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Anomalous Resistivity in Collisionless Plasma Shock Waves

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Anomalous Resistivity

in

Collisionless Plasma Shock Waves

by

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Abstract

The main aim of this dissertation is to highlight the fundamental processes involved in the concept of anomalous resistivity, incorporating them into an overall model for the dissipation mechanism in the earth's bow shock. The philosophy is to elucidate the physical understanding as clearly as possible, occassionally at the expense of more realistic models.

The instabilities, in particular the ion-acoustic and modified two-stream instabilities, that may grow in the shock layer are discussed in some detail. The interactions created by these instabilities are described by the quasilinear theory and particle trapping effects are employed to determine the saturated energy levels.

The expression for the anomalous resistance developed within the quasilinear framework is then applied to the earth's bow shock. After having established the necessity of dissipation for the existence of the shock it is attempted to demonstrate that the anomalous resistance produced by any of the various instabilities is also sufficient. The failure of the ion-acoustic instability to operate in the regime $T_i \gtrsim T_e$ is seen to disqualify it as a contender for the dissipation mechanism. However, calculations for the modified two-stream instability, an intuitively more realistic model since it necessitates passage of a current across a magnetic field, indicate an apparent excess of anomalous resistance. Such an excess may be due to over simplifications in the model but the results encourage further investigation.

Acknowledgements

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David C. Hill

Contents

Introduction

A fundamental problem and a topic of extensive research, in theoretical plasma physics, is how to explain the existence of the earth's bow shock. [5,12] Recent satellite missions (e.g. AMPTE - [15]) have provided a wealth of high-quality data enabling the theories to be verified more rigourously.

In ideal gas shocks the necessary dissipation required to balance the nonlinear wave steepening is provided by binary collisions. However, the mean free path between particles in the interplanetary plasma is of the order of the distance from the earth to the sun whilst the shock thickness is only of the order of one hundred kilometres; hence the term *collisionless*. The concept of a classical resistivity [19] produced by binary collisions can therefore not provide the dissipation.

The basic large-scale structure of the shock is assumed to be laminar, allowing a two-fluid model to be used. The incoming plasma wave (in the solar wind) grows due to nonlinear steepening effects and the resulting magnetic field gradients drive a current along the shock front. This relative streaming between the ions and electrons allows drift instabilities to grow in the shock layer. These instabilities produce rapidly varying electromagnetic fields which interact collectively with the individual particle motions (the so-called *wave-particle* interactions) until the waves are saturated to a level governed by a nonlinear effect known as *particle trapping*. An *anomalous* (i.e. not due to binary collisions) *resistivity* arises to restrict the growth of the incoming wave. This will then alter the instabilities generated in the shock layer, feeding back to produce a different value for the anomalous resistivity. Eventually the two opposing factors balance and the familiar shock profile is formed.

Chapter 1 simply outlines the common models used in plasma physics. In Chapter 2 we describe the analysis required to study perturbations about the field free equilibrium state. This is used to explain the concept of *Landau damping*, the essential wave-particle mechanism, which may result in driving certain waves unstable.

The *drift instabilities* (e.g. ion-acoustic, modified two-stream) that may occur in the shock layer are discussed in Chapter 3. These drift instabilities cause a large number of random collective interactions to be excited and the situation is referred to as *turbulent* in the literature. That is, there are now many waves present in the system and we assume it is possible to treat the phases of the waves as being random. A statistical formalism, called the *quasilinear theory*, is developed in Chapter 4 to describe this turbulence. The derivation of the expression for anomalous resistivity, within the framework of quasilinear theory, is then presented, including a definition of an *effective "collision" frequency* for use in Chapter 5. It becomes evident at this stage that an understanding of Chapters 1 and 2 is necessary to calculate this frequency. The quasilinear theory is also used to discuss the nonlinear effects of Landau damping, called quasilinear diffusion. The way in which the instabilities discussed in Chapter 3 saturate to a finite energy level is described and an estimate provided for this level which in turn enables the saturated collision frequency to be calculated.

A derivation of the equation determining the shock profile is given in Chapter 5. It is shown that the inclusion of dissipation in the two-fluid model transforms a soliton solution into a shock-type solution, thus demonstrating the necessity of dissipation. The dissipative term is modelled in such a way as to allow the "collision" frequency of Chapter 4 to be employed. A theoretical value for the shock thickness is then calculated in Chapter 6, enabling verification of the theory by comparison with the satellite data. [15, 20] The results and assumptions of the model are discussed in Chapter 7.

Note: Any reader unfamiliar with plasma physics notation is advised to consult Appendix 1.

Chapter 1

Governing Equations

There are basically three models used to describe plasma dynamics, depending on the scale of interest. The derivation of these models is somewhat lengthy and is not included here, instead the reader is referred to [3,6,9].

1.1 The Vlasov-Maxwell System

The first of these models, the Vlasov-Maxwell system is written below

$$\frac{\partial f_s}{\partial t} + \underline{v}.\nabla f_s + \frac{q_s}{m_s}(\underline{E} + \underline{v}_\wedge \underline{B}).\nabla_{\underline{v}} f_s = 0 \qquad 1.1.1a$$

$$\nabla.\underline{E} = \rho/\epsilon_o \qquad 1.1.1b$$

$$\nabla.\underline{B} = 0 \qquad 1.1.1c$$

$$\nabla_\wedge \underline{B} = \mu_o(\underline{j}+\epsilon_o\frac{\partial}{\partial t}(\underline{E})) \qquad 1.1.1d$$

$$\nabla_\wedge \underline{E} = -\frac{\partial}{\partial t}(\underline{B}) \qquad 1.1.1e$$

where $\rho = \sum_s q_s N_s \int f_s d\underline{v}$ is the charge density

and $\underline{j} = \sum_s q_s N_s \int \underline{v} f_s d\underline{v}$ is the current density.

The integral $\int d\underline{v}$ means integrate over all velocity space. The symbol $N_s=\overline{N}_s/V$ denotes the average density of particles of species s, where $\overline{N}_s$ is the total number of particles of species s and where V is the total plasma volume.

The function $f_s(\underline{x},\underline{v},t)$ is called a one-particle distribution function. It is such that $\frac{1}{V}f_s(\underline{x},\underline{v},t)\delta\underline{x}\delta\underline{v}$ is the probability of finding a particle of species s, within a position $(\underline{x},\underline{x}+\delta\underline{x})$ and with a velocity within $(\underline{v},\underline{v}+\delta\underline{v})$ at time t. It is normalised via $N_s\int f_s d\underline{x}d\underline{v} = \mathbb{N}_s$ where the integration is over all of six dimensional phase space.

This is actually a highly reduced description of a plasma since the probability of finding a charged particle at $\underline{x}$ is altered by the presence of another charged particle at $\underline{x}' \approx \underline{x}$. This information is not contained in f_s but would require a two particle distribution function. Similarly for three or more particle interactions. (See the BBGKY hierarchy in 6,9.) However,in a plasma,the range of the forces, $r_o \approx \lambda_D$, is much larger than the mean particle spacing, $n_s^{-1/3}$, (i.e. $1/N_D << 1$) so that the dominant forces on two adjacent particles are those from the many particles further away.

Moreover, the Vlasov equation is only valid on time scales much shorter than the mean time between particle collisions since binary collisions have been neglected. The interactions between particles appear through $\underline{E}(\underline{x},t)$, $\underline{B}(\underline{x},t)$ which are the average fields produced at a point $\underline{x}$ at time t by the particles and calculated self-consistently from Maxwell's equations.

1.2 The Two-Fluid Theory

The two fluid theory may be obtained by taking moments of the Vlasov equation [3,6,9] and assuming an equation of state to close the infinite chain of equations. The resulting equations are listed below.

Momentum equation

$$m_s n_s\left(\frac{\partial}{\partial t} + \underline{u}_s.\nabla\right)\underline{u}_s = q_s n_s(\underline{E}+\underline{u}_s{}_\wedge\underline{B}) - \nabla.\underline{\underline{P}}_s \qquad 1.2.1a$$

Continuity equation

$$\frac{\partial}{\partial t} + \nabla.(n_s \underline{u}_s) = 0 \qquad 1.2.1b$$

Equation of State

$$P_s = \kappa n_s T_s \qquad 1.2.1c$$

$$= C n_s^{\gamma_s}$$

where κ is Boltzmann's constant, γ_s is ratio of specific heat capacities and C is also a constant.

The suffix s denotes particle species. We always assume a Hydrogen plasma. That is, one fluid consists of electrons and the only other fluid consists of positive ions of equal but opposite charge. We write s = e for electrons and s = i for ions.

Together with Maxwell's equations we have 18 scalar equations for the 16 scalar unknowns $n_i, n_e, P_i, P_e, \underline{u}_i, \underline{u}_e, \underline{E}, \underline{B}$ but the divergence of 1.1.2f,g give 1.1.2d,e.

Maxwell's equations $\epsilon_o \nabla.\underline{E} = \sigma$ 1.2.1d

$$\nabla.\underline{B} = 0 \qquad 1.2.1e$$

$$\frac{1}{\mu_o} \nabla_\wedge \underline{B} = \underline{j} + \epsilon_o \frac{\partial(\underline{E})}{\partial t} \qquad 1.2.1f$$

$$\nabla_\wedge \underline{E} = -\frac{\partial(\underline{B})}{\partial t} \qquad 1.2.1g$$

where $\sigma = q_i n_i + q_e n_e$ is the charge density 1.2.1h

and $\underline{j} = q_i n_i \underline{u}_i + q_e n_e \underline{u}_e$ is the current density. 1.2.1i

The macroscopic variables are defined in terms of f_s by

Number density $n_s(\underline{x},t) = N_s \int f_s(\underline{x},\underline{v},t) d\underline{v}$

Fluid velocity $\underline{u}_s = \frac{N_s}{n_s} \int \underline{v} f_s d\underline{v}$

Pressure tensor $\underline{\underline{P}}_s = m_s N_s \int \underline{w}_s \underline{w}_s f_s d\underline{v}$

where $\underline{w}_s = \underline{v} - \underline{u}_s$, which reduces to the scalar

$$P_s = m_s N_s \int (\underline{v}-\underline{u}_s)^2 f_s d\underline{v} = n_s \kappa T_s$$

for the isotropic case.

Also, the flux of K.E. crossing unit area in phase space is

$$\underline{H}_s(\underline{x},t) = \frac{N_s m_s}{2} \int \underline{v}(\underline{v}.\underline{v}) f_s d\underline{v}$$

On larger scales the plasma may be thought of as a single fluid and the MHD equations may be used (Nicholson p193) but these equations will not concern us in this dissertation.

Chapter 2 Linear Theory

2.1 Perturbation Analysis

In this chapter we study small amplitude plasma waves that propogate as perturbations about some equilibrium state. We will then be able to discuss the important concept of Landau damping which is necessary for an understanding of anomalous resistivity.

We write

$$f_s(\underline{x},\underline{v},t) = f_{so}(\underline{x},\underline{v},t) + \epsilon f_{s1}(\underline{x},\underline{v},t) \qquad 2.1.1a$$

$$\underline{E}(\underline{x},t) = \underline{E}_o(\underline{x},t) + \epsilon \underline{E}_1(\underline{x},t) \qquad 2.1.1b$$

$$\underline{B}(\underline{x},t) = \underline{B}_o(\underline{x},t) + \epsilon \underline{B}_1(\underline{x},t) \qquad 2.1.1c$$

Substituting into 1.1.1 yields:

Zeroth order equations

$$\left(\frac{\partial}{\partial t} + (\underline{v}.\nabla) + \frac{q_s}{m_s}(\underline{E}_o+\underline{v}_\wedge\underline{B}_o).\nabla_{\underline{v}}\right)f_{so} = 0 \qquad 2.1.2a$$

$$\epsilon_o \nabla.\underline{E}_o = \sum_s q_s N_s \int f_{so} d\underline{v} \qquad 2.1.2b$$

$$\frac{1}{\mu_o} \nabla_\wedge \underline{B}_o = \sum_s q_s N_s \int \underline{v} f_{so} d\underline{v} + \epsilon_o \frac{\partial(\underline{E}_o)}{\partial t} \qquad 2.1.2c$$

Order ϵ equations

$$\left(\frac{\partial}{\partial t} + (\underline{v}.\nabla) + \frac{q_s}{m_s}(\underline{E}_o+\underline{v}_\wedge\underline{B}_o).\nabla_{\underline{v}}\right)f_{s1} = -\frac{q_s}{m_s}(\underline{E}_1+\underline{v}_\wedge\underline{B}_1).\nabla_{\underline{v}}f_{so} \qquad 2.1.3a$$

$$\epsilon_o \nabla.\underline{E}_1 = \sum_s q_s N_s \int f_{s1} d\underline{v} \qquad 2.1.3b$$

$$\frac{1}{\mu_o} \nabla_\wedge \underline{B}_1 = \sum_s q_s N_s \int \underline{v} f_{s1} d\underline{v} + \epsilon_o \frac{\partial(\underline{E}_1)}{\partial t} \qquad 2.1.3c$$

$$\nabla_\wedge \underline{E}_1 = -\frac{\partial(\underline{B}_1)}{\partial t} \qquad 2.1.3d$$

The distribution f_{so} and self consistent fields $\underline{E}_o,\underline{B}_o$, when they are all independent of t, represent a stationary plasma state whilst f_{s1} represents the development of the initial perturbation. The stationary states may be constructed from the constants of the motion. (See Chapter 4) This perturbation analysis can also be used with a two-fliud model but the Vlasov-Maxwell system contains information not found in fluid theory. (e.g. Landau damping)

For simplicity, consider a spatially uniform field-free plasma that obeys the equilibrium Vlasov-Maxwell equations.

i.e. $\underline{E}_o = \underline{0} = \underline{B}_o \; ; \; f_{so} = f_{so}(\underline{v})$ 2.1.4

2.1.2 become

$$\sum_s q_s N_s \int f_{so} d\underline{v} = 0$$

and

$$\sum_s q_s N_s \int \underline{v} f_{so} d\underline{v} = 0$$

We assume the perturbation to be electrostatic. (i.e.$\underline{B}_1 = \underline{0}$) [This is certainly true if the perturbed charge density varies only in one dimension since $\underline{E}$ is then necessarily in the form $\underline{E} = E(x)\underline{i}$ and so $\nabla_\wedge \underline{E} = 0$. Maxwell's equation then yields $\frac{\partial}{\partial t}(\underline{B}_1) = \underline{0}$ which implies $\underline{B}_1 = \underline{0}$ since the arbitrary constant of integration is absorbed into $\underline{B}_o$]

In this case $\underline{E}_1 = -\nabla\phi_1$, so that f_{s1} is given by

$$\left(\frac{\partial}{\partial t} + \underline{v}.\nabla\right) f_{s1} = \frac{q_s}{m_s} \nabla\phi_1 . \nabla_{\underline{v}} f_{so} \qquad 2.1.5$$

and

$$\epsilon_0 \nabla^2 \phi_1 = - \sum_s q_s N_s \int f_{s1} d\underline{v} \qquad 2.1.6$$

We use the method of integral transforms to solve these partial differential equations in the context of an Initial Value Problem. The equations are reduced to algebraic equations by taking their Fourier transform with respect to spatial variables and their Laplace transform with respect to time. The necessary inversions then solve the problem.

Define Fourier transform by

$$f_{sk}(\underline{k},\underline{v},t) = \frac{1}{(2\pi)^3}\int_V f_{s1}(\underline{x},\underline{v},t)\exp(-i\underline{k}.\underline{x})d\underline{x}$$

with inversion

$$f_{s1}(\underline{x},\underline{v},t) = \frac{1}{(2\pi)^3}\int_V f_{sk}(\underline{k},\underline{v},t)\exp(i\underline{k}.\underline{x})d\underline{x}$$

Define Laplace transform by

$$\hat{f}_{sk}(\underline{k},\underline{v},p) = \int_0^\infty f_{sk}(\underline{k},\underline{v},t)\exp(-pt)dt \qquad (p_r \geq p_o)$$

where suffices $_r$ and $_i$ shall, throughout the text, denote real and imaginary parts.

