no idea if Dirac went through the reasoning of equations (12) to (14) that Sunday night, but it seems to me he might have done. He was very familiar with the Hamiltonian formalism of classical mechanics and had already published a paper that made use of it [5]. He had to wait impatiently for the libraries to open next morning, but '..I still think that my confidence grew during the course of the night' [2]. Next morning he looked up Poisson Brackets in Whittaker's Analytical Dynamics and '…found that they were just what I needed' [2].

This was just the kind of connection that Dirac was looking for: in place of a strange-looking multiplication rule and the mathematically somewhat fuzzy Correspondence Principle, the Hamiltonian formalism was mathematically precise, elegant, and powerful. Of

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[5] Hamilton's equations are an elegant and powerful formulation of classical mechanics. Part of their power resides in the fact that they apply for any pair (*q, p*) of canonically conjugate variables: $dq/dt=\partial H/\partial p$ and $dp/dt=-\partial H/\partial q$. For the simple case of a 1-d harmonic oscillator, $H(p,q)=p^2/2m+kq^2/2$, where *p* is the usual momentum and *q* is the usual spatial coordinate.

course he had only *proved* the connection in a particular limit, using, ironically, the Correspondence Principle. So he made a leap. In his paper 'The Fundamental Equations of Quantum Mechanics' [6] he wrote: 'We make the fundamental assumption that *the difference between the Heisenberg products of two quantum quantities is equal to* $ih/2\pi$ *times their Poisson bracket expression*.' (Dirac's italics) So he assumed the equality not just in some limit of large quantum numbers, but always! With this assumption results simply pour out.

Since the difference between Heisenberg products turns up so often, it is convenient to define for quantum quantities:

$$[X,Y] \equiv XY - YX \qquad (16)$$

which is referred to as the *commutator* of X and Y. Choose X and P_x as the canonically conjugate variables, then Dirac's fundamental assumption for quantum quantities u and v is:

$$[u,v] = i\hbar(u,v)_{PB} = i\hbar\left\{\frac{\partial u}{\partial X}\frac{\partial v}{\partial P_X} - \frac{\partial v}{\partial X}\frac{\partial u}{\partial P_X}\right\}. \quad (17)$$

We are free to choose $u = X$ and $v = P_x$, which gives immediately:

$$[X, P_X] = i\hbar\left\{\frac{\partial X}{\partial X}\frac{\partial P_X}{\partial P_X} - \frac{\partial P_X}{\partial X}\frac{\partial X}{\partial P_X}\right\} = i\hbar, \quad (18)$$

which is *the* defining equation of quantum mechanics.

From equations (14) and (15), the Hamiltonian formalism of classical mechanics gives:

$$\frac{du}{dt} = (u,H)_{PB}. \quad (19)$$

We obtain the quantum version from equation (17):

$$\frac{du}{dt} = \frac{[u,H]}{i\hbar} \quad \text{or} \quad i\hbar\,\frac{du}{dt} = [u,H], \quad (20)$$

which is the fundamental equation of motion in quantum mechanics. Setting $u=H$ we obtain:

$$i\hbar\frac{dH}{dt} = 0, \quad (21)$$

since, by definition, the commutator of anything with itself is zero. So the Hamiltonian doesn't change with time in quantum mechanics: in other words energy is conserved, just as in classical mechanics. Hence energy conservation, which Heisenberg obtained only after laborious calculation, falls out with complete generality from Dirac's approach.

All this, and much more, Dirac showed to his supervisor Fowler, who realised the importance of what his graduate student had done. Dirac's paper [6] was 'communicated' to the Royal Society by Fowler (who was an FRS) and published in the 1st December 1925 issue of the Proceedings of the Royal Society, just ten weeks after Heisenberg's own paper had been published in Z. Phys.[6] Dirac sent Heisenberg a manuscript-copy of his paper in November, and Heisenberg wrote back almost by return: 'I have read your extraordinarily beautiful paper on quantum mechanics with the greatest interest, and there can be no doubt at all that all your results are correct as far as one believes at all in the newly proposed theory.' [8] The unknown Cambridge graduate student, just 23 years old, had announced himself as a physicist of the first rank. His paper is a plausible candidate for the greatest theoretical-physics paper ever written by a graduate student.

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[6] A paper 'communicated' to the Royal Society by an FRS could be published at the discretion of the appropriate officer of the Royal Society without the need for an external referee's opinion, provided the paper was not more than 24 pages long [7]. Dirac's paper was only 12 pages.

3. The blossoming of quantum theory: 1926-1927

Heisenberg's 1925 paper opened the floodgates. Dirac was not the only person to see the key idea it contained. In Göttingen Max Born saw the multiplication rule, thought 'matrix', and, with his recent graduate student Pascual Jordan, derived the key equations $[X, P_X] = i\hbar$ and $i\hbar.du/dt = [u, H]$. Born and Jordan based their approach explicitly on a matrix representation: the equations they obtained were matrix equations [9]. There was considerable overlap with the results obtained by Dirac, but Dirac's approach was, characteristically, more general. He was thinking of a general 'algebra' which quantum quantities, 'q-numbers' as he called them, must obey.

The new quantum mechanics was developed rapidly in a series of papers by Dirac and by Born and co-workers, notably in the 'Dreimännerarbeit' paper of Born, Jordan and Heisenberg, [10]. Pauli [11] (and, almost simultaneously, Dirac [12]) tackled the hydrogen atom using the matrix approach, obtaining full agreement with the results of the 'old' quantum theory of Bohr and Sommerfeld, a key result in establishing confidence in the new methods. Dirac went to Copenhagen to work with Niels Bohr, who presided paternalistically over the revolution he had long hoped for. Max Born went on a lecture tour of the USA, and spread the new faith to large audiences of American physicists. In early 1926 any self-respecting theoretical physicist would have been thinking that he or she would just have to buckle down and learn about matrices.

And then in early 1926 there was a remarkable development: an Austrian physicist, Erwin Schrödinger, somewhat older than the youthful trio of Dirac, Heisenberg and Jordan, published an alternative theory couched in the familiar terms of a differential equation [13]. He solved the equation, the Schrödinger equation, for the hydrogen atom obtaining the well-established result for the energy spectrum. Schrödinger, and probably quite a few other theorists, hoped he had killed off matrices and 'quantum jumps' and the like, and restored the traditional mathematics of differential equations to its rightful pre-eminence in theoretical physics.

Years later, in a lecture at Edinburgh in 1981 [14], Dirac recalled, 'When I first heard about the Schrödinger theory I did not like it. The reason was that I was perfectly happy with the Heisenberg theory'. Heisenberg's reaction was, not surprisingly, similar. Then Schrödinger himself demonstrated that the two approaches were in fact equivalent [15]: it was easy to take the solutions of the Schrödinger equation, the 'wave function', and construct quantities which were precisely the elements which filled the rows and columns of Heisenberg's matrices. Mathematically the issue was settled, but the struggle for the precise *meaning* of the new quantum theory, of the wave function and the quantum jumps, was only just beginning. Schrödinger went to Copenhagen in the fall of 1926 to discuss the situation with Niels Bohr; the discussions were so intense that Schrödinger retreated to his sick-bed.

For many of the protagonists, and above all for Bohr and Einstein, this issue of the meaning, or the interpretation, of quantum theory was a major theme for the rest of their professional lives. Not for Dirac. He had a very clear and pragmatic, even an 'engineering', view of how to use the new quantum theory. As far as he was concerned he said all that needed to be said about 'interpretation' in a paper he published at the end of 1926 entitled 'The Physical Interpretation of the Quantum Dynamics' [16]. He said later that this paper was his proudest achievement, which is a remarkable statement given the competition from his other achievements! In it Dirac developed, with characteristic generality and elegance, his 'transformation theory' which showed how to transform quantum quantities from one 'representation', for example based on the position, to another, for example based on energy.

Almost as a by-product he showed that the Schrödinger equation was just one particular representation of quantum theory, albeit a particularly useful one, and that the wave function was nothing but the transformation function needed to transform the Hamiltonian into its 'energy' representation. He also showed that the square (strictly the modulus squared) of the transformation functions gave the physically meaningful probabilities, as Born had postulated in mid-1926 for Schrödinger's wave function. He showed further that the basic equation of quantum theory should be 'first-order' in the energy. Dirac's faith in his transformation theory would lead to a huge advance a year later, as will be discussed in the next section. It was also in this paper that Dirac introduced his famous 'delta function'. [7]