The Laplace inversion is given by

$$f_{sk}(\underline{k},\underline{v},t) = \frac{1}{2\pi i}\int_{p_o-i\infty}^{p_o+i\infty} \hat{f}_{sk}(\underline{k},\underline{v},p)\exp(pt)dp$$

where the constant p_o is chosen large enough for $\hat{f}_{sk}$ to converge. Remembering that f_{so} is independent of $\underline{x}$ and t, taking Fourier and then Laplace transforms of 2.1.5 and 2.1.6 gives

$$(p+i\underline{k}.\underline{v})\hat{f}_{sk} = f_{sk}(\underline{k},\underline{v},t=0) + \frac{q_s}{m_s}(i\underline{k}.\nabla_{\underline{v}}f_{so})\hat{\phi}_k \qquad 2.1.7$$

and $$\epsilon_0 k^2\hat{\phi}_k = \sum_s q_s N_s\int \hat{f}_{sk}d\underline{v} \qquad 2.1.8$$

We can now eliminate $\hat{f}_{sk}$ to obtain

$$k^2\hat{\phi}_k = \frac{1}{D(\underline{k},\omega)}\sum_s q_s N_s\int\frac{f_{sk}(\underline{k},\underline{v},t=0)}{p+i\underline{k}.\underline{v}}d\underline{v} \qquad (p_r \geq p_o) \qquad 2.1.9$$

where the denominator of this equation has been identified as the dialectric function of a field free plasma for electrostatic waves of frequency $\omega = ip$, wave number $\underline{k}$. Namely,

$$D(\underline{k},\omega) = 1+\sum_s\frac{q_s^2 N_s}{\epsilon_0 m_s k^2}\int\frac{\underline{k}.\nabla_{\underline{v}}f_{so}}{ip-\underline{k}.\underline{v}}d\underline{v} \qquad 2.1.10$$

We now simplify the above velocity integrals by choosing a co-ordinate system in which $\underline{k}$ lies along one of the axes (i.e. $\underline{k} = k\underline{i}$) and then define F(u) as the integral of $f(\underline{v})$ over the other two velocity co-ordinates v_y and v_z.

$$\text{i.e. } F_{so}(u) = \int f_{so}(u,v_y,v_z)dv_y dv_z$$

$$\text{and } \hat{F}_{sk}(u) = \int \hat{f}_{sk}(u,v_y,v_z)dv_y dv_z$$

Then 2.1.7,2.1.9,2.1.10 become

$$\hat{f}_{s1}(\underline{k},\underline{v},p) = \frac{1}{p+i\underline{k}.\underline{v}}\left(f_{sk}(\underline{k},\underline{v},t=0) + \frac{q_s}{m_s}(i\underline{k}.\nabla_{\underline{v}} f_{so})\hat{\phi}_k\right) \qquad 2.1.11$$

$$\hat{\phi}_k(p) = \frac{-i}{|\underline{k}|^3 D(\underline{k},ip)}\sum_s q_s N_s \int \frac{F_{sk}(u,t=0)}{u-ip/|\underline{k}|}du \qquad 2.1.12$$

$$D(\underline{k},ip) = 1 - \sum_s \left(\frac{\omega_{ps}}{k}\right)^2 \int \frac{\frac{\partial F_{so}}{\partial u}(u)}{u-ip/|\underline{k}|}du \qquad (p_r \geq p_o) \qquad 2.1.13$$

We obtain the time dependence of $\phi(\underline{k})$ by inverting the Laplace transform to obtain

$$k^2\phi_k(t) = \frac{1}{2\pi i}\int_{p_o-i\infty}^{p_o+i\infty} \frac{1}{D(\underline{k},ip)}\sum_s q_s N_s \int \frac{F_{sk}(u,t=0)}{p+i|\underline{k}|u}du \exp(pt)dp \qquad 2.1.14$$

where p_o is chosen such that all poles of $\hat{\phi}(p)$ lie to the left of $p_r = p_o$.

In general this integral can not be evaluated analytically except for a few special $F_{so}(u)$ and $F_{sk}(u,t=0)$. However, the long time solution may be obtained for a wide class of equilibrium distributions. We shall see that this assymptotic behaviour is determined by the normal modes of the plasma oscillations rather than by the details of the initial perturbation.

Note that $\hat{\phi}_k(p)$ is only defined by 2.1.12 for $p_r \geq p_o$. For convenience in performing the p integration in 2.1.14 we define a function $\Phi_k(p)$ that is identical with $\hat{\phi}_k(p)$ for $p_r \geq p_o$ whilst for $p_r < p_o$, $\Phi_k(p)$ is defined as the analytic continuation of $\hat{\phi}_k(p)$. (See appendix 2) Inspection of 2.1.12 reveals that analytic continuation of $\hat{\phi}_k(p)$ requires a statement about the analytic properties of $F_{so}, \hat{F}_{sk}$ as well as about analytic continuation of velocity integrals of the form

$$H(p) = \int_{-\infty}^{+\infty} \frac{h(u)}{u-ip/|\underline{k}|}du \qquad (p_r \geq p_o)$$

to values of $p_r < p_o$.

We limit ourselves to functions h(u) that are analytic for all finite u. (e.g.Maxwellian.)

It is trivial to analytically continue (See Appendix 2) H(p) to the half plane $p_r > 0$ since we simply have

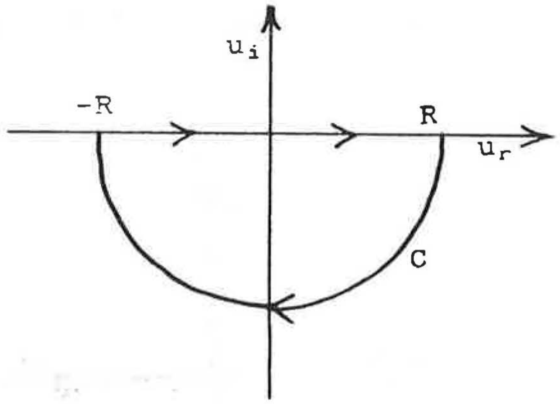
$$H(p) = \int_{-\infty}^{+\infty} \frac{h(u)}{u-ip/|\underline{k}|}du \qquad \text{for all } p_r > 0.$$

The only problem that arises is when we try to analytically continue H(p) across the line of integration in the u-plane. Consider,

$$H(p) = \int_{C} \frac{h(u)}{u-ip/|\underline{k}|}du$$

where C is the contour in Fig.1 as R→∞.

Fig.1



Cauchy's residue theorem states that

$$H(p) = -2\pi i\sum \text{res}\left(\frac{f(u)}{u-ip/|\underline{k}|}\right)du$$

where the summation is over all isolated singularities of $\frac{f(u)}{u-ip/|\underline{k}|}$. The singularity at $ip/|\underline{k}|$ will either be inside or outside C depending on the sign of p_r.

Defining

$$H(p) = \int_{-\infty}^{+\infty} \frac{h(u)}{u-ip/|\underline{k}|}du \qquad p_r \geq 0 \qquad 2.1.15a$$

$$H(p) = \int_{-\infty}^{+\infty} \frac{h(u)}{u-ip/|\underline{k}|}du + 2\pi i h(ip/|\underline{k}|) \qquad p_r \leq 0 \qquad 2.1.15b$$

we see that $H(p)$ defined by 2.1.15ab is the analytic continuation of our original expression for $H(p)$ which was only defined for $p_r \geq p_o$.

The value of $H(p)$ as $p_r \to 0$ is

$$H(ip_i) = \int_{-\infty}^{+\infty} \frac{h(u)}{u+ip_i/|\underline{k}|}du + \pi i h(-p_i/|\underline{k}|)$$

by the Plemelj formulae for Cauchy integrals (See Appendix 5) applied to either of the expressions 2.1.15ab.

Alternatively we can use the function

$$H(p) = \int_L \frac{h(u)}{u-ip/|\underline{k}|}du$$

where L is the Landau contour sketched in Fig.2, since this is also analytic for all p.

Fig.2a

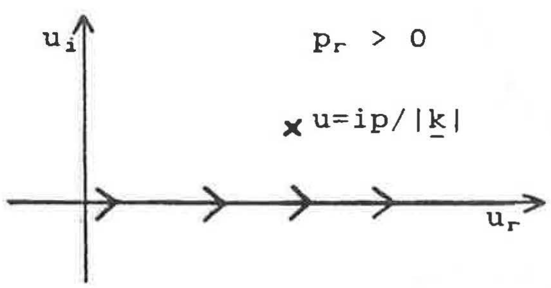


Fig.2b

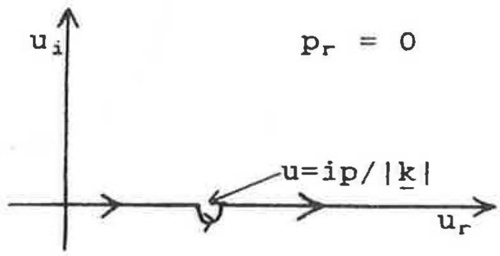


Fig.2c

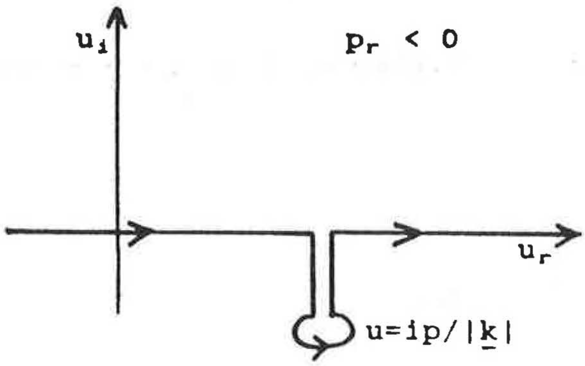
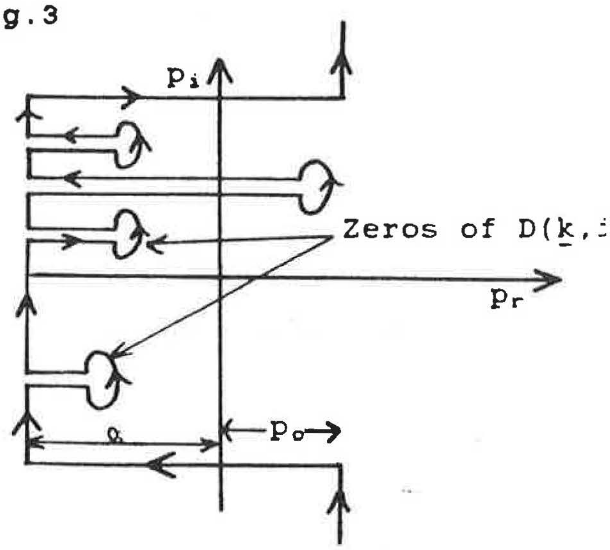


Fig.3



Thus,

$$\hat{\Phi}_k(p) = \frac{\dfrac{-i}{|\underline{k}|^3}\sum q_s N_s \displaystyle\int_L \frac{F_{sk}(u,t=0)}{u-ip/|\underline{k}|}du}{1 - \displaystyle\sum_s \left(\frac{\omega_{ps}}{k}\right)^2 \int_L \frac{\dfrac{\partial F_{so}}{\partial u}(u)}{u-ip/|\underline{k}|}du} \qquad 2.1.16$$

where the velocity integrals are defined by the Landau contour or alternatively, the Landau prescription 2.1.15 is used as the analytic continuaton of $\hat{\phi}_k(p)$. The contour in the Laplace inversion 2.1.14 can be deformed to any other path (because of Cauchy's theorem) provided that the poles (namely, $D(\underline{k},ip) = 0$) are not crossed. Fig.3 shows a deformed path of integration for a possible set of poles of $\hat{\Phi}_k(p)$. Thus,

$$\Phi_k(t) = \underset{(a)}{\sum_j R_j \exp(p_j(k)t)} + \underset{(b)}{\frac{1}{2\pi i}\int_{-i\infty+p_o}^{-i\infty-\alpha} \hat{\Phi}_k(p)\exp(pt)dp} +$$

$$\underset{(c)}{\frac{1}{2\pi i}\int_{-\alpha-i\infty}^{-\alpha+i\infty} \hat{\Phi}_k(p)\exp(pt)dp} + \underset{(d)}{\frac{1}{2\pi i}\int_{i\infty-\alpha}^{i\infty+p_o} \hat{\Phi}_k(p)\exp(pt)dp}$$

where $D(\underline{k},ip_j(\underline{k})) = 0$ locates the poles p_j of $\hat{\Phi}_k(p)$ and R_j is the residue

$$R_j = \lim_{p \to p_j} \left((p-p_j)\hat{\Phi}_k(p) \right)$$

[N.B. We have assumed all singularities of $D(\underline{k},ip) = 0$ are isolated simple poles.]

Consider term (b)

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{-iR+p_o}^{-iR-\alpha} \hat{\Phi}_k(p_r - iR)\exp(p_r t + Rt)dp_r$$

Assuming $\hat{\Phi}_k(p_r - iR) \to 0$ faster than $\exp(Rt)$ as $R \to \infty$ this expression [and term (d)] vanishes.

Term (c) involves integration of the form

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{iR-\alpha}^{iR-\alpha} \hat{\Phi}_k(-\alpha + ip_i)\exp(-\alpha t + ip_i t)dp_i$$

which decays as $t\to\infty$ due to the $\exp(-\alpha t)$ factor. [It decays faster than $\exp(p_{jr}t)$ since α is arbitrarily large and so (c) is negligible compared with (a) as $t\to\infty$]

Thus,

$$\Phi_k \longrightarrow \sum_j R_j \exp(p_j(k)t) \quad \text{for large } t.$$

We note that poles to the left of $p_r = 0$ are damped whilst those to the right create growing (unstable) fields.

Define the frequency, $\omega = ip$.

Then, $$\Phi_k(t) = \sum_j R_j \exp(-i\omega_j t) \qquad 2.1.17$$

where $\omega_j = \omega_{jr} + i\omega_{ji}$ satisfies $D(k,\omega_j) = 0$, the dispersion relation, with D given by 2.1.13.

Thus the non-transient response is determined by the normal modes (the zeros of the dialectric) of phase velocity ω/k and group velocity $\frac{\partial\omega}{\partial k}$. It is important to note that the interpretation of the zeros of D being the normal modes is only valid in the time assymptotic limit.

[N.B.It can easily be shown that [6] by assuming f_1, Φ_1 have dependence $\exp(i(\underline{k}.\underline{x}-\omega t))$, the same dispersion relation 2.1.13 results.]

The normal modes are those wave-like disturbances that persist long after any transients associated with the initial disturbance have died out. The eigenfrequency ω is almost purely real since if ω_i were large (assuming $\omega_i < 0$, a stable plasma) then the wave would be damped out quickly and would not be called a normal mode. This assumption of small ω_i simplifies the integration in 2.1.13 since a Taylor expansion about $\omega_i = 0$ may be used.

Thus,

$$\int_{-\infty}^{\infty} \frac{\frac{\partial F_{so}(u)}{\partial u}}{u-\omega/|\underline{k}|} du = \left(1 + i\omega_i \frac{\partial}{\partial \omega_r}\right)\left(\int_{-\infty}^{\infty} \frac{\frac{\partial F_{so}(u)}{\partial u}}{u-\omega/|\underline{k}|} du\right)\Bigg|_{\omega=\omega_r}$$

so 2.1.13 becomes

$$1 - \sum_s \left(\frac{\omega_{ps}}{k}\right)^2 \left(1+i\omega_i \frac{\partial}{\partial \omega_r}\right)\left[\int_{-\infty}^{\infty} \frac{\frac{\partial F_{so}(u)}{\partial u}}{u-\omega_r/|\underline{k}|} du + \pi i \frac{\partial F_{so}(u)}{\partial u}\Bigg|_{u=\frac{\omega_r}{|\underline{k}|}}\right] = 0 \qquad 2.1.18$$

using the Plemelj formulae. (See Appendix 5)

It is important to remember that the dialectric 2.1.13 and 2.1.18 are only valid for $\underline{E}_o = \underline{0} = \underline{B}_o$. More complicated equilibrium states have different dialectric properties. However, the general procedure is the same. Namely to derive the plasma dialectric, to locate its zeros and identify those zeros with plasma waves, which in the time assymptotic limit represent the normal modes of the system.

2.2 Solving the Dispersion Relation

Approximate solutions of 2.1.18 may be found for phase velocities in certain ranges. First, we assume $\omega_r/k >> v_{th}$ where v_{th} is the thermal speed of the particles. (See Appendix 1.) Then the Cauchy Principle Value integral in 2.1.18 may be evaluated via a Binomial expansion in u. This is because contributions to the integral outside of the range $-v_p < u < v_p$ are negligible. (See Appendix 7)

We obtain

$$\left(\frac{\omega_{pe}}{k}\right)^2 C\!\!\int_{-\infty}^{\infty} \frac{\frac{\partial F_{eo}}{\partial u}}{u-\omega_r/|\underline{k}|} du = \frac{\omega_{pe}^2}{\omega_r^2} + \frac{3\omega_{pe}^4 k^2 \lambda_D^2}{\omega_r^4} + \ldots \qquad 2.2.1$$

Substituting this into 2.1.18, neglecting ion terms since they are smaller by a factor m_e/m_i, we obtain

$$\omega_r^2 = \omega_{pe}^2(1 + 3k^2\lambda_D^2) \qquad 2.2.2a$$

$$\omega_i = -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_{pe}}{|k^3\lambda_D^3|} \exp-\left(\frac{1}{2k^2\lambda_D^2} + \frac{3}{2}\right) \qquad 2.2.2b$$

For small $k\lambda_D$ we have

$$\omega_r \approx \omega_{pe}(1 + 3k^2\lambda_D^2/2)$$

which is the familiar result for Langmuir waves obtained from fluid theory [6,9]. The Vlasov theory, however, did not need to assume an equation of state, unlike fluid theory; rather it *reveals* the necessary equation of state needed in two-fluid theory.

Note our solution 2.2.2 does indeed satisfy our two assumptions $\omega_r/|\underline{k}| >> v_{th}$ and $\omega_i << \omega_r$, provided $k\lambda_D << 1$. A most important property predicted by Vlasov and not fluid theory is the collisionless damping of the electrostatic potentials of the normal modes. This is discussed below.

2.3 Landau Damping

The damping is characterised by the small negative quantity ω_i given by 2.2.2b. We can find ω_r, ω_i in a more general way as follows.