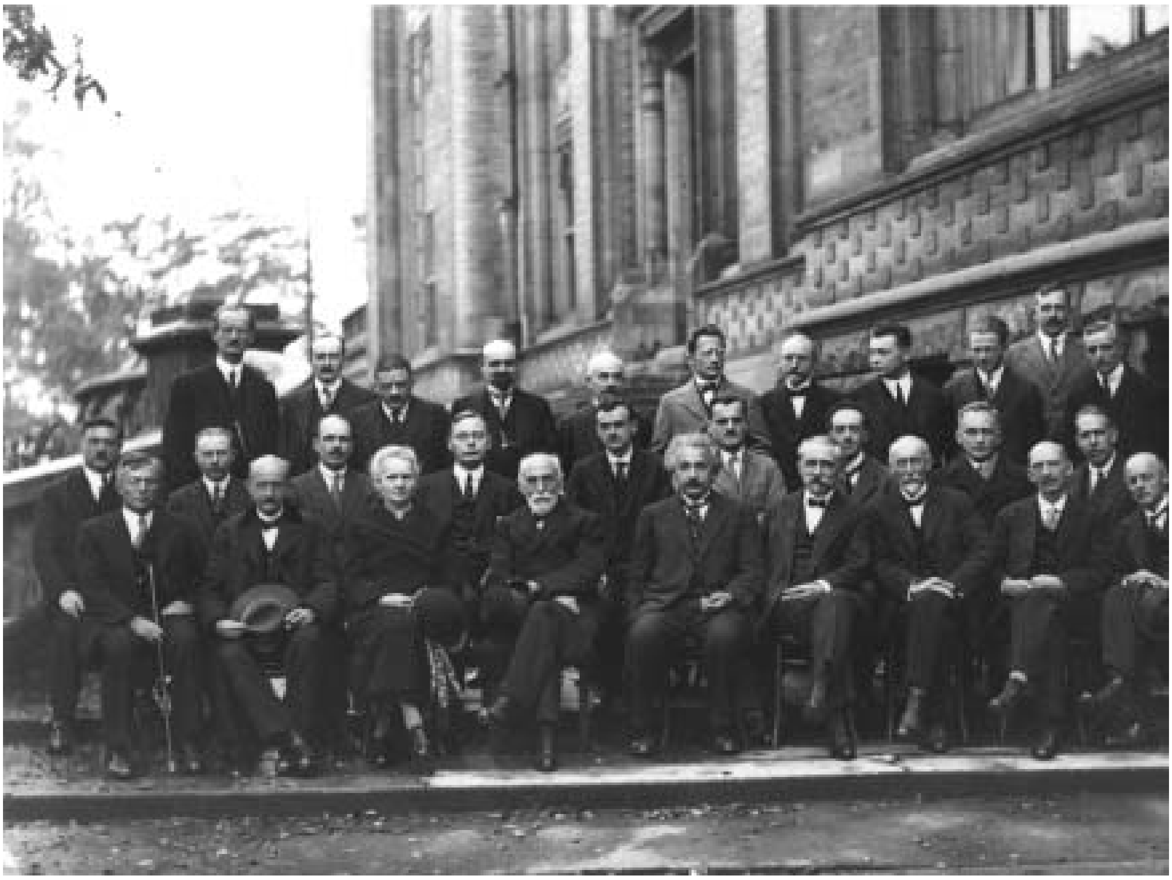

Figure 2. The 5th Solvay Conference, October 1927. Dirac is in the middle of the second row, with Einstein and Lorentz seated in front of him. To Dirac's left are Compton, de Broglie, Born, and Bohr. Schrödinger (with spectacles and bow tie) is standing in the back row behind Dirac's left shoulder. To Schrödinger's left are Verschaffelt, Pauli, Heisenberg, Fowler, and Brillouin. Of the twenty-nine participants, seventeen were past or future Nobel Prize winners. Acknowledgement: The Living Archive, Manchester University.

In October 1927 Dirac was the youngest delegate to the 5th Solvay conference. Figure 2 is the famous photograph of that meeting, in which Dirac stands right in the centre of the middle row[8]. The conference was dominated by quantum theory and is noteworthy for the first serious discussions on interpretation between Bohr and Einstein. Bohr asked Dirac what he was working on currently, and Dirac said that he was thinking about how to bring Einstein's special relativity into quantum theory. Bohr replied that that was all sorted out already by Klein. Dirac tried to explain to Bohr why that was not the case, but the start of a lecture interrupted them [17]. Three months later Dirac had solved the problem to his own satisfaction, making one of the landmark discoveries in 20th century physics.

[7] With a typographical error in the last equation on p. 625 of [16]!

[8] Actually there are two photographs extant. The most obvious difference is that in figure 2 Pauli (back row, fourth from right) is looking to his right, and in the other photograph he is looking much more to the front.

4. The Dirac Equation: 1928

Schrödinger's approach to quantum theory started from the expression for the total non-relativistic energy of a particle of mass m in a potential V:

$$\frac{p^2}{2m} + V = E. \quad (22)$$

Replacing p_x (we consider just one dimension) by the differential operator $-i\hbar\partial/\partial x$, and introducing a function (the 'wave function') for the operator to operate on gives:

$$\begin{aligned} &\frac{1}{2m}(-i\hbar\frac{\partial}{\partial x})(-i\hbar\frac{\partial}{\partial x})\psi + V\psi = E\psi \\ &\therefore -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi, \end{aligned} \quad (23)$$

which is the (time-independent) Schrödinger equation in one dimension. Generalising in an obvious way to three dimensions and setting V to the electrostatic Coulomb potential, Schrödinger then solved for the hydrogen atom, and obtained the same energy levels as the old Bohr theory, which agreed, roughly, with experiment.[9]

In developing his theory in the winter of 1925/26 Schrödinger had been more ambitious. He had started from the relativistic expression:

$$p^2c^2 + m_o^2c^4 = E^2. \quad (24)$$

He then introduced the electromagnetic potential (in a relativistically covariant way, the details of which are not important here), replaced p by the appropriate differential operator, and solved the resultant differential equation. He abandoned this approach only because it gave an energy spectrum that disagreed with the experimental results: of course since he was using a relativistic approach he was seeking closer agreement with experiment than the simple non-relativistic Bohr theory.

Dirac considered that any attempt to develop relativistic quantum theory directly from equation (24) was doomed from the outset: not because it is an incorrect relativistic equation (it isn't!), but because it transgressed his transformation theory of which he was so proud. That theory required that one must use an equation that is first order in E. Very well:

$$\sqrt{p^2c^2 + m_o^2c^4} = E.$$

But this is unsatisfactory from the perspective of 'relativity mathematics' which requires that E and p should appear in an equation in the 'same way', reflecting the way in which relativity treats time and space on an equal footing mathematically. What Dirac really wanted was an equation like:

$$pc + m_oc^2 = E, \quad (25)$$

which satisfies all Dirac's mathematical requirements, but is, unfortunately, wrong!

The essence of Dirac's problem was simply that:

$$\sqrt{p^2c^2 + m_o^2c^4} \neq pc + m_oc^2,$$

and, in general: $\sqrt{a^2 + b^2} \neq a + b$ (for non-zero a and b), as everyone learns (or should learn) at school.

At which point mere mortals give up. Geniuses try harder.

---

[9] The essence of the equivalence between the Schrödinger and Heisenberg theories lies in the fact that, for any $f(x)$, $x(-i\hbar\partial/\partial x)f(x) - (-i\hbar\partial/\partial x)\big(x.f(x)\big) = i\hbar f(x)$, which is obviously analogous to $xp_x - p_xx = i\hbar$.

Let us introduce 'numbers' d_1 and d_2 ('d' for Dirac, say) and consider:

$$(d_1a+d_2b)^2=(d_1a+d_2b)(d_1a+d_2b)=d_1^2a^2+d_1d_2ab+d_2d_1ba+d_2^2b^2$$
$$=d_1^2a^2+(d_1d_2+d_2d_1)ab+d_2^2b^2.$$

Now let us suppose that these d 'numbers' obey:

d_1^2 ="1", d_2^2 ="1" and $d_1d_2+d_2d_1$ ="0", then indeed:

$(d_1a+d_2b)^2$ $="1"a^2+"1"b^2$ i.e. $\sqrt{"1"a^2+"1"b^2}=d_1a+d_2b$.

Thus, similarly to Heisenberg in his breakthrough paper [4], Dirac needed a funny 'multiplication rule' so that $d_1d_2\neq d_2d_1$, but rather $d_1d_2+d_2d_1="0"$. And, just as in Heisenberg's case, matrices will do the job. For example, we can choose:

$$d_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \text{ and } d_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

so that:

$$d_1^2=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\times\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}="1",\quad d_2^2=\begin{pmatrix}0&1\\1&0\end{pmatrix}\times\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}="1", \text{ and}$$

$$d_1d_2+d_2d_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\times\begin{pmatrix}0&1\\1&0\end{pmatrix}+\begin{pmatrix}0&1\\1&0\end{pmatrix}\times\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}+\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}="0".$$

So, in summary:

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}pc+\begin{pmatrix}0&1\\1&0\end{pmatrix}m_0c^2=\begin{pmatrix}1&0\\0&1\end{pmatrix}E \quad (26)$$

has the mathematical structure Dirac wanted (equation (25)), and satisfies:

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}p^2c^2+\begin{pmatrix}1&0\\0&1\end{pmatrix}m_0^2c^4=\begin{pmatrix}1&0\\0&1\end{pmatrix}E^2, \quad (27)$$

as required by relativity, equation (24).

In fact that isn't quite general enough because the momentum p is a vector quantity with three components p_x, p_y, p_z so that $p^2 = p_x^2 + p_y^2 + p_z^2$. Hence Dirac's problem was actually to find four matrices, $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ such that:

$$(\alpha_1p_xc+\alpha_2p_yc+\alpha_3p_zc+\alpha_4m_0c^2)^2=("1"p_x^2c^2+"1"p_y^2c^2+"1"p_z^2c^2+"1"m_0^2c^4)="1"E^2.$$

So the matrices must satisfy $\alpha_i^2="1"$ and $\alpha_i\alpha_j+\alpha_j\alpha_i="0"$ for $i\neq j$.

It turns out that this can't be done using '2x2' (2 rows, 2 columns) matrices. In later years Dirac always said that he was slightly embarrassed that it took him some time to find the answer: you have to use '4x4' matrices, for example:

$$\alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix};\alpha_2=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix};\alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix};\alpha_4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

which satisfy the conditions on the α's, as can be checked by straightforward matrix multiplication. Hence we can indeed write:

$$\alpha_1p_xc+\alpha_2p_yc+\alpha_3p_zc+\alpha_4m_0c^2="1"E, \quad (28)$$

where '1' is the 4x4 unit matrix.

To obtain the quantum mechanical equation, we replace p_x by $-i\hbar\partial/\partial x$, and similarly for p_y and p_z, and give the differential operators a wave function to operate on:

$$-i\hbar c(\alpha_1 \frac{\partial\psi}{\partial x} + \alpha_2 \frac{\partial\psi}{\partial y} + \alpha_3 \frac{\partial\psi}{\partial z}) + \alpha_4 m_0 c^2 \psi = "1" E\psi \quad (29)$$

which is the (time-independent) Dirac equation.

But what sort of equation is this? To be meaningful the wave function, ψ, must be not only a function of *x, y*, and *z* but must also cater somehow for the matrices, α. If we take:

$$\psi = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} f(x,y,z) = (a) f(x,y,z)$$

then the α matrices multiply the column vector (*a*) to give another column vector, and the Dirac equation is finally revealed as a set of four (coupled) partial differential equations.

So the mathematics is now meaningful; but what is the *physics* of these matrices and column vectors? To answer this we recall yet another development of that remarkable period in physics. In order to understand atomic spectra in ever greater detail, two Dutch physicists, Goudsmit and Uhlenbeck, had proposed in 1925 that the electron had an 'intrinsic' two-valued quantum number, which could be thought of as a kind of intrinsic 'spin' in that it combined with the orbital angular momentum quantum number in a way which was similar to the combination of classical angular momenta.

Pauli had shown that the mathematics of this 'spin' was conveniently handled using '2x2' matrices. Dirac knew this, indeed had probably discovered these spin matrices for himself [18], and he showed that the solutions of his equation described this 'spin' in a completely natural way. Before the discovery of the Dirac equation the spin had to be grafted on to theory in a rather ad hoc fashion. It is important to emphasise, however, that Dirac did not set out to describe spin. As described above, he was driven by the requirements of his transformation theory and relativity. In his 1981 Edinburgh lecture Dirac recalled: 'It was a

Figure 3. The plaque commemorating Dirac in Westminster Abbey. The Dirac equation is written in a very compact notation, and in units in which $\hbar = c = 1$.

great surprise to me when the spin turned up in that way. I was just trying to get a satisfactory relativistic theory for a particle'[14]. Indeed; but as soon as he found himself playing around with matrices he must surely have started thinking that this would have something to do with spin in the end.