Assuming $\omega_i << \omega_r$ we have

$$0 = D(\underline{k},\omega) \approx D(\underline{k},\omega_r) + i\omega_i \frac{\partial D(\underline{k},\omega_r)}{\partial \omega_r} \qquad 2.3.1$$

where $D(\underline{k},\omega_r) = D_r(\underline{k},\omega_r) + iD_i(\underline{k},\omega_r)$

Equating real and imaginary parts in 2.3.1 ⇒

$$D_r(\underline{k},\omega_r) = 0 \qquad 2.3.2a \qquad \text{and} \quad \omega_i = \frac{-D_i(k,\omega_r)}{\frac{\partial D_r(\underline{k},\omega_r)}{\partial \omega_r}} \qquad 2.3.2b$$

where
$$D_i = -\pi \sum_s \frac{\omega_{pe}^2}{k^2} \frac{\partial F_{so}}{\partial u}\bigg|_{u=\omega_r/|\underline{k}|} \qquad 2.3.3a$$

and
$$D_r = 1 - \sum_s \left(\frac{\omega_{ps}}{k}\right)^2 C\!\!\int_{-\infty}^{+\infty} \frac{\frac{\partial F_{so}}{\partial u}}{u - \omega_r/|\underline{k}|} du \qquad 2.3.3b$$

from 2.1.18.

So we have $\phi_1 \propto \exp(\omega_i t)$ where $\omega_i \propto \frac{\partial F_{eo}}{\partial u}\Big|_{u=\omega_r/|\underline{k}|}$. Thus, Landau damping is a resonant effect due to particles moving with velocity close to the phase velocity of the waves. Physically, Landau damping is explained by noting that for $\frac{\partial F_{eo}}{\partial u} < 0$ there are more particles travelling slightly slower than the wave than there are faster so, if the slower particles are accelerated by the wave, the wave loses energy and is damped.

From 2.2.2b it follows that

$$1/\omega_i >> 2\pi/\omega_{pe} \text{ for } k\lambda_D \to 0$$

i.e. Landau damping time >> plasma osillation period.

As the wavelength λ decreases and approaches λ_D the damping increases and then the oscillation can no longer be considered a normal mode.

2.4 Ion Acoustic Waves

Had the ion term in $D(\underline{k},\omega)$ been retained the Langmuir frequency would only have altered by a small amount. However, if the electrons are warm ($T_e >> T_i$) we shall see that there exists an electrostatic wave for which the ions play a significant role in the range

$$v_{thi} = \left(\frac{KT_i}{m_i}\right)^{1/2} < \frac{\omega}{k} < \left(\frac{KT_e}{m_e}\right)^{1/2} = v_{the} \qquad 2.4.1$$

For the ions we perform a similar method to that used in deriving 2.2.1 (See Appendix 7) but for the electron integral a different approach is required.

For the electrons we have

$$-\,\mathcal{C}\!\!\int_{-\infty}^{\infty} \frac{\frac{\partial F_{eo}}{\partial u}(u)}{u-\omega_r/|\underline{k}|}\,du = \lim_{\delta\to 0^+}\left\{\left[-\ln(u-v_p)\frac{\partial F_{eo}}{\partial u}\right]_{v_p+\delta}^{\infty} + \int_{v_p+\delta}^{\infty}\ln(u-v_p)\frac{\partial^2 F_{eo}}{\partial u^2}du + \left[-\ln(v_p-u)\frac{\partial F_{eo}}{\partial u}\right]_{-\infty}^{v_p-\delta} + \int_{-\infty}^{v_p-\delta}\ln(v_p-u)\frac{\partial^2 F_{eo}}{\partial u^2}du\right\}$$

The terms in square brackets cancel and if F_{eO} is an even function (e.g. Maxwellian) then

$$\int_{-\infty}^{v_p-\delta}\ln(v_p-u)\frac{\partial^2 F_{eo}}{\partial u^2}du = \int_{v_p+\delta}^{\infty}\ln(u-v_p)\frac{\partial^2 F_{eo}}{\partial u^2}du$$

But,

$$\int_{v_p+\delta}^{\infty}\ln(u-v_p)\frac{\partial^2 F_{eo}}{\partial u^2}du = \int_{v_p+\delta}^{\infty}\ln(u)\frac{\partial^2 F_{eo}}{\partial u^2} - \frac{v_p}{u}\frac{\partial^2 F_{eo}}{\partial u^2} + \ldots du$$

(The expansion is valid because $u > v_p$ throughout the integral.)

So,

$$C\int_{-\infty}^{\infty}\frac{\frac{\partial F_{eo}(u)}{\partial u}}{u-\omega_r/|\underline{k}|}\,du = 2\lim_{\delta\to 0^+}\int_{v_p+\delta}^{\infty}\ln(u)\frac{\partial^2 F_{eo}}{\partial u^2}du$$

i.e.

$$C\int_{-\infty}^{\infty}\frac{\frac{\partial F_{eo}(u)}{\partial u}}{u-\omega_r/|\underline{k}|}\,du \approx 2\int_{0}^{\infty}\ln(u)\frac{\partial^2 F_{eo}}{\partial u^2}du$$

since v_p is small. (i.e. $\omega_{pe}/|k| \ll v_{the}$)

Let
$$I = \int_0^\infty \ln(u)\frac{\partial^2 F_{eo}}{\partial u^2}du$$
$$= -2bc\int_0^\infty (1-2u^2c)\exp(-cu^2)\ln u\,du$$

for F_{eo} = $b\exp(-cu^2)$ which is in the form of a Maxwellian.

Consider
$$J = \int_0^\infty u^2\exp(-cu^2)\ln u\,du$$
$$= \left[\frac{-1}{2c}\exp(-cu^2)u\ln u\right]_0^\infty + \frac{1}{2c}\int_0^\infty(1+\ln u)\exp(-cu^2)du$$

The first term vanishes so

$$-I/2bc = K - 2cJ \qquad \text{where } K = \int_0^\infty \exp(-cu^2)\ln u\,du$$

Then
$$I/2bc = \int_0^\infty \exp(-cu^2)du \qquad \text{by simple substitution.}$$
$$= \frac{1}{2}\left(\frac{\pi}{c}\right)^{1/2} \qquad \text{using the } \Gamma \text{ function.}$$

Finally,

$$-C\int_{-\infty}^{\infty}\frac{\frac{\partial F_{eo}(u)}{\partial u}}{u-\omega_r/|\underline{k}|}\,du = \frac{1}{k^2\lambda_D^2} \qquad 2.4.3$$

since $b = \left(\frac{m_e}{2\pi\kappa T_e}\right)^{1/2}$, $c = \frac{m_e}{2\kappa T_e}$ for a Maxwellian.

We now substitute 2.4.2 and 2.4.3 into 2.3.3a,b and employ 2.3.2a,b to obtain

$$\omega_r^2 = k^2 c_s^2/(1+k^2\lambda_D^2) \qquad 2.4.4a$$

and

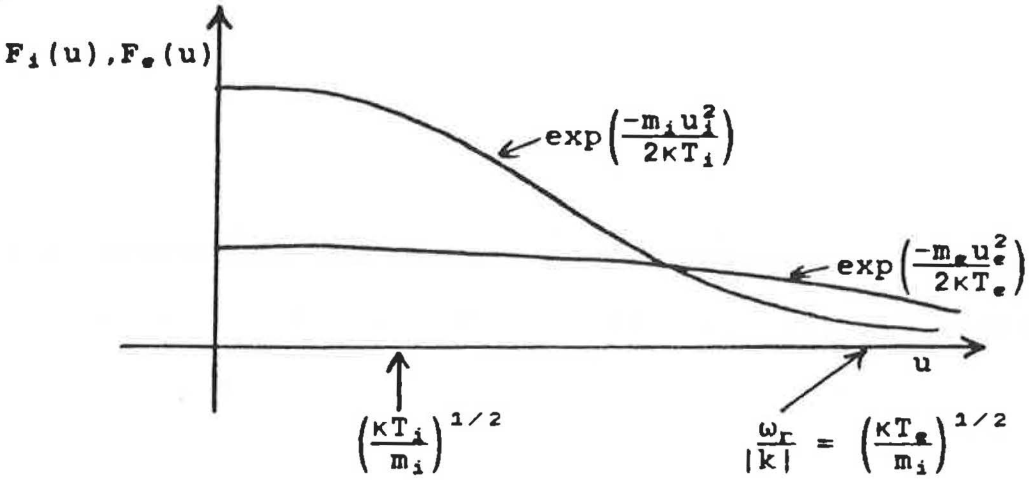
$$\omega_i = -\left(\frac{\pi}{8}\right)^{1/2} \frac{|\omega_r|}{(1+k^2\lambda_D^2)^{3/2}} \left[\left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(\frac{-T_e/T_i}{2(1+k^2\lambda_D^2)}\right) + \left(\frac{m_e}{m_i}\right)^{1/2}\right] \qquad 2.4.4b$$

which are valid provided $\left(\frac{\kappa T_i}{m_i}\right)^{1/2} < \frac{\omega_r}{|k|} < \left(\frac{\kappa T_e}{m_e}\right)^{1/2}$ is true.

These waves are called ion-acoustic waves since they are like sound waves in the sense that all wavelengths propogate at the same speed c_s if $k\lambda_D << 1$. This is different to Langmuir waves which have the same frequency for all wavelengths ($k\lambda_D << 1$). We see from 2.4.4b that in order for $\omega_i << \omega_r$ we require $T_e >> T_i$. From 2.4.4b one might also think that $T_i >> T_e \Rightarrow \omega_i << \omega_r$ but this is not true since 2.4.4b is only valid for $T_i < T_e$ because the assumption $\frac{\omega_r}{|k|} > \left(\frac{\kappa T_i}{m_i}\right)^{1/2}$ is just $\frac{\omega_r}{|k|} = c_s = \left(\frac{\kappa T_e}{m_i}\right)^{1/2} > \left(\frac{\kappa T_i}{m_i}\right)^{1/2}$ which is actually $T_e > T_i$.

The electron damping term $\left(\frac{m_e}{m_i}\right)^{1/2}$ in 2.4.4b is always small for ion waves which is because, although there are a lot of electrons with approximately the same speed as the ion-sound wave, the slope of the electron distribution is small. (i.e. there are nearly as many electrons going faster than the wave as there are slower so the Landau effect is negligible, see Fig.4)

Fig.4



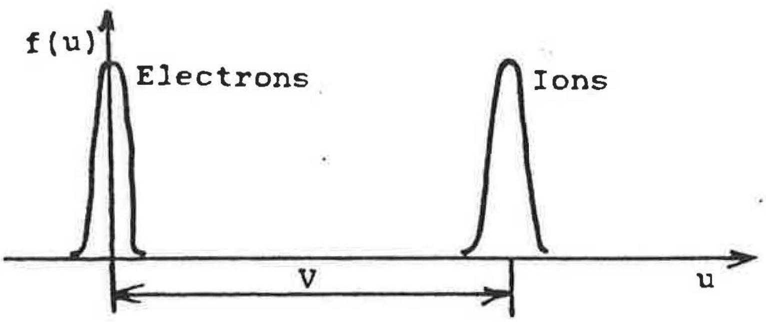
The two waves (Langmuir and ion-acoustic) are actually the only two modes favoured by the field free isotropic plasma.

Chapter 3

Plasma Stability

We have shown that small perturbations about an equilibrium state grow under certain conditions - this is called instability. We shall study two types of instability with a view to using the growth rates in a calculation of an effective collision frequency. We have already seen that $\omega_i \propto \left.\frac{df}{du}\right|_{u=\omega_r/|\underline{k}|}$ which implies that a distribution with a positive slope may be driven unstable by resonant particle (Landau) effects. (See the gentle bump instability in [6].) However, another type of instability that is not due to these wave-particle effects is the *two-stream instability* which occurs when the plasma consists of two streams drifting relative to each other with a constant velocity, $\underline{V}$, much greater than the thermal spread of the beams. (See Fig.5) Such beams and currents are very common in both space and laboratory plasmas.

Fig 5



3.1 The Two-stream Instability

Consider a field-free plasma with equilibrium distribution

$$f_{io}(\underline{v}) = \delta(\underline{v}-\underline{V})$$

$$f_{eo}(\underline{v}) = \delta(\underline{v})$$

i.e. $$F_{io}(\underline{v}) = \delta(u - \hat{\underline{k}}.\underline{V}) \qquad 3.1.1a$$

$$F_{eo}(\underline{v}) = \delta(u) \qquad 3.1.1b$$

where $\hat{\underline{k}}$ is the unit vector in the $\underline{k}$ direction.

Assuming that the perturbations are electrostatic we have shown (see 2.1.13) that the resulting oscillations are of frequency ω given by

$$D(\underline{k},\omega) = 1-\sum_s \left(\frac{\omega_{ps}}{k}\right)^2 \int_L \frac{\frac{\partial F}{\partial u}}{u-\omega/|\underline{k}|} du = 0$$

Now,

$$\int_L \frac{1}{u-\omega/|\underline{k}|}\frac{\partial}{\partial u}\left[\delta(u-\hat{\underline{k}}.\underline{V})\right]du = \left[\delta(u-\hat{\underline{k}}.\underline{V})\frac{1}{u-\omega/|\underline{k}|}\right]_{-\infty}^{\infty} +$$

$$\int_{-\infty}^{\infty} \delta(u-\hat{\underline{k}}.\underline{V})\,\frac{1}{(u-\omega/|\underline{k}|)^2}du$$

by parts.

Thus,

$$D(\underline{k},\omega) = 1 - \left(\frac{\omega_{pe}}{\omega}\right)^2 - \left(\frac{\omega_{pi}}{\omega-\underline{k}.\underline{V}}\right)^2 \qquad 3.1.2$$

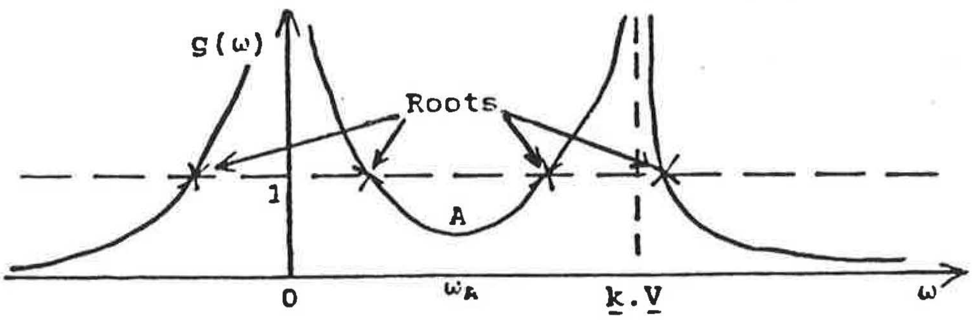
using the well known result

$$\int f(x)\,\delta(x-x_o)\,dx = f(x_o)$$

[N.B. This result 3.1.2 may also be obtained directly from fluid theory.]

Suppose we plot $g(\omega) = \left(\frac{\omega_{pe}}{\omega}\right)^2 - \left(\frac{\omega_{pi}}{\omega-\underline{k}.\underline{V}}\right)^2$ as a function of ω

Fig.6



When the value, A, of the minima is < 1, there exist 4 real roots but if A > 1 we have a complex conjugate pair with one root corresponding to instability.

The point A is at $\omega_A = \underline{k}.\underline{V}\left(\frac{\mu}{\mu+1}\right)$ 3.1.3

where $\mu \equiv \left(\frac{\omega_{pe}}{\omega_{pi}}\right)^{2/3}$

Thus ω_A is always less than $\underline{k}.\underline{V}$. i.e. the waves that are driven unstable exist between the two 2 streams and therefore can not be due to Landau effects.

No real stream will have a δ function shape but thermal effects may be ignored if ω_A is not within the thermal spread of the distributions.

Thus,

$$\left|\frac{\omega}{k} - \hat{\underline{k}}.\underline{V}\right| >> \left(\frac{\kappa T_i}{m_i}\right)^{1/2} \ ; \ V_o >> \left(\frac{\kappa T_e}{m_e}\right)^{1/2} \qquad 3.1.4$$

must be valid.

The question that immediately arises is "What is the physical mechanism for the instability if it is not Landau effects?" Consider a local decrease in charge density of electrons, say. (This could correspond to an electrostatic wave.) This would induce a charge perturbation in the stream passing over the decreased density region. Ions passing over this "hole of charge" would be slowed down and then the conservation of charge equation

$$\frac{1Dn}{nDt} = - \frac{\partial v}{\partial x}$$

implies that n increases so that the stream feeds the hole, creating an instability.

3.2 Ion Acoustic Instability

We now tie up some of the ideas discussed in this and the preceding chapter. In chapter 2 we saw that ion-acoustic waves are weakly Landau damped if $T_e >> T_i$ and strongly damped if $T_e = T_i$. The findings on the two-stream instability would appear to indicate that these ion waves might be driven unstable by a somewhat weak drift if the electrons are sufficiently hot that Landau effects are overriden by the growth from the drift.

Thus, consider the distribution

$$F_{eo} = \left(\frac{m_e}{2\pi K T_e}\right)^{1/2} \exp\left(\frac{-m_e(u-u_o)^2}{2\kappa T_e}\right) \qquad 3.2.1a$$

$$F_{io} = \left(\frac{m_i}{2\pi K T_i}\right)^{1/2} \exp\left(\frac{-m_i u^2}{2\kappa T_i}\right) \qquad 3.2.1b$$

where u_o is the constant drift velocity.

N.B. Here we are in the ion rest frame whereas in section 3.1 we were in the electron rest frame.

Ion waves exist in the range

$$\left|\frac{\omega_r}{k}\right| >> \left(\frac{\kappa T_i}{m_i}\right)^{1/2}$$

$$\left|\frac{\omega_r}{k} - u_o\right| << \left(\frac{\kappa T_e}{m_e}\right)^{1/2}$$

according to section 2.4.

Substituting 3.2.1 in the field-free dialectric of Chapter 2 and solving as in section 2.2 implies that we obtain a contribution $-\frac{\omega_{pi}^2}{\omega_r^2}$ to D_r from the ion term since $\left|\frac{\omega_r}{k}\right| >> \left(\frac{\kappa T_i}{m_i}\right)^{1/2}$ and we obtain a contribution $\frac{1}{\bar{k}^2 \lambda_D^2}$ from the electron term since $\left|\frac{\omega_r}{k} - u_o\right| << \left(\frac{\kappa T_e}{m_e}\right)^{1/2}$.

Thus,

$$D_r(\underline{k},\omega) = 1 - \left(\frac{\omega_{pi}}{\omega_r}\right)^2 + \frac{1}{\bar{k}^2 \lambda_D^2} = 0$$

So,

$$\omega_r^2 = \frac{k^2 c_s^2}{1+k^2\lambda_D^2} \qquad 3.2.2a$$

Now, using 2.3.2b and 2.3.3a yields

$$\omega_i = \frac{\sqrt{\pi}|\omega_r|k^2c_s^2}{k^2(1+k^2\lambda_D^2)}\left\{\left(\frac{m_i}{m_e}\right)\lambda_e{}^{3/2}\left(\frac{\omega_r}{|k|} - u_o\right)\exp\left[-\lambda_e\left(\frac{\omega_r}{|k|} - u_o\right)^2\right] + \lambda_i{}^{3/2}\frac{\omega}{|k|}\exp\left[-\lambda_i\frac{\omega_r^2}{|k|^2}\right]\right\}$$

where we have put $\lambda_s = \frac{m_s}{2\kappa T_s}$ for convenience.

Now,

$$\lambda_e\left(\frac{\omega_r}{|k|} - u_o\right)^2 = \frac{m_e}{2KT_e}\left(\frac{\omega_r}{|k|} - u_o\right)^2 << 1$$

because otherwise the waves would not exist.

So,

$$\exp\left[-\lambda_e\left(\frac{\omega_r}{|k|} - u_o\right)^2\right] \approx 1$$

Simple algebra yields

$$\frac{\lambda_i\omega_r^2}{k^2} = \frac{T_e}{2T_i(1+k^2\lambda_D^2)}$$

and

$$c_s^2\lambda_i{}^{3/2}\omega_r/k = \frac{1}{2\sqrt{2}}\left(\frac{T_e}{T_i}\right)^{3/2}\frac{1}{(1+k^2\lambda_D^2)^{1/2}}$$

Altogether this implies that ω_i becomes

$$\omega_i = -|\omega_r|\left(\frac{\pi}{8}\right)^{1/2}\frac{1}{(1+k^2\lambda_D^2)^{3/2}}\left\{\underbrace{\left(\frac{T_e}{T_i}\right)^{3/2}\exp\left(\frac{-T_e}{2T_i(1+k^2\lambda_D^2)}\right)}_{(a)} + \underbrace{\left(\frac{m_e}{m_i}\right)^{1/2}\left(1 - \frac{u_o}{c_s}(1+k^2\lambda_D^2)^{1/2}\right)}_{(b)}\right\} \qquad 3.2.2b$$

When u_o=0 we do indeed get the result 2.4.4b but if u_o is large enough the sign of ω_i may reverse to give instability. Term (a) is due to the ion damping whilst (b) is the growth due to the electron drift.

Instability occurs if

$$u_o > \left(\frac{c_s^2}{1+k^2\lambda_D^2}\right)^{1/2} = \left(\frac{\kappa T_e}{m_i(1+k^2\lambda_D^2)}\right)^{1/2}$$

i.e. if $$u_o > \left(\frac{KT_i}{m_i}\right)^{1/2}$$

since we also require $T_e >> T_i$ for (a) to be negligible. Under these conditions

$$\omega_i = k\left(\frac{\pi m_e}{8m_i}\right)^{1/2}\frac{(u_o-\omega_r/k)}{(1+k^2\lambda_D^2\ )^{3/2}} \qquad 3.2.3$$

This can be differentiated w.r.t. k to find the maximum growth rate needed in the calculation of the "collision" frequency of Chapter 4.

3.3 The Modified Two-Stream Instability

We now extend the idea of the two-stream instability to include a magnetic field $\underline{B}_o$ in the equilibrium state. [8] The instabilities created by the relative streaming of ions and electrons across a magnetic field play a crucial role in the concept of anomalous resistance. (See Chapters 4,6.) This is particularly relevant to a collisionless shock wave since a current does exist along the shock front. (See Chapter 5.)

Let the ions drift relative to electrons with speed V. We assume the electrostatic assumption is valid, so $\underline{E}=-\nabla\phi$.

McBride et al initially start from kinetic theory but we derive the dispersion relation from two-fluid theory which is valid provided resonant particle effects are unimportant.

i.e. valid if $$k\rho_e << 1$$

$$kv_{thi} << |\omega_r - \underline{k}.\underline{V}|$$

$$k_z v_{the} << |\omega_r|$$

where ρ_e is the Larmor radius for electrons.

In the electron drift frame the perturbation potential gives rise to a perturbation $\underline{v}_e$ in electron velocity given by the following equations where we have assumed $\exp i(\underline{k}.\underline{x}-\omega t)$ dependence for all perturbation quantities.

The linearised electron momentum equations are

$$i\omega v_{ex} = -\frac{iek_x\phi}{m_e} - \omega_{ce}v_{ey} \qquad 3.3.1a$$

$$i\omega v_{ey} = -\frac{iek_y\phi}{m_e} + \omega_{ce}v_{ex} \qquad 3.3.1b$$

$$i\omega v_{ez} = -\frac{iek_z\phi}{m_e} \qquad 3.3.1c$$

where $\omega_{cs} = \frac{|q|B}{m_s}$ for s=e,i is the cyclotron frequency of species s.

The linearised electron continuity equation ⇒

$$n_{1e} = -\frac{n_o\nabla.\underline{v}_e}{i\omega} = \frac{n_o}{\omega}(k_xv_{ex}+k_yv_{ey}+k_zv_{ez}) \qquad 3.3.2$$

Now, $k_x(3.3.1b)-k_y(3.3.1a)$ ⇒

$$-k_yv_{ex}+k_xv_{ey} = \frac{\omega_{ce}}{i\omega}(k_yv_{ey}+k_xv_{ex}) \qquad 3.3.3$$

and $k_x(3.3.1a)+k_y(3.3.1b)$ ⇒

$$k_xv_{ex}+k_yv_{ey} = \frac{-e}{m_e\omega}(k_x^2+k_y^2)\phi + \frac{i\omega_{ce}}{\omega}(k_xv_{ey}-k_yv_{ex}) \qquad 3.3.4$$

Substitute 3.3.3 in 3.3.4 ⇒

$$k_xv_{ex}+k_yv_{ey} = \frac{-e}{m_e\omega}\left(\frac{k_x^2+k_y^2}{1-\omega_{ce}^2/\omega^2}\right)\phi$$

So 3.3.2 ⇒

$$-\frac{en_{e1}}{\epsilon_o} = \frac{e^2n_o\phi}{\epsilon_om_e\omega^2}\left(\frac{k_x^2+k_y^2}{1-\omega_{ce}^2/\omega^2} + k_z^2\right) \qquad \text{(using 3.3.1c)}$$

$$= \omega_{pe}^2k^2\phi\left(\frac{\sin^2\theta}{\omega^2-\omega_{ce}^2} + \frac{\cos^2\theta}{\omega^2}\right) \qquad 3.3.5$$

where $\omega_{ps} = \left(\frac{n_oe^2}{m_s\epsilon_o}\right)^{1/2}$ is the plasma frequency for species s and where θ is the angle between $\underline{k}$ and $\underline{B}_o$ so that $\cos\theta=k_z/k$.

For the ions $\underline{v}_i' = \underline{V}+\underline{v}_i$ is the actual ion velocity where $\underline{V}$ is the constant drift velocity and $\underline{v}_i$ the perturbation.

So

$$(\underline{v}_i'.\nabla)\underline{v}_i' = (\underline{V}+\underline{v}_i).\nabla(\underline{V}+\underline{v}_i)$$

$$= -i(\underline{k}.\underline{V})\underline{v}_i \qquad \text{after linearising.}$$

So

$$\left(\frac{\partial}{\partial t} + (\underline{v}.\nabla)\right)\underline{v} = i(\omega-\underline{k}.\underline{V})\underline{v}$$

Consider the relative sizes of the terms

$$\frac{\partial(\underline{v})}{\partial t} : e\underline{v}_i{}_\wedge\underline{B}$$

$$|i\omega v_i| : \left|\frac{ev_iB_o}{m_i}\right|$$

i.e. $\omega : \omega_{ci}$

We consider the case $\omega_{ce} >> \omega >> \omega_{ci}$ so we neglect $\underline{v}_i{}_\wedge\underline{B}_o$ in the ion equation but retain $\underline{v}_e{}_\wedge\underline{B}_o$ in the electron equation. The algebra for the ions is now the same as for the electrons only with

ω replaced by $\Omega = \omega-\underline{k}.\underline{V}$

m_e " m_i

$-e$ " e

ω_{pe} " ω_{pe}

and ω_{ce} " ω_{ci}

Thus 3.3.5, by analogy $\Rightarrow$

$$\frac{en_{i1}}{\epsilon_o} = \frac{\omega_{pi}^2 k^2\phi}{\Omega^2} \qquad 3.3.6$$

Now, $\nabla.\underline{E} = -\nabla^2\phi = \frac{e(n_{i1}-n_{e1})}{\epsilon_o}$ is Poisson's equation.

So 3.3.5,3.3.6 yield the dispersion relation

$$1 - \frac{\omega_{pi}^2}{(\omega-\underline{k}.\underline{V})^2} - \frac{\omega_{pe}^2\sin^2\theta}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pe}^2\cos^2\theta}{\omega^2} = 0 \qquad 3.3.7$$

For the case $k_y=0$ and remembering $\omega_{ce}>>\omega$ this is seen to be equation (5) in McBride et al.

We then have

$$1 - \frac{\omega_{pi}^2}{(\omega-\underline{k}.\underline{V})^2} - \frac{\omega_{pe}^2\sin^2\theta}{\omega^2-\omega_{ce}^2} - \frac{\omega_{pe}^2\cos^2\theta}{\omega^2} = 0 \qquad 3.3.8$$

(a) (b) (c) (d)

where $\cos\theta=k_z/k$, $\sin\theta=k_x/k$ and $k^2=k_x^2+k_z^2$.

3.4 Solving the Dispersion Relation

We shall now solve 3.3.8 for the special case of $\cos\theta = \left(\frac{m_e}{m_i}\right)^{1/2}$. Compare 3.3.8 with the dispersion relation 3.1.2 for the two-stream instability. (i.e. $\underline{B}_o=\underline{0}$) Terms (a),(b) are the same. Term (c) is due to the adiabatic polarisation drift of electrons across the magnetic field. Term (d) indicates that electrons behave as if they have an effective mass $\overline{m}_e = k^2m_e/k_z^2$. This can be large for $k^2/k_z^2 >> 1$ (i.e.$\underline{k}$ nearly perpendicular to $\underline{B}_o$) and we shall in fact study the case $\theta = \theta_o$ where $\cos\theta = \left(\frac{m_e}{m_i}\right)^{1/2}$ since we then have $\overline{m}_e = m_i$. [N.B.McBride's θ is our $\frac{\pi}{2}-\theta$]

At this angle 3.3.8 $\Rightarrow$

$$1 - \frac{\omega_{pi}^2}{(W-\underline{k}.\underline{V}/2)^2} + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{\omega_{pi}^2}{(W+\underline{k}.\underline{V}/2)^2} = 0 \quad (\text{Uses } \omega_{pi}^2 << \omega_{pe}^2) \qquad 3.4.1$$

where we have put $W=\omega-\underline{k}.\underline{V}/2$ so that expressing this equation over a common denominator leads to a quadratic in W^2 whose solutions are found to be

$$W^2 = (\underline{k}.\underline{V})^2/4 + \omega_{LH}^2 \pm \omega_{LH}(\omega_{LH}^2 + (\underline{k}.\underline{V})^2)^{1/2} \qquad 3.4.2$$

where $\omega_{LH} = \left(\frac{\omega_{pi}^2\omega_{ce}^2}{\omega_{ce}^2+\omega_{pe}^2}\right)^{1/2}$ is the lower hybrid frequency. The positive root implies W^2 is always positive so instability is not possible whilst W^2 may be negative if we take the negative root.

Let $W^2 = -\alpha$ where α is some positive real.

Then $W_i = \omega_i = \surd\alpha$

Now, $\frac{d\omega_i}{dk} = \frac{1}{(2\surd\alpha)}\frac{d\alpha}{dk}$

$= 0$ when $\frac{d\alpha}{dk} = 0$

Simple algebra shows this occurs when

$$(\underline{k}.\underline{V})^2 = 3\omega_{LH}^2$$

whereupon $\alpha = \omega_{LH}^2/4$.

Thus

$$W_i = \omega_i = \surd\alpha = \omega_{LH}/2 \qquad 3.4.3$$

is the maximum growth rate for $\theta=\theta_o$. Assuming $\underline{V}$ is in the x direction then this growth rate occurs at a wavenumber given by

$$kV = \surd 3\omega_{LH}. \qquad 3.4.4$$

Also we have

$$\omega_r = W_r+kV/2 = \surd 3\omega_{LH}/2 \qquad 3.4.5$$

Thus

$$\omega_r \sim \omega_i \sim \omega_{LH} \qquad 3.4.6$$

so this instability occurs at the lower hybrid frequency, which is why the modified two-stream instability is often referred to as the lower hybrid instability. (Also as the Buneman instability, see [2])

Since $\omega_r = kV/2$ then $|\omega-kV|>kv_{ti}$ becomes approximately $V\gtrsim v_{ti}$ whilst $|\omega|>k_z v_{te}$ becomes $V\gtrsim\left(\frac{T_e}{m_i}\right)^{1/2}$ for the case $\theta=\theta_o$.

The main differences between the modified and ordinary two-stream instabilities are that

1) the instability threshold is $V\gtrsim v_{ti}$ rather than $V>v_{te}$

2) the electrons behave as if they have an effective mass much larger than m_e.

Unlike many other electron-ion instabilities (e.g.ion acoustic) that mainly heat the electrons the modified 2-stream instability results in comparable electron (parallel to $\underline{B}_o$) and ion (perpendicular to $\underline{B}_o$) heating. [8] Moreover the instability is insensitive to the ratio T_e/T_i, unlike the ion acoustic instability and may therefore operate in regimes where the ion acoustic instability is inoperative. We will show in Chapter 4 how the growth rates calculated here may give some estimate of the saturated energy level needed to calculate an effective "collision" frequency. A comparison of the relative importance of the two instabilities in the anomalous resistivity of a collisionless shock wave can then be made.

Chapter 4

Nonlinear Effects

A weakly nonlinear theory (i.e. nonlinear theory treated by perturbation methods) is now presented in order to provide a basis for the explanation of plasma phenomena beyond the scope of linear theory. Examples of such nonlinear properties are as follows:

a) Linear theory predicts growth/damping of amplitudes of plasma waves and the corresponding change in wave energy must be balanced by a change in energy of the particles. Energy conservation theorems are nonlinear since wave energy is proportional to E^2.

b) In linear theory the change in the distribution function due to the growth of the waves is given by

$$f_s = f_{so} + \int \exp(i\underline{k}.\underline{x}) f_{sk} d\underline{k}$$

Each Fourier component vanishes on averaging so changes in the average plasma properties (e.g. temperature) only appear in nonlinear theory.

c) Waves of finite amplitude exhibit properties which depend on products of wave amplitudes and are necessarily nonlinear.

There are two cases for which weakly nonlinear theory is tractable

a) Where there are only a few waves of finite amplitude it is possible to treat each wave individually - the *theory of weak coherent waves*. [1,13]

b) Where there are so many waves present that a statistical approach is needed to determine the features that do not depend on the initial phase of the waves - the *theory of weak turbulence* or *quasilinear theory*.

Both theories fail when the wave amplitudes become so large that

a) the perturbation series fails to converge.

b) the particle orbits become so distorted by the wave fields that $f_s \approx f_{so}$ can no longer be used to calculate the *linear* wave properties. (e.g. particle trapping - see 4.5)

In quasilinear theory the finite wave amplitudes are considered to be small enough that the wave propogation can be treated by linear theory but nonlinear theory is needed to determine the long term effect of many waves on the background distribution function f_o. (N.B. The term quasilinear is not to be confused with the usual mathematical definition associated with partial differential equations. [18])

4.1 Quasilinear Theory for the General Equilibrium State.

We use the Vlasov-Maxwell system for a collisionless plasma. By collisionless we mean one in which transport properties are dominated by collective interactions (instabilities) rather than by short range binary collisions. We follow the work done by Liewer and Krall [7].