Dirac's paper 'The Quantum Theory of the Electron' was completed just before Christmas 1927, and published in January 1928 [19]. The paper contained the derivation of the equation itself, along the lines described above (although much more succinct!), proof of the Lorentz covariance of the equation, and the demonstration that the equation implies directly and naturally all the previously obtained relativistic and spin-dependent corrections to the hydrogen energy spectrum. This paper is rightly considered one of the very greatest in all physics. The Dirac equation (in compact notation) is inscribed on the plaque commemorating Dirac in Westminster Abbey, see figure 3. It is the only equation in the abbey.

Electron 'spin' has several intriguing properties. As already mentioned, it can take on only two values, which is what is meant by saying it has spin '½', the two possible values being $+1/2\hbar$ and $-1/2\hbar$. Mathematically the spin has another surprising property: you have to rotate by *two* complete revolutions (4π) in order to get back where you started. This is in contrast to normal experience of everyday objects that only have to be rotated by one complete revolution (2π) to return to the original state. In fact it is possible to construct macroscopic examples that have this 4π property: what is required is that an object is connected in a certain way to its surroundings. A demonstration can be seen at http://heplocal.rl.ac.uk/McCubbin/public/Rotate.mpg.

A good way to think about this is that each 2π revolution multiplies the electron mathematics by –1. This factor of –1 has utterly profound significance: if one considers the mathematics of *two* electrons, the operation of interchanging them is the same as the operation of a 2π revolution on one of them. A macroscopic example can be viewed at http://heplocal.rl.ac.uk/McCubbin/public/Interchange.mpg. So the two-electron mathematics is multiplied by –1 if you interchange the two electrons. This has the direct consequence that two electrons cannot occupy the same (quantum) state, which is the Pauli Exclusion Principle[10]. So an energy level in an atom (strictly an energy level with orbital quantum number 0) can be occupied by at most two electrons: one with spin $+1/2\hbar$ and one with $-1/2\hbar$. If a state could hold an arbitrary number of electrons the lowest-energy state of any atom would consist of all the electrons in the lowest energy state, and there would be no chemistry, no elements, and no life, which is what I mean by 'utterly profound significance'. This property of electrons (and of all spin – ½ particles) was used by Dirac in his next great advance.

[10] Let $\psi_\alpha(1)$ denote the wave function for the first electron, (1), having quantum numbers denoted by 'α'. We require that the total wave function for two electrons, $\Psi(1,2)$, is multiplied by –1 when the two electrons are interchanged, i.e. $\Psi(2,1) = -\Psi(1,2)$. This requirement is satisfied by $\Psi(1,2) = \psi_\alpha(1)\psi_\beta(2) - \psi_\beta(1)\psi_\alpha(2)$. But if the two electrons have the same quantum numbers (i.e. α=β), $\Psi(1,2)$ is identically zero. Thus two electrons cannot have the same quantum numbers, which is the Pauli Exclusion Principle. This is the argument, based on the 'anti-symmetry' of the wave function, as given by Dirac [20]. The energy distribution for a gas of particles requiring this anti-symmetry was obtained a little earlier by Fermi [21]. Such particles are said to obey Fermi-Dirac statistics. In fact Jordan had developed the relevant theory before either Fermi or Dirac, and had sent a paper to Born, as editor of Zeitschrift für Physik, at the end of 1925. Born was about to leave for his lecture tour in the USA and packed Jordan's paper in his suitcase…… where, to his great embarrassment, he found it several months later, by which time Fermi and Dirac had done their work [22].

5. Anti-matter: 1931

In the introductory remarks of the paper on the Dirac equation [19], the author noted that previous work to incorporate relativity into quantum theory (the work which Bohr felt had sorted it out) had two problems: the equation used wasn't first-order in the energy, and the solutions allowed both positive and negative energies. As Dirac emphasised, negative energies could not be ignored in quantum theory as they could be in classical physics, because transitions between energy levels were the very essence of quantum theory, and so one had to face up to transitions to a bottomless pit of negative-energy levels.

As discussed above, the Dirac equation is, triumphantly, first-order in the energy. But the problem of negative energies remains. The origin of the problem is simple. In taking the square root of the relativistic equation:

$$p^2c^2 + m_0^2c^4 = E^2$$

both positive and negative energies are allowed, just as the number 25 has two square roots: +5 and –5.

Despite his brilliant manipulations to obtain a first-order equation for *E*, Dirac did not avoid this second problem. His (matrix) equation relating *E*, *p* and *m* was indeed constructed to ensure that it was consistent with $p^2c^2 + m_0^2c^4 = E^2$, and so it is not too surprising that the Dirac equation yields solutions with both positive and negative energies. In 1930 Dirac proposed a solution [23]: if each negative-energy state were filled by two electrons then the Pauli exclusion principle would forbid any transition to negative energy, since each negative-energy state would already hold its maximum number of electrons.

This solves the problem, but at a price. Nature's ground state, the vacuum, now becomes a 'sea' of filled negative-energy states, with infinite (negative) electric charge. Dirac argued this would not pose a problem for the equations of electromagnetism provided charge in those equations was interpreted as *change of charge* with respect to this infinite sea.