We can not write $f_s = f_{so}+f_{s1}$, where f_{so} is the initially spacially averaged or ensemble averaged distribution, since f_{s1} would then contain the difference between the final and the initial distributions. This difference may be as large as

f_{so} itself in the vicinity of the instability and would cause the expansion to break down.

Instead, we write

$$f_s(\underline{x},\underline{v},t) = f_{so}(\underline{x},\underline{v},t) + f_{s1}(\underline{x},\underline{v},t) \qquad 4.1.1a$$

$$\underline{E}(\underline{x},t) = \underline{E}_o(\underline{x},t) + \underline{E}_1(\underline{x},t) \qquad 4.1.1b$$

$$\underline{B}(\underline{x},t) = \underline{B}_o(\underline{x},t) \qquad 4.1.1c$$

where the suffix $_o$ denotes a smooth, average over the microscopic scale and $_1$ represents the turbulent oscillatory part. Note that we have assumed the waves to be electrostatic so $\underline{B}_1 = \underline{0}$.

We have $\quad f_{so} = \langle f_s \rangle \quad \underline{E}_o = \langle \underline{E} \rangle \quad \underline{B}_o = \langle \underline{B} \rangle$

and $\quad \langle f_{s1} \rangle = 0 \quad \langle \underline{E}_1 \rangle = \underline{0}$

where $\langle . \rangle$ is an average over some microscopic scale.

Substituting 4.1.1 in the Vlasov equation and averaging over the microscopic scale ⇒

$$\left(\frac{\partial}{\partial t} + \underline{v}.\nabla + \frac{q_s}{m_s}(\underline{E}_o + \underline{v}_\wedge \underline{B}_o).\nabla_{\underline{v}}\right) f_{so} = -\frac{q_s}{m_s}\left\langle \underline{E}_1 . \nabla_{\underline{v}} f_{s1} \right\rangle \qquad 4.1.2$$

which shows the development of the macroscopic distribution function f_{so} due to the turbulent term on the R.H.S. The R.H.S. acts as a "collision" term since, as we shall see, it allows momentum and energy transfer between particles of different species. This nonlinear R.H.S. is the only nonlinear part of quasilinear theory, the rest is completely linear.

We now take velocity moments of 4.1.2.

Integrating 4.1.2 over $\underline{v}$ ⇒

$$\int \frac{\partial f_{so}}{\partial t} d\underline{v} + \int \underline{v}.\nabla f_{so} d\underline{v} + \frac{q_s}{m_s}\int (\underline{E}_o + \underline{v}_\wedge \underline{B}_o).\nabla_{\underline{v}} f_{so} d\underline{v} = \frac{-q_s}{m_s}\int \langle \underline{E}_1 . \nabla_{\underline{v}} f_{s1} \rangle d\underline{v} \qquad 4.1.3$$

Now,

$$\int \frac{\partial f_{so}}{\partial t} d\underline{v} = \frac{\partial}{\partial t}\int f_{so} d\underline{v} = \frac{1}{N_s}\frac{\partial n_s}{\partial t} \qquad 4.1.4$$

where we have defined $\quad n_s = N_s \int f_{so} d\underline{v}$

Also,

$$\int \underline{v}.\nabla f_{so} d\underline{v} = \nabla.\int \underline{v} f_{so} d\underline{v} \quad \text{(since } \nabla \text{ independent of } \underline{v}\text{)}$$

But,

$$\int \underline{v} f_{so} d\underline{v} = \frac{n_s}{N_s}\underline{u}_s$$

where we have defined $\underline{u}_s = \frac{N_s}{n_s}\int \underline{v} f_{so} d\underline{v}$

So,

$$\int \underline{v}.\nabla f_{so} d\underline{v} = \frac{1}{N_s}\nabla.(n_s \underline{u}_s) \qquad 4.1.5$$

Now,

$$\int \langle \underline{E}_1.\nabla_{\underline{v}} f_{s1}\rangle dv = \left\langle \int \underline{E}_1.\nabla_{\underline{v}} f_{s1} d\underline{v} \right\rangle$$

and

$$\begin{aligned}\int \underline{E}_1.\nabla_{\underline{v}} f_{s1} d\underline{v} &= \int \nabla_{\underline{v}}.(\underline{E}_1 f_{s1}) d\underline{v} \quad \text{(since } \underline{E} \text{ independent of } \underline{v}\text{)} \\ &= \int_{S_\infty} f_{s1}\underline{E}_1.d\underline{S}\end{aligned}$$

where S_∞ is a surface at ∞ in $\underline{v}$-space.

This vanishes provided $f \to 0$ faster than $1/v^2$ as $v \to \infty$ which is indeed true for any distribution with finite energy since energy is proportional to $\int v^2 f dv$.

i.e. $$\int \langle \underline{E}_1.\nabla_{\underline{v}} f_{s1}\rangle d\underline{v} = 0 \qquad 4.1.6$$

Similarly,

$$\int \underline{E}_o.\nabla_{\underline{v}} f_{so} d\underline{v} = 0 \qquad 4.1.7$$

Finally,

$$\begin{aligned}\int \underline{v}_\wedge \underline{B}_o.\nabla_{\underline{v}} f_{so} d\underline{v} &= \int \nabla_{\underline{v}}.(f_{so}\underline{v}_\wedge \underline{B}_o) d\underline{v} - \int f_{so}\nabla_{\underline{v}}.(\underline{v}_\wedge \underline{B}_o) d\underline{v} \\ &= \int_{S_\infty} f_{s1}\underline{v}_\wedge \underline{B}_o.d\underline{S} - \int f_{so}\nabla_{\underline{v}}.(\underline{v}_\wedge \underline{B}_o) d\underline{v} \\ &= 0 \qquad 4.1.8\end{aligned}$$

since the first integral vanishes by the above argument and the second integral vanishes because $\nabla_{\underline{v}}.\underline{v}_\wedge\underline{B}_o = \underline{0}$

Altogether the results 4.1.4,5,6,7,8 ⇒ 4.1.3 becomes the continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla.n_s\underline{u}_s = 0 \qquad 4.1.9$$

Now multiply 4.1.2 by $m_s\underline{v}$ and integrate over $\underline{v}$ to obtain

$$\underbrace{m_s\int \underline{v}\frac{\partial f_{so}}{\partial t}d\underline{v}}_{(a)} + \underbrace{m_s\int \underline{v}(\underline{v}.\nabla)f_{so}d\underline{v}}_{(b)} + \underbrace{q_s\int \underline{v}(\underline{E}_o+\underline{v}_\wedge\underline{B}_o).\nabla_{\underline{v}}f_{so}d\underline{v}}_{(c)} =$$

$$\underbrace{-q_s\int \underline{v}\langle\underline{E}_1.\nabla_{\underline{v}}f_{s1}\rangle d\underline{v}}_{(d)} \qquad 4.1.10$$

Term (a) is

$$m_s\int \underline{v}\frac{\partial f_{so}}{\partial t}d\underline{v} = \frac{m_s}{N_s}\frac{\partial}{\partial t}(n_s\underline{u}_s) \qquad 4.1.11$$

by the same argument as above.

Term (b) is

$$m_s\int \underline{v}(\underline{v}.\nabla f_{so})d\underline{v} = m_s\int \nabla.(f_{so}\underline{v}\underline{v})d\underline{v}$$

We use the identity (see appendix 4)

$$\nabla.(\underline{a}\underline{b}) = \underline{a}.(\nabla\underline{b}) + \underline{b}\nabla.\underline{a} \qquad \forall \text{ vectors } \underline{a},\underline{b}$$

to see that

$$\nabla.(f_{so}\underline{v}\underline{v}) = f_{so}\underline{v}.\nabla\underline{v} + \underline{v}\nabla.(f_{so}\underline{v})$$

The first term vanishes since $\underline{v}$ is independent of $\underline{x}$

Thus, $$\nabla.(f_{so}\underline{v}\underline{v}) = \underline{v}(f_{so}\nabla.\underline{v} + \underline{v}.\nabla f_{so})$$

$$= \underline{v}(\underline{v}.\nabla)f_{so}$$

So,

$$m_s\int \underline{v}(\underline{v}.\nabla f_{so})d\underline{v} = m_s\nabla.\int f_{so}\underline{v}\underline{v}d\underline{v}$$

Now,

$$m_sN_s\int f_{so}\underline{v}\underline{v}d\underline{v} = m_sN_s\left\{\int f_{so}(\underline{v}-\underline{u}_s)(\underline{v}-\underline{u}_s)d\underline{v} + 2\underline{u}_s\int \underline{v}f_{so}d\underline{v} - \underline{u}_s\underline{u}_s\int f_{so}d\underline{v}\right\}$$

$$= \underline{\underline{P}} + m_sn_s\underline{u}_s\underline{u}_s$$

But,

$$\nabla.(n_s\underline{u}_s\underline{u}_s) = n_s\underline{u}_s(\nabla\underline{u}_s) + \underline{u}_s\nabla.(n_s\underline{u}_s)$$

So we have

$$m_sN_s\int\underline{v}(\underline{v}.\nabla)f_{so}d\underline{v} = \nabla.\underline{\underline{P}} + m_sn_s\underline{u}_s(\nabla\underline{u}_s) + m_s\underline{u}_s\nabla.(n_s\underline{u}_s) \qquad 4.1.12$$

Two of the terms in 4.1.11 and 4.1.12 add to give

$$\frac{m_s}{N_s}\left\{\frac{\partial}{\partial t}(n_s\underline{u}_s) + \nabla.(n_s\underline{u}_s\underline{u}_s)\right\} = \frac{m_s}{N_s}\left\{\frac{\partial}{\partial t}(n_s\underline{u}_s) + n_s\underline{u}_s.(\nabla\underline{u}_s) + \underline{u}_s\nabla.(n_s\underline{u}_s)\right\}$$

$$= \frac{m_s}{N_s}\left\{\underline{u}_s\left(\frac{\partial n_s}{\partial t} + \nabla.n_s\underline{u}_s\right) + n_s\left(\frac{\partial u_s}{\partial t} + (\underline{u}_s.\nabla)\underline{u}_s\right)\right\}$$

$$= \frac{m_s}{N_s}n_s\frac{D}{Dt}(\underline{u}_s) \qquad 4.1.13$$

using the continuity equation 4.1.9. [N.B. $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\underline{u}_s.\nabla)$]

Consider the identity

$$\nabla_{\underline{v}}.(f\underline{F}\underline{v}) = f\underline{F}.(\nabla_{\underline{v}}\underline{v}) + \underline{v}(\nabla_{\underline{v}}.f\underline{F})$$

$$= f\underline{F}.\underline{\underline{I}} + \underline{v}(f\nabla_{\underline{v}}.\underline{F} + \underline{F}.\nabla_{\underline{v}}f)$$

where $\underline{\underline{I}}$ is the unit tensor, $\underline{F}$ is any vector function and where f is any scalar function.

But, $\nabla_{\underline{v}}.\underline{F} = \underline{0}$

for F = $\underline{E}_o + \underline{v}_\wedge\underline{B}_o$ since $\underline{E}_o,\underline{B}_o$ are independent of $\underline{v}$.

So term (c) is

$$q_s\int\underline{v}\underline{F}.\nabla_{\underline{v}}f_{so}d\underline{v} = q_s\left\{\int\nabla_{\underline{v}}.(f_{so}\underline{F}\underline{v})d\underline{v} - \int f\underline{F}d\underline{v}\right\}$$

The first integral on the R.H.S. vanishes by the same argument involving S_∞ already used.

So,

$$q_s\int\underline{v}\underline{F}.\nabla_{\underline{v}}f_{so}d\underline{v} = -q_s\int f_{so}(\underline{E}_o+\underline{v}_\wedge\underline{B}_o)d\underline{v}$$

$$= -q_s\underline{E}_o\int f_{so}d\underline{v} + q_s\underline{B}_{o\wedge}\int f_{so}\underline{v}d\underline{v}$$

$$= -\frac{q_s}{N_s}n_s(\underline{E}_o+\underline{u}_{s\wedge}\underline{B}_o) \qquad 4.1.14$$

Term (d) is

$$q_s\int\underline{v}\langle\underline{E}_1.\nabla_{\underline{v}}f_{s1}\rangle d\underline{v} = q_s\left\langle\int\underline{v}(\underline{E}_1.\nabla_{\underline{v}}f_{s1})d\underline{v}\right\rangle$$

$$= -q_s\left\langle\int\underline{E}_1f_{s1}d\underline{v}\right\rangle \qquad \text{by a similar argument.}$$

$$= \frac{-q_s}{N_s}\left\langle n_{s1}\underline{E}_1\right\rangle \qquad 4.1.15$$

where $n_{s1} = N_s\int f_{s1}d\underline{v}$

Altogether 4.1.12,13,14,15 ⇒ 4.1.10 becomes

$$m_s n_s\left(\frac{\partial}{\partial t} + (\underline{u}_s.\nabla)\right)\underline{u}_s = q_s n_s(\underline{E}_o+\underline{u}_{s\wedge}\underline{B}_o) - \nabla.\underline{\underline{P}}_s + q_s\langle n_{s1}\underline{E}_1\rangle \qquad 4.1.16$$

This is essentially the momentum equation of Chapter 1 but with the anomalous term $q_s\langle n_{s1}\underline{E}_1\rangle$ (the so called anomalous resistivity) allowing momentum exchange between ordered particle motions and fluctuating fields and also allowing momentum exchange between species.

(N.B.Multiplying 4.1.2 by $m_s\underline{u}_s{}^2/2$ and integrating over $\underline{v}$ will yield an heat equation with an anomalous heating term. [7])

We continue to follow Liewer and Krall in order to obtain an expression for the anomalous resistivity. Subtracting 4.1.2 from the Vlasov equation and dropping second order terms ⇒

$$\left(\frac{\partial}{\partial t} + \underline{v}.\nabla + \frac{q_s}{m_s}(\underline{E}_o+\underline{v}_{\wedge}\underline{B}_o).\nabla_{\underline{v}}\right)f_{s1} = -\frac{q_s}{m_s}\underline{E}_1.\nabla_{\underline{v}}f_{so} \qquad 4.1.17$$

More generally, the right hand side of 4.1.17 would be $-\frac{q_s}{m_s}(\underline{E}_1+\underline{v}_{\wedge}\underline{B}_1).\nabla_{\underline{v}}f_{so}$ and we now illustrate the procedure to solve this equation by the method of characteristics, often referred to as "integrating along unperturbed orbits" in the plasma physics literature.

Define $$\frac{d}{dt'}(\underline{x}') = \underline{v}' \qquad 4.1.18a$$

and $$\frac{d}{dt'}(\underline{v}') = \frac{q_s}{m_s}\left(\underline{E}_o(\underline{x}',t')+\underline{v}'_{\wedge}\underline{B}_o(\underline{x}',t')\right) \qquad 4.1.18b$$

with boundary conditions

$$\underline{x}'(t'=t) = \underline{x} \qquad 4.1.19a$$

$$\underline{v}'(t'=t) = \underline{v}(t) \qquad 4.1.19b$$

The solution curves to the ordinary differential equations

4.1.18 are called characteristics and t' is the parameter along the characteristics. We solve 4.1.17 for $f_{s1}(\underline{x},\underline{v},t)$ by considering the function $f_{s1}(\underline{x}'(t'),\underline{v}'(t'),t')$ which is a function of time t' and satisfies the equation

$$\frac{Df_{s1}}{Dt'} \equiv \left(\frac{\partial}{\partial t'} + \frac{dx'}{dt'}\cdot\nabla_{x'} + \frac{dv'}{dt'}\cdot\nabla_{v'}\right)f_{s1}(\underline{x}',\underline{v}',t')$$

$$= \frac{-q_s}{m_s}\left(\underline{E}_1(\underline{x}',t')+\underline{v}'{}_\wedge\underline{B}_1(\underline{x}',t')\right)\cdot\nabla_{\underline{v}'}f_{so}(\underline{x}',\underline{v}') \qquad 4.1.20$$

which is valid along the characteristics.

Now, because of boundary conditions 4.1.19

$$f_{s1}(\underline{x}',\underline{v}',t') = f_{s1}(\underline{x},\underline{v},t) \text{ at time } t' = t$$

so the solution to 4.1.20 is also a solution to 4.1.17 at time t' = t.

The L.H.S. of 4.1.20 is just the exact differential $\frac{Df_{s1}}{Dt'}$.

Integrating 4.1.20 from t' = $-\infty$ to t' = t $\Rightarrow$

$$f_{s1}(\underline{x},\underline{v},t) = -\frac{q_s}{m_s}\int_{-\infty}^{t}\left(\underline{E}_1(\underline{x}',t')+\underline{v}'{}_\wedge\underline{B}_1(\underline{x}',t')\right)\cdot\nabla_{\underline{v}'}f_{so}(\underline{x}',\underline{v}')dt'$$

$$+ f_{s1}(\underline{x}'(-\infty),\underline{v}'(-\infty),t'=-\infty) \qquad 4.1.21$$

Assume

$$f_{s1}(\underline{x},\underline{v},t) = f_{sk}(\underline{v})\exp i(\underline{k}.\underline{x}-\omega t) \qquad 4.1.22a$$

$$\underline{E}_1(\underline{x},t) = \underline{\overline{E}}_k\exp i(\underline{k}.\underline{x}-\omega t) \qquad 4.1.22b$$

$$\underline{B}_1(\underline{x},t) = \underline{\overline{B}}_k\exp i(\underline{k}.\underline{x}-\omega t) \qquad 4.1.22c$$

where $\underline{\overline{E}}_k,\underline{\overline{B}}_k$ are constant vectors.

Then 4.1.21 $\Rightarrow$

$$f_{sk}(\underline{v})\exp i(\underline{k}.\underline{x}-\omega t) = f_{sk}(\underline{v}')\exp i(\underline{k}.\underline{x}'-\omega t')\Big|_{t'=-\infty}$$

$$- \frac{q_s}{m_s}\int(\underline{\overline{E}}_k+\underline{v}'{}_\wedge\underline{\overline{B}}_k)\exp i(\underline{k}.\underline{x}'-\omega t')\cdot\nabla_{\underline{v}'}f_{so}(\underline{x}',\underline{v}')dt'$$

We assume that the first term on the right hand side of this equation vanishes (i.e. no perturbation at t' = $-\infty$).

Define $\qquad \tau = t'-t \quad$ and $\quad \underline{X} = \underline{x}'-\underline{x}$

so that $\qquad d\tau = dt'$

So,

$$f_{sk}(\underline{v})\exp i(\underline{k}.\underline{x}-\omega t) =$$

$$-\frac{q_s}{m_s}\exp i(\underline{k}.\underline{x}-\omega t)\int_{-\infty}^{0}(\overline{E}_k+\underline{v}'_\wedge\overline{B}_k)\exp i(\underline{k}.\underline{X}-\omega\tau).\nabla_{\underline{v}}\cdot f_{so}(\underline{x}',\underline{v}')d\tau$$

This is because $\underline{x},t$ are independent of the integration variable and may be taken outside the integral.

Thus,

$$f_{sk}(\underline{v}) = -\frac{q_s}{m_s}\int_{-\infty}^{0}(\overline{E}_k+\underline{v}'_\wedge\overline{B}_k)\exp i(\underline{k}.\underline{X}-\omega\tau).\nabla_{\underline{v}}\cdot f_{so}(\underline{x}',\underline{v}')d\tau \qquad 4.1.23$$

This holds for $\omega_i > 0$, analytic continuation being used for $\omega_i < 0$.

The variables $\underline{v}',\underline{x}',\underline{X}$ can all be expressed in terms of τ by finding the unperturbed orbits from 4.1.18, with boundary conditions 4.1.19, for specified external field configurations $\underline{E}_o,\underline{B}_o$. Incidentally, the fact that the characteristics are the trajectories under the action of the unperturbed fields explains the phrase "integrating along unperturbed orbits" with reference to integration of 4.1.20.

The equilibrium distribution $f_{so}(\underline{x},\underline{v})$ satisfies

$$\left(\underline{v}.\nabla + \frac{q_s}{m_s}(\underline{E}_o+\underline{v}_\wedge\underline{B}_o).\nabla_{\underline{v}}\right)f_{so} = 0 \qquad 4.1.24$$

A way to generate solutions for $f_{so}(\underline{x},\underline{v})$ is as follows:

Suppose

$$a_1(\underline{x}'(t'),\underline{v}'(t')),a_2(\underline{x}'(t'),\underline{v}'(t')),\ldots$$

are constants of the motion (for example, energy or components of momentum) of the particle under the action of the equilibrium fields $\underline{E}_o,\underline{B}_o$. [N.B. boundary conditions 4.1.19 still apply.]

Then,

$$\frac{da_1}{dt'} = \underline{v}'.\nabla_{\underline{x}}\cdot a_1 + \frac{d\underline{v}'}{dt}.\nabla_{\underline{v}}\cdot a_1 = 0$$

$$\frac{da_2}{dt} = \underline{v}'.\nabla_{\underline{x}}\cdot a_2 + \frac{d\underline{v}'}{dt}.\nabla_{\underline{v}}\cdot a_2 = 0$$

so any function

$$f_{so} = f_{so}(a_1(\underline{x}',\underline{v}'),a_2(\underline{x}',\underline{v}'),\ldots)$$

satisfies 4.1.24 at time t. This is easily verified by substituting f_{so} into 4.1.24 and using 4.1.19.

i.e. $f_{so} = f_{so}(a_1(\underline{x},\underline{v}),a_2(\underline{x},\underline{v}),\ldots)$ 4.1.25

is a stationary state solution of the Vlasov equation.

For cases where

$$\rho = \sum_s q_s N_s \int f_{so} d\underline{v} = 0 \text{ (no net charge in the plasma)}$$

$$\underline{j} = \sum_s q_s N_s \int \underline{v} f_{so} d\underline{v} = 0 \text{ (no net current in the plasma)}$$

in the equilibrium state, f_{so} is controlled by the external fields and the constants of the motion for a particle in the fields $\underline{E}_o,\underline{B}_o$ are used to construct stationary states.

Altogether we have demonstrated that 4.1.23 (with 4.1.22a) provides us with a solution for $f_{s1}(\underline{x},\underline{v},t)$.

Note: It is worthwhile pointing out that the order ϵ Maxwell equations

$$\nabla_\wedge \underline{E}_1 = -\frac{\partial(\underline{B}_1)}{\partial t}$$

$$\epsilon_o \nabla.\underline{E}_1 = \sum_s q_s N_s \int f_{s1} d\underline{v}$$

$$\frac{1}{\mu_o}\nabla_\wedge \underline{B}_1 = \sum_s q_s N_s \int \underline{v} f_{s1} d\underline{v} + \epsilon_o \frac{\partial(\underline{E}_1)}{\partial t}$$

when transformed will, upon substituting 4.1.23, result in 6 linear equations for $\overline{E}_k,\overline{B}_k$. We can then eliminate $\overline{B}_k$ in terms of $\overline{E}_k$ to give an equation $\underline{\underline{D}}.\underline{E} = 0$ where D_{ij} (i,j = 1,2,3) is the dispersion tensor. The equation has solutions only if $\det(D) = 0$ which is the dispersion relation.

However, we restrict ourselves to the electrostatic assumption ($\underline{\overline{B}}_k = 0$) so that we only require Poisson's equation (the rest of Maxwell's equations become redundant) in order to obtain our dispersion relation.

Poisson's equation is

$$i\epsilon_o\underline{k}.\underline{E}_1 = \sum_s q_s N_s \int f_{s1} d\underline{v} \qquad 4.1.26$$

Using 4.1.22, 4.1.23 this states

$$i\epsilon_o\underline{k}.\underline{\overline{E}}_k = -\sum_s \frac{q_s^2 N_s}{m_s} \int\int_{-\infty}^{0} \underline{\overline{E}}_k \exp i(\underline{k}.\underline{X}-\omega\tau).\nabla_{\underline{v}}\cdot f_{so}(\underline{x}',\underline{v}')d\tau d\underline{v}$$

Now, $\underline{\overline{E}}_k$ is parallel to $\underline{k}$ so

$$k^2\underline{\overline{E}}_k = (\underline{k}.\underline{\overline{E}}_k)\underline{k}$$

and we then have the dispersion relation

$$D(\underline{k},\omega) = 1 - \sum_s \epsilon_s(\underline{k},\omega) = 0 \qquad 4.1.27a$$

where $$\epsilon_s = \frac{i}{\epsilon_o}\sum_s \frac{q_s^2 N_s}{m_s k^2} \int\int_{-\infty}^{0} \underline{k} \exp i(\underline{k}.\underline{X}-\omega\tau).\nabla_{\underline{v}}\cdot f_{so}(\underline{x}',\underline{v}')d\tau d\underline{v} \qquad 4.1.27b$$

is the contribution of species $_s$ to the plasma dialectric and is often called the plasma susceptibility.

Now,

$$n_{s1}(\underline{x},t) = N_s\int f_{s1}d\underline{v}$$

by definition and n_{sk} is defined by

$$n_{s1}(\underline{x},t) = n_{sk}\exp i(\underline{k}.\underline{x}-\omega t)$$

Using 4.1.23 and 4.1.27b this becomes

$$n_{sk} = \frac{i\epsilon_o}{q_s}(\underline{k}.\underline{\overline{E}}_k)\epsilon_s \qquad 4.1.28$$

[N.B. This is equation 9 in Liewer and Krall]

We are now in a position to find an expression for the anomalous term

$$q_s\langle \underline{E}_1 n_{s1}\rangle = \frac{q_s}{V}\int_V n_{s1}\underline{E}_1 d\underline{x}$$

where $\int_V d\underline{x}$ means integrate over the microscopic scale. (Typically, the volume V is the Debye sphere.)

So,

$$\begin{aligned} q_s\langle \underline{E}_1 n_{s1}\rangle &= \frac{q_s}{V}\int\left(\frac{1}{(2\pi)^3}\int \underline{E}_1(\underline{m})\exp i\underline{m}.\underline{x}d\underline{m}\right)\left(\frac{1}{(2\pi)^3}\int n_1(\underline{k})\exp i\underline{k}.\underline{x}d\underline{k}\right)d\underline{x} \\ &= \frac{q_s}{V(2\pi)^6}\int\left(\int\int \underline{E}_1(\underline{m})n_1(\underline{k})\exp(i(\underline{k}+\underline{m}).\underline{x})d\underline{m}d\underline{k}\right)d\underline{x} \\ &= \frac{q_s}{V(2\pi)^6}\int\int \underline{E}_1(\underline{m},t)n_1(\underline{k},t)\left(\int \exp(i(\underline{k}+\underline{m}).\underline{x})d\underline{x}\right)d\underline{m}d\underline{k} \end{aligned}$$

But,

$$\lim_{l\to\infty}\frac{\sin la}{\pi a} = \delta(a) \quad \text{which} \Rightarrow \quad \int_{-\infty}^{\infty}\exp(iax)dx = 2\pi\delta(a)$$

So

$$\begin{aligned} \int \exp i\underline{K}.\underline{x}d\underline{x} &= \int \exp iK_x dx\int \exp iK_y dy\int \exp iK_z dz \\ &= (2\pi)^3\delta(K_1)\delta(K_2)\delta(K_3) \\ &\equiv (2\pi)^3\delta(\underline{K}) \qquad \forall \text{ vectors } \underline{K} = (K_x,K_y,K_z) \end{aligned}$$

Thus,

$$\begin{aligned} q_s\langle \underline{E}_1 n_{s1}\rangle &= \frac{q_s}{V(2\pi)^3}\int\left(\int \underline{E}_1(\underline{m},t)\delta(\underline{m}+\underline{k})d\underline{m}\right)n_1(\underline{k},t)d\underline{k} \\ &= \frac{q_s}{V(2\pi)^3}\int \underline{E}_1(-\underline{k},t)n_1(\underline{k},t)d\underline{k} \qquad\qquad 4.1.29 \\ &= \frac{i\epsilon_o}{V(2\pi)^3}\int \underline{E}_1(-\underline{k},t)\epsilon_s(\underline{k}\cdot\underline{E}_k)d\underline{k} \end{aligned}$$

using 4.1.28.

Now,

$$(\underline{k}\cdot\underline{E}_1(\underline{k},t))\underline{E}_1(-\underline{k},t) = |\underline{E}_1(-\underline{k},t)||\underline{E}_1(\underline{k},t)|\underline{k}$$

since $\underline{E}_1$ is parallel to $\underline{k}$ for electrostatic waves.

But,

$$\underline{E}_1(\underline{k},t) = \underline{E}_1^*(-\underline{k},t)$$

from Appendix 3.

So we have

$$(\underline{k}.\underline{E}_1(\underline{k},t))\underline{E}_1(-\underline{k},t) = |\underline{E}_1(\underline{k},t)^2|\underline{k}$$

Thus,

$$q_s\langle\underline{E}_1 n_{s1}\rangle = \frac{i\epsilon_o}{V(2\pi)^3}\int|\underline{E}_1(\underline{k},t)|^2\epsilon_s(\underline{k},\omega)\underline{k}d\underline{k}$$

Defining the spectral energy density by

$$\zeta(\underline{k},t) = \frac{\epsilon_o}{2V(2\pi)^3}\,\underline{E}_1(\underline{k},t).\underline{E}_1(-\underline{k},t) \qquad 4.1.30$$

so that the electrostatic energy is just

$$\left\langle\frac{\epsilon_o E^2}{2}\right\rangle = \int\zeta(\underline{k},t)d\underline{k}$$

implies

$$q_s\langle\underline{E}_1 n_{s1}\rangle = -2\int\zeta(\underline{k},t)\epsilon_{si}(\underline{k},\omega)\underline{k}d\underline{k} \qquad 4.1.31$$

[N.B. The imaginary part ϵ_{si} appears because the L.H.S. must be real.]

4.2 The Effective Collision Frequency

Thus, with a knowledge of ϵ_s, the plasma susceptibility, and a knowledge of ζ, the electrostatic energy density, we can calculate the anomalous resistivity term $q_s\langle\underline{E}_1 n_{s1}\rangle$. This enables us to model the wave-particle interactions by defining an effective "collision" frequency, ν_s, via

$$-\nu_s = \frac{q_s}{n_s V_d^2}\underline{V}_d.\langle\underline{E}_1 n_{s1}\rangle \qquad 4.2.1$$

where $\underline{V}_d$ is the drift velocity between the ions and electrons. This can also be written, using 4.1.15

$$\nu_s = \frac{q_s N_s}{n_s V_d^2}\underline{V}_d.\int\underline{v}\langle\underline{E}_1.\nabla_{\underline{v}}f_{s1}\rangle d\underline{v}$$

This is from a dimensional argument since the momentum equation is now

$$m_s\left(\frac{\partial}{\partial t} + (\underline{u}_s.\nabla)\right)\underline{u}_s = q_s(\underline{E}_o+\underline{u}_s{}_\wedge\underline{B}_o) - \frac{1}{n_s}\nabla.\underline{\underline{P}} - \nu_s\underline{V}_d \qquad 4.2.2$$

(i.e. $\nu_s\underline{V}_d$ acts as a damping term.)

The application of this collision frequency is discussed in Chapter 5 on "collisionless" shock waves.

As stated previously, the only part of quasilinear theory that is nonlinear is the term $\langle E_1 n_{s1}\rangle$; the rest is purely linear. Consequently, we use linear theory, namely 2.1.13, to find $\epsilon_s(\underline{k},\omega)$. In fact assuming $\omega_i \ll \omega_r$, we have from 2.3.3a (remember $D = 1 - \sum_s \epsilon_s$)

$$\epsilon_{si}(\underline{k},\omega) = \frac{\pi\omega_{ps}^2}{k^2}\frac{\partial f_{so}}{\partial u}\bigg|_{u=\omega_r/|\underline{k}|} \qquad 4.2.3$$

To find ζ recall

$$\zeta(\underline{k},t) = \frac{\epsilon_o}{2V(2\pi)^3}\,\underline{E}_1(\underline{k},t).\underline{E}_1(-\underline{k},t)$$

where $\qquad \underline{E}_1(\underline{k},t) = \underline{\overline{E}}_k \exp i(\underline{k}.\underline{x}-\omega t)$

Now,

$$\underline{E}_1(\underline{k},t).\underline{E}_1(-\underline{k},t) = \underline{\overline{E}}_k\underline{\overline{E}}_{-k}\exp -i(\omega(\underline{k},t)+\omega(-\underline{k},t))t$$

But,

$$\omega(\underline{k},t) = -\omega^*(-\underline{k},t)$$

from Appendix 3.

Hence,

$$\underline{E}_1(\underline{k},t).\underline{E}_1(-\underline{k},t) = \underline{\overline{E}}_k\underline{\overline{E}}_{-k}\exp(2\omega_i(\underline{k},t)t)$$

Thus,

$$\begin{aligned}\frac{\partial\zeta}{\partial t}(\underline{k},t) &= \frac{\epsilon_o}{2V(2\pi)^3}\underline{\overline{E}}_k\underline{\overline{E}}_{-k}\frac{\partial}{\partial t}\Big(\exp(2\omega_i(\underline{k},t)t)\Big)\\ &= \frac{\epsilon_o}{2V(2\pi)^3}\underline{\overline{E}}_k\underline{\overline{E}}_{-k}\Big(2\omega_i(\underline{k},t) + 2t\frac{\partial\omega_i}{\partial t}(\underline{k},t)\Big)\exp(2\omega_i t)\\ &= 2\omega_i(\underline{k},t)\zeta(\underline{k},t)\end{aligned} \qquad 4.2.5$$

upon neglecting the second term, since the wave energy saturates to a constant level after a few periods. (See 4.4)

So,

$$\zeta(\underline{k},t) = \zeta(\underline{k},0)\exp\Big(2\int_o^t\omega_i(\underline{k},T)dT\Big) \qquad 4.2.6$$

upon integrating.

4.3 Quasilinear Diffusion

We now derive the equations determining the time development of the equilibrium distribution f_{so} for the more simple equilibrium state with $\underline{E}_o = 0 = \underline{B}_o$. Moreover we assume f_{so} is independent of $\underline{x}$ so 4.1.2 becomes

$$\frac{\partial f_{so}}{\partial t} = -\frac{q_s}{m_s}\langle \underline{E}_1 . \nabla_{\underline{v}} f_{s1}\rangle \qquad 4.3.1$$

A similar argument to that used in the derivation of 4.1.29 ⇒

$$\frac{\partial f_{so}}{\partial t} = -\frac{q_s}{m_s V(2\pi)^3}\nabla_{\underline{v}} . \int \underline{E}_1(-\underline{k},t) f_{s1}(\underline{k},t)d\underline{k}$$

But $$f_{s1}(\underline{k},t) = -\frac{q_s}{m_s i(\underline{k}.\underline{v}-\omega)}\underline{E}(\underline{k},t).\nabla_{\underline{v}} f_{so} \qquad 4.3.2$$

from *linear* theory.