But this picture of the vacuum raises another question: what was there to stop a deposit of energy raising one of the electrons in a negative-energy state into a positive-energy state? Certainly there's an energy gap to be overcome: the positive-energy states start at $+m_0c^2$ and the negative states at $-m_0c^2$, but a photon of energy $2m_0c^2$ (or greater) has enough energy to overcome this gap. The answer is that, given enough energy, nothing can stop such a process, and an electron can be 'lifted' out of the negative-energy sea into a positive-energy state. What then are we to make of the negative-energy sea when it isn't *quite* filled, but when there is an unfilled 'hole' in the negative-energy states? The negative-energy states have now one more unit of *positive* charge (since the electron charge is negative) than when completely filled. Dirac showed that such an unfilled sea did indeed behave like a positively charged particle in his equation, see figure 4.

But what positively charged particle?

In 1930 there was only one known fundamental particle with the opposite charge to the electron: the proton. Dirac therefore suggested that a 'hole' in the negative-energy sea was in fact a proton. The proton mass is, however, some 2000 times larger than the electron mass, and Dirac had no explanation for such a gross asymmetry. However, there was a certain appealing economy in this interpretation: the two known (at the time) elementary particles, the electron and the proton, were both described by the Dirac equation.

In the first edition (1930) of his classic textbook 'The Principles of Quantum Mechanics', Dirac wrote, 'We assume that these unoccupied negative-energy states are the protons.' In the second (and subsequent) editions he changed exactly one word.

For all its economy, nobody liked very much the proton interpretation. A clinching argument was provided by Oppenheimer [24] and Tamm [25], independently, who showed that the hydrogen atom, consisting of an electron and negative-energy 'hole' (the proton), would annihilate essentially immediately into photons, in striking contradiction to experimental observation!

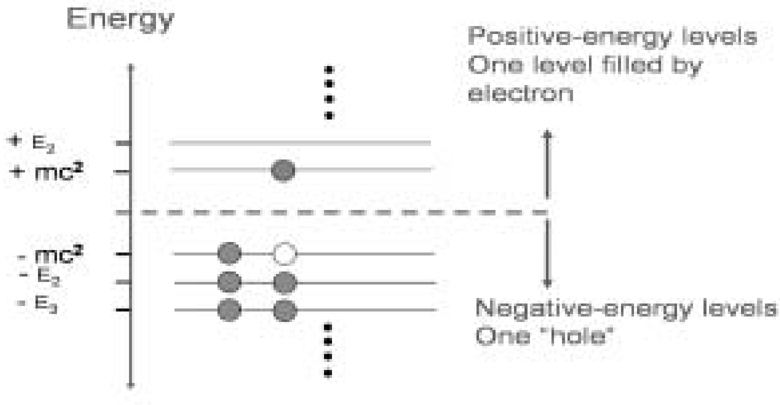

Figure 4. Sea of filled negative-energy levels with one 'hole', which behaves like a positively-charged particle.

In 1931 Dirac followed the inexorable logic of his mathematics: the natural prediction of his equation was that the positively charged particle arising from a 'hole' in the negative-energy states should have the same mass as the electron. In a paper entitled 'Quantized Singularities in the Electromagnetic Field' [26], he proposed 'a new kind of particle, unknown to experimental physics, having the same mass and opposite charge to an electron'. This particle, the anti-electron or positron, did not remain unknown to experimental physics for long: it was discovered by the American physicist Anderson in 1932. In making his proposal Dirac was of course doubling the number of particles in nature, since this idea of anti-particle partner should presumably apply to protons, as Dirac stated explicitly, and as has subsequently been abundantly confirmed by experiment for all particles.

In 1933 Dirac was awarded, jointly with Schrödinger, the Nobel Prize in physics for '..the discovery of new and fruitful forms of atomic theory'. The Nobel committee had made no award in physics for 1932, delaying until 1933 when they awarded the 1932 prize to Heisenberg. Thus the three pioneers of the revolution in quantum theory were all in Stockholm together to receive their prizes. It was the first time the Nobel committee made awards for work that was fundamentally theoretical in nature, rather than for work in experimental physics or for theoretical work which bore very directly on experiment. Famously, Einstein won the prize in 1921 for '..services to Theoretical Physics, and especially the discovery of the law of the photo-electric effect'. Not for relativity![11]

Whilst no one questioned that the winners deserved their prizes, there was some discontent that certain others, principally Born and Pauli, had not also been recognised. Heisenberg in particular was very unhappy that Born had been overlooked. Amends were made when Pauli won in 1945, for the Exclusion Principle, and Born in 1954, for his contributions to quantum mechanics [28].

The Nobel Prize was the most important, but not the first, official recognition Dirac received: he was elected to the Royal Society in 1930 (on the first occasion he was a candidate for election, which is extremely rare), and he had been appointed to the Lucasian

[11] Indeed, the secretary of the Swedish Academy of Sciences wrote to Einstein to emphasise that the Nobel committee had awarded him the prize '..without taking into account the value which will be accorded your relativity and gravitation theories after these are confirmed in the future' [27].

Professorship in mathematics at Cambridge in 1932, in succession to Sir William Larmor. (The Lucasian Professorship was established in 1664. The second holder was Sir Isaac Newton. The current, seventeenth, professor is Stephen Hawking.)