[N.B. We only need linear theory here since R.H.S. is already second order.]

Hence,

$$\frac{\partial f_{so}}{\partial t} = \frac{q_s^2}{m_s^2 V(2\pi)^3}\nabla_{\underline{v}} . \int \frac{1}{i(\underline{k}.\underline{v}-\omega)}\underline{E}_1(-\underline{k},t)(\underline{E}_1(\underline{k},t).\nabla_{\underline{v}} f_{so})d\underline{k}$$

Using $\underline{E}(\underline{k}) = |\underline{E}(\underline{k})|\underline{k}/|\underline{k}|$ and 4.1.30 this becomes

$$\frac{\partial f_{so}}{\partial t} = \frac{2q_s^2}{\epsilon_o m_s^2}\int \zeta \underline{k}.\nabla_{\underline{v}}\left(\frac{1}{ik^2(\underline{k}.\underline{v}-\omega)}\underline{k}.\nabla_{\underline{v}} f_{so}\right)d\underline{k}$$

which may be written in the form of a diffusion equation

$$\frac{\partial f_{so}}{\partial t} = \nabla_{\underline{v}} . \underline{\underline{H}} . \nabla_{\underline{v}} f_{so} \qquad 4.3.2a$$

where $$\underline{\underline{H}} = \frac{2q_s^2}{\epsilon_o m_s^2}\int \frac{1}{ik^2(\underline{k}.\underline{v}-\omega)}\zeta(\underline{k},t)\underline{k}\underline{k}d\underline{k}$$

is the diffusion tensor.

This may be simplified since

$$\frac{-i}{(\omega-\underline{k}.\underline{v})}\zeta(\underline{k},t) = -i\left(\frac{\Omega_r - i\omega_i}{\Omega_r^2+\omega_i^2}\right)\zeta(\underline{k},t)$$

where we have put $\Omega_r = \omega_r - \underline{k}.\underline{v}$ for convenience.

Using

$$\omega_r(-\underline{k},t) = -\omega_r(\underline{k},t) \qquad \text{(See Appendix 3)}$$

and $$\zeta(\underline{k},t) = -\zeta(-\underline{k},t)$$

we see that the imaginary part is an odd function and so produces no contribution to the integral.

Thus,

$$\underline{\underline{H}} = \frac{2q_s^2}{\epsilon_o m_s^2}\int\frac{1}{k^2}\left(\frac{1}{(\omega_r-\underline{k}.\underline{v})^2+\omega_i^2}\right)\zeta(\underline{k},t)\omega_i(\underline{k},t)\underline{k}\underline{k}d\underline{k} \qquad 4.3.2b$$

Recall

$$\frac{\partial\zeta}{\partial t}(\underline{k},t) = 2\omega_i(\underline{k},t)\zeta(\underline{k},t) \qquad 4.3.2c$$

Equations 4.3.2a,b,c constitute the quasilinear diffusion equations. For small ω_i, the dominant contribution to $\underline{\underline{H}}$ is from the particles with velocity approximately equal to $\omega_r/|k|$, the so called resonant particles. See [9] for a special case of 4.3.2. In linear theory, the resonant particles cause damping of the waves and in quasilinear theory these particles are diffused (in velocity space) by the wave fields which is the essential nonlinear feedback mechanism of quasilinear theory.

Quasilinear theory can be shown to conserve particles, momentum and energy; unlike linear theory. [6] Indeed, an interesting account of the effects of quasilinear theory on Landau damping, instabilities, etc is to be found in [6].

4.4 Particle Trapping

We now consider situations where the resonant wave-particle interactions play a significant role in the nonlinear evolution of the plasma. Linear Landau damping has shown that there can be an effective energy exchange between particles and waves when the particle velocity is approximately equal to the phase velocity of the wave. The particles experience an electric field which is approximately constant in time.

However, in nonlinear theory, when the amplitude of the field is finite the particle orbits are modified considerably. Purely as an illustration consider the trajectory of an electron moving in an electric field

$$E(x,t) = E_o \sin(kx-\omega t) \qquad -\infty < x < \infty,\ E_o \text{ constant}$$

[N.B. The assumption E_o is constant means that we have neglected the fact that the field is calculated self-consistently. However, for our present purposes this will suffice to obtain an understanding of the relevant physics.]

The orbit $x(t)$ is given by

$$\frac{d^2x}{dt^2} = -\frac{e\bar{E}}{m_e}\sin(kx(t)-\omega t) \qquad 4.4.1$$

In the limit of $\bar{E}\to 0$ we have

$$x(t) = x_o+v_o t$$

where $\quad x_o = x(0)$ and $v_o = v(0)$

Define

$$x_1(t) = x(t)-\omega t/k$$

as the displacement of the electron relative to the wave frame.

Then

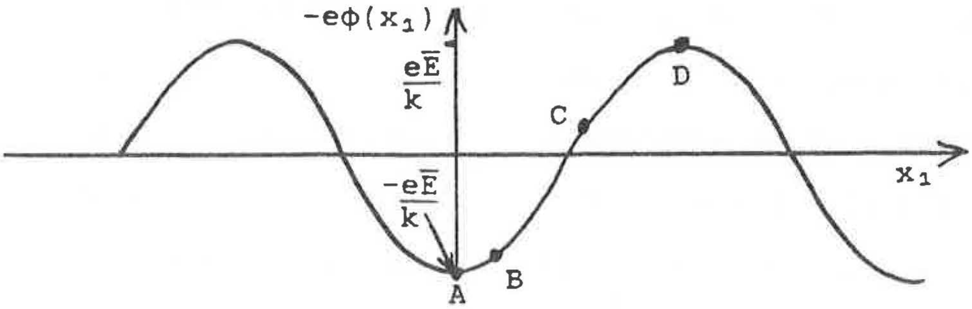
$$\frac{d^2x_1}{dt^2} + \frac{e\bar{E}}{m_e}\sin kx_1(t) = 0 \qquad 4.4.2$$

For small kx_1 this is just

$$\frac{d^2x_1}{dt^2} + \omega_B{}^2 x_1 = 0$$

where $\omega_B = \left(\frac{e\bar{E}k}{m_e}\right)^{1/2}$ is called the bounce frequency. It is the frequency of oscillation of electrons trapped near the bottom of the potential well of the finite amplitude wave (at B in Fig.7)

Fig.7



Integrating 4.4.2 ⇒

$$W = \frac{m_e}{2}\left(\frac{dx_1}{dt}\right)^2 - e\phi(x_1) = \text{constant}$$

is the total energy of the electron where $\phi(x_1) = \frac{\bar{E}}{k}\cos kx_1$.

If $-e\bar{E}/k < W < e\bar{E}/k$ the electron is trapped in the potential well, whilst if $W > |e\bar{E}/k|$ the electron accelerates and decelerates as it passes over the crests and troughs but remains untrapped. For $kx_1 \approx 0, \pm 2\pi, \ldots$ and $W \approx -e\bar{E}/k$ the particles exhibit simple harmonic motion with frequency ω_B.

We define characteristic Landau time scales of oscillation and damping respectively by

$$\tau_o = 1/|\omega_r| \quad \text{and} \quad \tau_d = 1/|\omega_i|$$

where ω_r, ω_i are calculated from linear theory.

Recall

$$\omega_i \approx -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_{pe}}{(|k|\lambda_D)^3} \exp\left(\frac{-1}{2k^2\lambda_D^2} - \frac{3}{2}\right)$$

from 2.2.2b for Langmuir waves.

So for $|k\lambda_D| << 1$ we see that $\tau_o << \tau_d$.

The fact that τ_d can be so long means that it is possible for

$$\tau_t \equiv 1/\omega_B << \tau_d \quad (\tau_t \text{ is the trapping time scale.})$$

so trapped particle effects must be taken into account as the wave is not sufficiently damped. Linear theory is only valid for $t \lesssim \tau_t$.

Note also that

$$\tau_o{}^2/\tau_t{}^2 \approx \frac{ek\bar{E}}{\omega_{pe}^2 m_e} = \frac{k\bar{E}}{4\pi n_o e} \approx \frac{n_{1m}}{n_o} << 1$$

where n_{1m} is the maximum perturbation in n.

(The approximation is from Poisson's equation.) This inequality means that the wave makes many oscillations in the time of one single particle oscillation in the well.

The question arises as to which particles are actually trapped. Consider particles with velocity v = 0 at t = 0 in the wave frame. The maximum velocity obtained by any of these particles will be the trapping speed, v_t, given by

$$m_e v_t^2/2 = 2|e\phi_m| \qquad (= \text{depth of the well})$$

where ϕ_m is the maximum value of ϕ.

Any particle with a velocity $> v_t$ will not be trapped.

Thus particles with velocities in the range

$$\omega/k - v_t < V < \omega/K + v_t$$

are trapped.

We might also consider the concept of particle trapping by considering particle orbits in the v-x phase plane.(Fig.8) The reader is referred to the standard text [17] on phase plane analysis. We have from 4.4.2 the equation

$$\frac{dy_1}{dt} + a\sin kx_1 = 0$$

where we have put $a = \frac{e\bar{E}}{m_e}$ and where we have defined

$$y_1 = \frac{dx_1}{dt}$$

Thus,

$$\frac{dy_1}{dx_1} = -\frac{a\sin(kx_1)}{y_1} \qquad 4.4.3$$

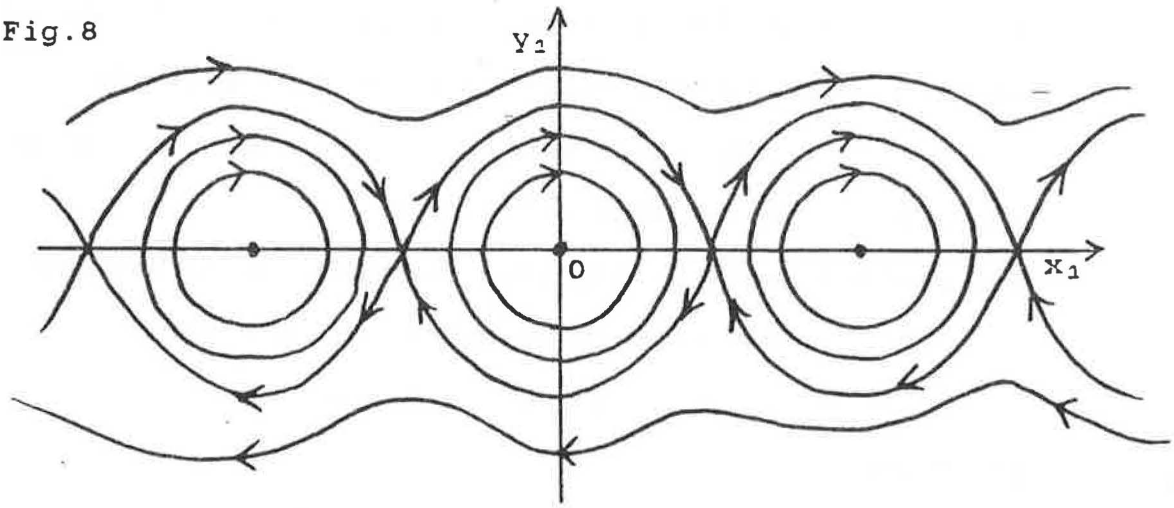
The equilibrium points are therefore given by

$$y_1 = 0, \quad kx_1 = n\pi \qquad (n \text{ integer})$$

The standard linear approximation method for phase planes [17] shows that for n even the equilibrium points are centres whilst for n odd they are saddle points. See Fig.8.

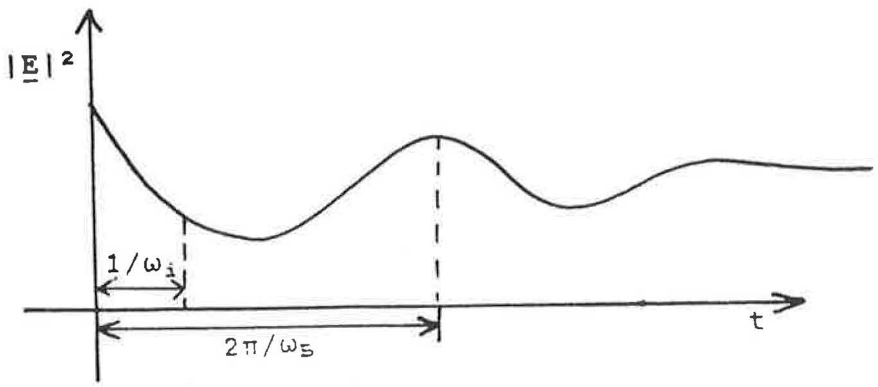
The contours indicated show particle trajectories for different initial conditions and therefore also represent contours of constant energy. The closed contours represent trapped particles as they are seen to represent oscillatory behaviour.

Fig.8



For the period up to $t = O(\tau_t)$ linear theory is valid and particles exchange energy with the wave according to Landau damping/growth. For $t > \tau_t$ the wave amplitude has risen to such a level that trapped particle effects must be considered. Moreover, neighbouring trapped particles have different bounce frequencies and therefore become out of phase. At $t \approx \pi/\omega_B$ the trapped particles have experienced half a bounce and may now put energy back into the wave but the reversal of energy will not be complete because the particles are no longer in phase. Thus, in reality we expect the wave amplitude to vary schematically as in Fig.9. The case shown here is for a damped oscillation but the same arguments apply to a growing wave which will eventually saturate to some higher energy level. (See 4.5)

Fig.9



It must be noted that equation 4.4.2 does not take account of any energy exchange between the particles and the wave since it is a conservative equation. It is merely used to illustrate the basic concepts involved and to derive an approximate time scale for trapping to occur. We shall assume in section 4.5 that the effects of particle trapping are sufficiently weak for quasilinear theory to be invoked. The energy exchange between the waves and particles can then be assumed to be due to the resonant particles oscillating in the troughs of the waves. The continuous growth/damping predicted by linear Landau theory is then modified by the fact that the particles become out of phase causing a saturation in the energy level. A more rigorous account of particle trapping is to be found in Davidson Chapter 4 and [10]. However it will suffice for our purposes to find an estimate of the saturated energy level.

4.5 Saturation Level

We now develop the above ideas to derive an estimate of the saturated energy level of the waves, needed in the calculation of ν_s. We follow the work done by McBride et al.[8]

The wave potential energy in the electron frame of reference required to commence trapping the ions is

$$2e\phi_i = m_i(V_d-\omega_r/k)^2/2 \qquad 4.5.1$$

where V_d is the relative drift velocity. McBride suggests the electrons trap on a similar time scale to the ions and has shown $\phi_i \approx \phi_e$. We may therefore use this result to find an estimate of the energy saturation level for either ions or electrons.

Define $\qquad W = \epsilon_o|\underline{E}|^2$

which is, using $\underline{E} = -\nabla\phi$ and 4.5.1,

$$W = \frac{\epsilon_o k^2 m_i^2}{16e^2}(V_d-\omega_r/k)^4 \qquad 4.5.2$$

where we take the fastest growing mode (since this causes trapping first) to find ω_r and k. These are found from linear theory (See Chapter 3).

The above estimate of the saturation level assumes that

a) the nonlinear effects are sufficiently weak for linear theory to be invoked to find ω_r/k in 4.5.2.

b) the initially fastest growing mode remains the fastest growing mode in the nonlinear situation.

Another approach might be to try to extend Davidson's [4] work on particle trapping to apply it to the cases of the ion acoustic and modified two stream instabilities. This could provide the basis for future research.

Chapter 5

Collisionless Shock Waves

In ordinary collision dominated gases the shock thickness occurs in a distance of the order of a few collision mean free paths. Since collisions are so frequent, velocity distributions on both sides of the shock are Maxwellian, and the various species of particles obtain the same temperature.

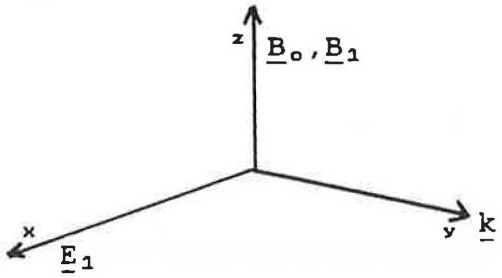
However, in a collision-free plasma energy and momentum can be transferred from particles to electromagnetic fields (see 4.1) by instabilities (see Chapter 3) so the conservation laws are now more complicated. Moreover, the ions and electrons are affected differently by these instabilities so there is no reason to believe their temperatures are equal. We shall be discussing physics that occurs on time-scales much shorter than classical (binary) collision times so we may use the Vlasov-Maxwell equations or the two-fluid model derived from these equations.

We shall be dealing with magnetosonic shocks. A magnetosonic wave is a low frequency perpendicular electromagnetic wave travelling across an equilibrium field B_o with

$$\omega^2 = \frac{k^2 V_A^2}{1+V_A^2/c^2}$$

where V_A is the Alfven speed. Fig.10 indicates the relative directions of vectors.