6. Dirac and the birth of Quantum Field Theory: 1927

By 1925 it was clear that there was *something* particle-like about light, i.e. about the electromagnetic field. That the energy of light had a discrete aspect, $E = \hbar\omega$, had been the starting-point for quantum theory at the turn of the century in the work of Planck on black-body radiation (1901) and of Einstein on the photo-electric effect (1905). However, the idea of a 'particle of light', with both energy and momentum, took some time to emerge. Einstein was fully convinced only in 1917 when he showed, from an analysis of the fluctuations of gas molecules emitting and absorbing radiation, that the light quanta must also be fully directional, i.e. carry momentum as well as energy. (As Pais has noted, it is striking that the father of relativity took twelve years before he was prepared to publish $p = \hbar\omega / c$ alongside $E = \hbar\omega$ [29]). Not everyone was as convinced as Einstein, but in 1923 the American physicist Arthur Compton obtained conclusive experimental verification of the light quantum, or photon, as a carrier of both energy and momentum from a study of the scattering of light by electrons (the Compton effect). So it was of course hoped that the new quantum ideas of Heisenberg et al. would have something to say about light and the photon.

As a first step, Born and Jordan showed, at the end of their first paper [9], that Maxwell's equations of electrodynamics could be written and manipulated in matrix form. In the Born, Heisenberg, Jordan 'Dreimännerarbeit' [10], the last section, written by Jordan, was devoted to an application of the new quantum mechanics to a vibrating string. The authors considered it 'particularly encouraging' that the result obtained for the mean-square fluctuation in the energy contained both 'wave' and 'particle' terms, exactly as Einstein had obtained for radiation using an argument based on thermodynamic equilibrium and Planck's formula for black-body radiation. The authors may have found this 'particularly encouraging', but in later years Jordan commented ruefully that he felt that nobody had read this section of the 'Dreimännerarbeit', and those who did didn't want to believe it [30].

Dirac turned his attention to radiation and the new quantum mechanics during his stay at the Niels Bohr Institute in Copenhagen in 1926. He had just worked out his 'transformation theory' [16], which, as noted above, he viewed as a general and powerful formulation of quantum theory, including how to extract physically meaningful probabilities from the theory. It was of course axiomatic that the sum of all those probabilities for the system under consideration should be one. But suppose one considered, instead of a single system, an ensemble of N particles. Any observation of the ensemble would find n_1 particles in state 1, n_2 in state 2, etc., such that $n_1 + n_2 + ... = N$. Of course $n_1, n_2, ...$ must be integers. Could this be guaranteed in the new quantum mechanics formalism? Dirac showed how this question could indeed be answered in the affirmative, using an almost magical inter-weaving of all his quantum ideas: the analogy with the Hamiltonian formalism of classical mechanics, commutation relations, and his transformation theory. It is of course clear that, in order to guarantee that $n_1, n_2, ...$ are always integers, one must ensure that any changes in these *occupation numbers* take place in units of 1. So Dirac introduced *creation* and *annihilation operators,* which increase or decrease an occupation number by 1.

Dirac then applied this formalism to an ensemble of photons (i.e light) perturbed by an atom that emits or absorbs photons. In order to treat this problem he introduced the idea of a 'ground state' containing an infinite number of non-observable photons and which acts as a 'source' of emitted photons or as a 'sink' for absorbed ones. (This idea clearly anticipates the 'sea' of electrons filling the negative-energy states that he used a few years later in interpreting the Dirac equation.) He then obtained expressions for the probability that the atom would absorb or emit a photon. Dirac's formalism showed that photon emission could either be induced by an electromagnetic field ('stimulated emission'), or could take place when no field was initially present ('spontaneous emission'). Ten years earlier Einstein had shown that both emission processes must indeed be present in order to maintain thermal equilibrium between the atoms and radiation. Dirac's work showed how quantum theory explained Einstein's results, and further how the intrinsic atomic transition probabilities could actually be calculated. (Einstein's theory left these transition probabilities undetermined.) From the perspective of quantum theory, the existence of spontaneous emission can be traced directly to the commutation relation between the creation and annihilation operators, which is in turn a direct consequence of the fundamental relation $[X, P_X] = i\hbar$. Working through this provides a moment when even the ordinary practitioner can know some of the thrill which creators like Dirac must have experienced. (See, for example, [31].)

Dirac's paper entitled 'The quantum theory of the emission and absorption of radiation' was 'communicated' to the Royal Society (by Bohr, a foreign member) at the end of 1926, and published in 1927 [32]. In effect, he had 'quantised' the electromagnetic field, and his paper marks the start of quantum field theory and, in particular, of quantum electrodynamics.

Quantum field theory was developed rapidly through the 1930's with contributions from a galaxy of theorists: Jordan, Pauli, Heisenberg, Fermi, Wigner, Klein, Weisskopf, and Dirac himself. However, serious and profound difficulties were encountered in the form of infinite results from the perturbative calculations. As is well known, the resolution of these infinities had to wait until after the second World War, and the pioneering work of Feynman, Schwinger, and Tomanaga, all of whom were inspired directly by some of the earlier work of Dirac. The whole story is told wonderfully by Schweber [33]. That resolution requires the procedure of 'renormalisation' in which, in effect, the infinities are absorbed into the *physical* values of mass and electric charge. Renormalisable quantum field theories now provide the theoretical framework for all of particle physics.

As is also well known, Dirac was profoundly unsympathetic to the whole renormalisation programme, viewing it as mathematical chicanery. Infinite results should be faced directly, and not removed by convenient definition. In his last published remarks in 1983 [34] he returned to this theme in a lecture entitled 'The inadequacies of quantum field theory'. Written in Dirac's precise and unemotional prose, it is, nevertheless, an impassioned and poignant lament about the way quantum field theory had evolved. He urged that the correct approach *must* be to find an appropriate Hamiltonian; that the remarkable agreement with experiment achieved by the renormalisation approach in quantum electrodynamics may, nevertheless, be illusory, just as Bohr's original quantum theory of the hydrogen atom was fundamentally flawed despite its impressive agreement with experiment; that the discovery of the Dirac equation showed the benefits of adherence to sound mathematical principles; and that he planned to continue *his* programme as long as he could. His audience listened, no doubt respectfully, but almost certainly disagreed. In the twenty years since that lecture the triumphant march of renormalisable quantum field theory has continued. But a quantum theory of gravity has resisted the renormalisable approach, and may well require something fundamentally different, like 'string theory'. So Dirac's intuition may yet prove to be correct.

7. Vignette

When asked, in 1955, to state his philosophy of physics, Dirac wrote on the blackboard, 'Physical Laws should have mathematical beauty'. The precision and conciseness of the statement are typical of him. He took great pains in what he wrote and how he wrote it. Notation matters: Dirac introduced the symbol $\hbar$ to save writing myriad factors of 2π, and, more importantly, his 'bra' and 'ket' notation for quantum states is now standard. His classic textbook, ‘The Principles of Quantum Mechanics’, is now in its 4th edition (1958), and has not been out of print since it was first published in 1930. Nor has the book been re-set, so the 4th edition looks very much like the first, and indeed significant passages are word-for-word identical. For example, the discussion of photons and interference in a set-up like Young’s double-slit is almost unchanged between the first and fourth editions. The first edition already contains the famous sentences: ‘Each photon then interferes only with itself. Interference between two different photons can never occur’.[12]

Of course Dirac did make certain changes. For the second edition in 1935 he changed the sentence ‘We assume that these unoccupied negative-energy states are the protons’ to ‘We assume that these unoccupied negative-energy states are the positrons’, and he added a chapter on quantum electrodynamics. In the third edition (1947) he introduced the ‘bra’ and ‘ket’ notation. For the fourth (1958) he re-wrote the chapter on quantum electrodynamics.

It is indeed the case that his Cambridge lecture course on Quantum Mechanics consisted in part of Dirac reading out sections of his book. Dirac’s logic was no doubt that he had taken great pains to explain quantum mechanics as clearly as he could in his book, so why should he use a less good explanation? This logical approach certainly carried over into aspects of his social interactions and has been the source of many ‘Dirac stories’ of which a reasonable selection can be found in [36]. As part of the celebrations of the centenary of his birth the IoP prepared a set of six posters about Dirac’s work in the style of Manga comics [http://education.iop.org/Schools/supteach/dirac.html]. At the Bristol celebration in August 2002 his daughter Margit was interviewed by the BBC and asked what her father would have thought of such a depiction of his work. She replied that he would have loved it, and added that when the weekly comic section of the newspaper was delivered to the Dirac household one was ill-advised to get in her father’s way in his dash to get it! It makes a charming counter-point to the usual picture of the rather taciturn theorist.

Dirac retired from the Lucasian chair at Cambridge in 1969, at the statutory age of 67. He held an emeritus professorship in Florida from 1972, and died there in 1984. A plaque was placed in Westminster Abbey in 1995, see figure 3. It has the simplicity, elegance and conciseness that he always sought.

In the first Dirac Memorial Lectures in 1986, Steven Weinberg, Nobel Prize winner for his work on the quantum field theory of weak and electromagnetic interactions, gave a lecture entitled ‘Towards the Final Laws of Physics’ in which he described some of the latest ideas on ‘string theory’. He closed by saying, ‘I don’t know, of course, whether Dirac would think that the mathematics of string theory is sufficiently beautiful to make it likely that it will survive as part of the final laws of physics. He might agree with that, and he might not agree with that, but I don’t think he would disapprove of what we are trying to do’ [37].

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[12] By the 4th edition the second sentence has been changed to ‘Interference between two different photons never occurs’, which is hardly a significant change. In fact it is interesting that Dirac did *not* add any further comments for the 4th edition, since he must have been preparing it around the time of the furore (among physicists) over the Hanbury-Brown Twiss experiments on intensity interferometry [35]. Dirac’s statements are perfectly correct in the context of Young’s double-slit and similar experiments, but need some qualification when ‘higher-order correlations’ are measured, as in the Hanbury-Brown Twiss experiment. On the other hand it is entirely typical of Dirac to avoid unnecessary verbiage: his statements, taken in context, were correct. If others chose to mis-interpret or mis-apply them, that was hardly his fault!

and to Robin Marshall of Manchester University who digitised the recording and provided several sound-bites. I am also grateful to Robin Devenish of Oxford University for the opportunity to peruse a first edition of Dirac's The Principles of Quantum Mechanics. I have leant heavily on two sources: van der Waerden's compilation of, translation of, and commentary on some of the key papers in his Sources of Quantum Mechanics [4,9,10,11], and the six-volume encyclopaedic work on The Historical Development of Quantum Theory by Mehra and Rechenberg [1,8,18]. For filling out aspects of Dirac, both his life and his work, I have often consulted Helge Kragh's Dirac: A Scientific Biography [38].

Finally, although it is not properly an acknowledgement, I want to mention the pleasure there is to be had from reading Dirac's original papers. They are not always easy to read - Dirac never uses a single superfluous word – but there are passages that have an almost poetic quality. Any physicist interested in Dirac's work should start there [39].

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