Fig.10



The drift of electrons and ions in the y direction are approximately equal but the ions have an extra component in the x direction due to the polarisation drift. This current in the x direction produces the fluctuating field $\underline{B}$ in the z direction. It is the current in the x direction that gives rise to the streaming instabilities (See Chapter 3).

We have shown (4.3) how these instabilities may give us an expression for an effective "collision" frequency ν_s and later in this chapter we shall demonstrate why this dissipation is necessary for shock solutions to exist. The value for ν_s may then be used to predict an estimate of the shock thickness, enabling satellite observations to test the theory.

5.1 Derivation of the Shock Profile Equations

We use the two-fluid model of Chapter 1 and follow the work done by Tidman and Krall. [12,5]

We seek constant profile waves in the wave frame so $\frac{\partial}{\partial t} \equiv 0$ and the magnetic field $\underline{B}$ is taken along the z axis. The electric field has x and y components. Specify

$$u_{sx}(-\infty) = \text{const} = u_1 \text{ for } s = e,i$$

(i.e. assume no external current)

All variables are assumed independent of y and z and we neglect the pressure term in the momentum equation. (The so called cold plasma approximation.)

1.1.2a reduces to $$u_{sx}\frac{du_s}{dx} = \frac{q_s}{m_s}(E_x+u_{sy}B, E_y-u_{sx}B, 0) \quad 5.1.1a$$

1.1.2b reduces to $\frac{d(n_s u_{sx})}{dx} = 0$ 5.1.1b

1.1.2d " $\epsilon_o \frac{dE_x}{dx} = \sum_s q_s n_s$ 5.1.1c

1.1.2f " $(0, -\frac{dB}{dx}, 0) = \mu_o \sum_s q_s n_s \underline{u}_s$ 5.1.1d

1.1.2g " $\frac{dE_y}{dx} = 0$ 5.1.1e

Integrate 5.1.1b ⇒

$$n_{sx} u_{sx} = \text{constant} = n_1 u_1$$

where $_1$ denotes the upstream value, not the perturbation.

So $$n_{sx} = \frac{n_1 u_1}{u_{sx}} \qquad 5.1.2$$

Integrate 5.1.1e ⇒

$$E_y = \text{constant}$$

Now the y component of 5.1.1a is

$$u_{sx} \frac{du_{sy}}{dx} = \frac{q_s}{m_s}(E_y - u_{sx} B)$$

Assume $\frac{du_{sy}}{dx} = 0$ at $x = -\infty$

so that $E_y = u_1 B_1$ at $-\infty$

Thus, $$E_y = u_1 B_1 \qquad 5.1.3$$

We assume quasineutrality

$$n_e \approx n_i = n, \text{ say.}$$

We already have

$$n_i u_{ix} = n_1 u_1 = n_e u_{ex}$$

So $$u_{ix} \approx u_{ex} = u_x, \text{ say} \qquad 5.1.4$$

m_e(y component of 5.1.1a for electrons)+m_i(y component of 5.1.1a for ions)⇒

$$m_e u_x \frac{du_{ey}}{dx} + m_i u_x \frac{du_{iy}}{dx} = 0$$

Integrating ⇒

$$m_e u_{ey} + m_i u_{iy} = \text{constant}$$

At $x = -\infty$ $u_{iy} = u_{ey} = 0$ so the constant is zero.

Thus $$u_{iy} = -\frac{m_e u_{ey}}{m_i} \qquad 5.1.5$$

This is the drift between electrons and ions referred to in the two-stream instability.

Subtracting the electron and ion x components of 5.1.1a and using 5.1.4 and 5.1.5 ⇒

$$E_x = -Bu_{ey} + O\left(\frac{m_e}{m_i}\right) \qquad 5.1.6$$

Whilst,

m_e(x comp of 5.1.1a for electrons) + m_i(x comp of 5.1.1a for ions) ⇒

$$(m_e+m_i)u_x\frac{du_x}{dx} = eB(u_{iy}-u_{ey}) \qquad 5.1.7$$

Recall 5.1.1d is

$$-\frac{dB}{dx} = -en\mu_o(u_{ey}-u_{iy})$$

So n(5.1.7) becomes

$$n(m_e+m_i)u_x\frac{du_x}{dx} = -\frac{B}{\mu_o}\frac{dB}{dx} \qquad 5.1.8$$

Using 5.1.2 we may integrate this to obtain

$$u_x = -\frac{1}{\lambda}(B^2-B_1^2) + u_1 \qquad 5.1.9$$

where we have put $\lambda = 2n_1u_1\mu_o m_i$

for convenience and where the constant of integration was found from the condition

$$u_x = u_1,\ B = B_1 \text{ at } x = -\infty$$

[N.B.We have neglected m_e in the factor m_e+m_i as $m_e << m_i$]

Now we find u_{ey} in terms of B.

Recall 5.1.1d is

$$\frac{dB}{dx} = \mu_o en(u_{ey}-u_{iy})$$

But $$u_{iy} = -\frac{m_e u_{ey}}{m_i}$$

from 5.1.5 so we may neglect u_{iy} here as it is $O\left(\frac{m_e}{m_i}\right)$.

i.e. $$\frac{dB}{dx} = \mu_o enu_{ey}$$

5.1.9 and 5.1.2 then give

$$u_{ey} = \left(1 - \frac{(B^2-B_1^2)}{\lambda u_1}\right)\frac{1}{\mu_o e n_1}\frac{dB}{dx} \qquad 5.1.10$$

We now seek an equation for B only

$u_x \frac{d}{dx}$ (5.1.1d) $\Rightarrow$

$$u_x\frac{d}{dx}\left(u_x\frac{dB}{dx}\right) = \mu_o e u_x \frac{d}{dx}(n u_x u_{ey})$$

$$= \mu_o n_1 u_1 e u_x \frac{du_{ey}}{dx} \quad \text{(from 5.1.2)} \qquad 5.1.11$$

The y component of 5.1.1a then $\Rightarrow$

$$u_x\frac{d}{dx}\left(u_x\frac{dB}{dx}\right) = \frac{\mu_o n_1 u_1 e^2}{m_e}(Bu_x - E_y) \qquad 5.1.12$$

But u_x is known as a function of B from 5.1.9 and $E_y=u_1B_1$ so we have an equation for B. Simple substitution yields

$$f(B)\frac{d}{dx}\left(f(B)\frac{dB}{dx}\right) = g(B) \qquad 5.1.13$$

where $f(B) = 1-\frac{(B^2-B_1^2)}{\lambda u_1}$ and $g(B) = \frac{\omega_{pe}^2}{c^2}(B-B_1)\left(1-B\frac{(B+B_1)}{\lambda u_1}\right)$

Now, $\frac{dB}{dx}$(5.1.13)$\Rightarrow$

$$\frac{1}{2}\frac{d}{dx}\left(f\frac{dB}{dx}\right)^2 = g\frac{dB}{dx}$$

And

$$\int g\,dB = \frac{\omega_{pe}^2}{2c^2}(B-B_1)^2\left(1-\frac{(B+B_1)^2}{2u_1\lambda}\right) + \text{constant}$$

Integrating we have an equation of the form

$$\frac{1}{2}\left(\frac{dB}{dx}\right)^2 + \phi(B) = 0 \qquad (A)$$

$$\text{where } \phi(B) = \frac{-\left(\frac{dB}{dx}\right)^2_{x_1} - \frac{\omega_{pe}^2}{c^2}(B-B_1)^2\left(1-\frac{(B+B_1)^2}{2u_1\lambda}\right)}{2\left(1-\frac{(B^2-B_1^2)}{\lambda u_1}\right)^2}$$

and where x_1 is the point at which $B = B_1$ (i.e. $x = -\infty$)

Equation (A) is in the form of a classical potential well problem and $\phi(B)$ is often called the Sagdeev potential [14] as he was the first to recognise that the equation was in this form.

The so called anomalous term

$$\int \underline{v}\left\langle \left(\underline{E}_1+\underline{v}_\wedge \underline{B}_1\right) \cdot \nabla_{\underline{v}} f_{s1} \right\rangle d\underline{v}$$

in Chapter 4 was omitted in the above theory. We now include this term and will show how it allows shock-like solutions to emerge.

Write

$$C_s = \frac{q_s}{m_s}\left\langle \left(\underline{E}_1+\underline{v}_\wedge \underline{B}_1\right) \cdot \nabla_{\underline{v}} f_{s1} \right\rangle$$

for convenience. Since, for magnetosonic waves the dominant drift between electrons and ions is in the y direction we *assume* this term has only an appreciable y component. The only equation that has changed in 5.1.1a-5.1.1e is 5.1.1a, which is now

$$u_{sx}\frac{du_s}{dx} = \frac{q_s}{m_s}(E_x+u_{sy}B, E_y-u_{sx}B, 0) + \left(0, \frac{1}{n}\int C_s v_{sy} d\underline{v}, 0\right) \qquad 5.1.14$$

Equation 5.1.3 is still valid since we assume the turbulence is negligible at $-\infty$ (i.e. $\frac{1}{n}\int C_s v_{sy} dv \approx 0$ at $-\infty$).

Now, m_e(y component of 5.1.14 for electrons) + m_i(y component of 5.1.14 for ions) $\Rightarrow$

$$\left(m_e\frac{du_{ey}}{dx} + m_i\frac{du_{iy}}{dx}\right)u_x = 0 + \frac{m_e}{n}\int C_e v_y d\underline{v} + \frac{m_i}{n}\int C_i v_y d\underline{v}$$

So

$$u_{iy} \approx \frac{-m_e u_{ey}}{m_i}$$

is only valid now provided that we assume the turbulence level is small (i.e. $\sum_s m_s \int C_s v_y d\underline{v} \approx 0$). Then all the algebra up to 5.1.11 is the same and the corresponding equation for 5.1.12 is now

$$u_x\frac{d}{dx}\left(u_x\frac{dB}{dx}\right) = \mu_o e n_1 u_1\left(\frac{-eE_y}{m_e} + \frac{eu_x B}{m_e} - \nu_e u_{ey}\right)$$

where the "collision" frequency ν_e is defined by

$$\nu_e u_{ey} = -\frac{1}{n}\int C_e v_y d\underline{v}$$

Again we can substitute for u_x from 5.1.9 and E_y from 5.1.3 to obtain an equation for B.

The new term on the R.H.S. is (using 5.1.9,5.1.10)

$$-\mu_o n_1 u_1 e \nu_e u_{ey} = -\nu_e u_x \frac{dB}{dx} \qquad 5.1.15$$

Our equation for B is just

$$\frac{u_x d}{u_1 dx}\left(\frac{u_x dB}{u_1 dx}\right) = \frac{\omega_{pe}^2}{c^2}(B-B_1)\left(1 - \frac{B(B+B_1)}{\lambda u_1}\right) - \frac{\nu_e u_x}{u_1^2}\frac{dB}{dx} \qquad 5.1.16$$

Define a variable τ by $\frac{dx}{d\tau} = \frac{u_x}{u_1}$ 5.1.17

so that 5.1.16 becomes

$$\frac{d^2B}{d\tau^2} = -\frac{\partial\Phi}{\partial B} - \frac{\nu_e dB}{u_1 d\tau} \qquad (B)$$

where $\Phi(B) = \frac{\omega_{pe}^2}{c^2}\frac{(B-B_1)^2}{2}\left(\frac{(B+B_1)^2}{2\lambda u_1} - 1\right)$

which is in the form of a standard potential well problem with damping present.

5.2 Solutions of the Profile Equations

We must emphasise at this point that the following interpretation of the equations (A),(B) for B(x) are not intended to be rigorous mathematical proofs, but they give physical reasons for why the inclusion of dissipation is necessary for shock-like solutions to exist.

Equation (A) is in the form of equation for a particle moving in a potential well.

(i.e. in the form $\frac{d^2x}{dt^2} = -\frac{dV}{dx} \Rightarrow \left(\frac{dx}{dt}\right)^2 = -V(x) + \text{constant}$)

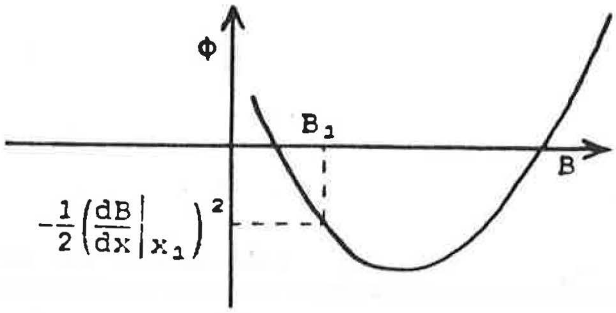
So B plays role of x in cassical well problem,

x " " " t " " " "

ϕ " " " V " " " "

Part of the function ϕ is sketched in Fig.11

Fig.11



By analogy with the well problem we see that infinite nonlinear wavetrain solutions exist for B(x), corresponding to "reflections" at the "walls" of the well.

Note that

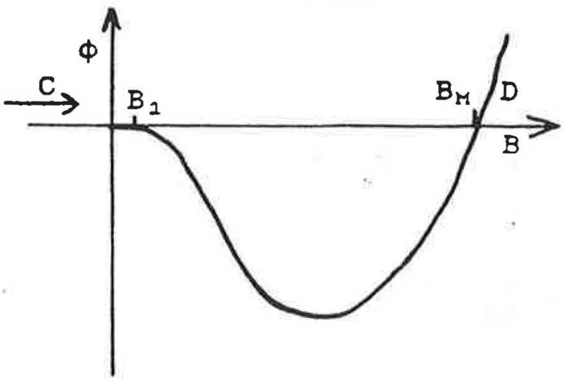
$$\frac{d\phi}{dB} = 0 \quad \text{at } B=B_1$$

Consider the case when

$$\frac{dB}{dx}\Big|_{x_1} = 0$$

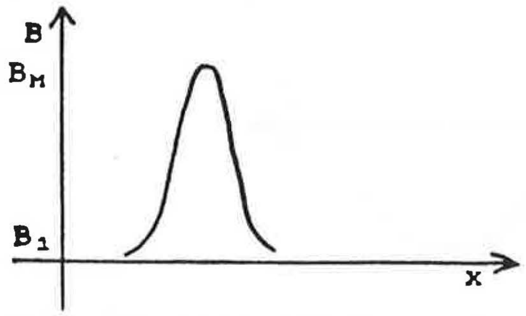
so that $\phi(B_1)$ is also zero. See Fig.12.

Fig.12 Case $\frac{dB}{dx}\Big|_{x_1} = 0$



By considering a particle in the classical well incident from C we see it will be reflected at D but will then never be reflected again. Moreover, as it passes over the well it's speed increases and decreases. This corresponds to a solitary wave solution. (Fig.13)

Fig.13



This physical argument shows that shock-like solutions do not exist in equation (A). The above is only actually valid in a certain range of the parameter, called the Alfven Mach number, M, defined as

$$M = U_1/U_{A1} \qquad \text{where } U_{A1} = B_1/\surd(n_1 m_i \mu_o).$$

This is true for the following reasons:

Let B_M denote the maximum value of B.

We have already assumed $\left.\frac{dB}{dx}\right|_{x_1} = 0.$

In order that $\phi(B)$ is a potential well and not a hump we require that the stationary point between B_1 and B_M is a minima and that this is the only stationary point in (B_1, B_M). Simple algebra shows that this entails $M > 1$.

Now,

$$\frac{dB}{dx} = 0 \quad \text{at } B = B_M$$

(from the analogy with the potential well)

So (A) $\Rightarrow$ $\qquad \phi(B_M) = 0$

which $\Rightarrow$ $\qquad U_1 = \frac{U_{A1}}{2}\left(\frac{B_M}{B_1} + 1\right) \qquad$ 5.2.1

Then (A) becomes

$$\left(1 - \frac{2(B^2-B_1^2)}{(B_M+B_1)^2}\right)^2\left(\frac{dB}{dx}\right)^2 = \frac{\omega_{pe}^2}{c^2}(B-B_1)^2\left(1 - \frac{(B+B_1)^2}{(B_M+B_1)^2}\right) \qquad 5.2.2$$

The R.H.S. is positive since $B < B_M$ in the "well" so in order for $\left(\frac{dB}{dx}\right)^2$ to be positive (i.e. in order that a solution exists) we must have

$$1 - \frac{2(B^2-B_1^2)}{(B_M+B_1)^2} > 0 \quad \text{for all } B \in (B_1, B_M)$$

which is actually just $u_x > 0$.

So we must have

$$B_M < 3B_1$$

which means $\qquad M < 2 \qquad$ 5.2.3

Altogether solitary wave solutions only exist if

$$1 < M < 2 \qquad 5.2.4$$

For situations where $M > 2$, we will have $u_x < 0$ which means that the particles are reflected back upstream at the shock.[5]

Similarly, with dissipation present (i.e. (B) valid) we still need a well not a hump and this again entails $M > 1$. Now the transformation 5.1.17 defining τ breaks down when $u_x = 0$, at which point

$$B = (\lambda u_1 + B_1{}^2)^{1/2} \qquad \text{from 5.1.9.}$$

The maximum value of B that can occur in the well is denoted by $B = B_M$ and occurs when $\phi = 0$. (See Fig.14.)

Thus,

$$B_M = -B_1 + 2u_1\surd(n_1 m_i \mu_o) \qquad 5.2.5$$

from (B).

Fig.14

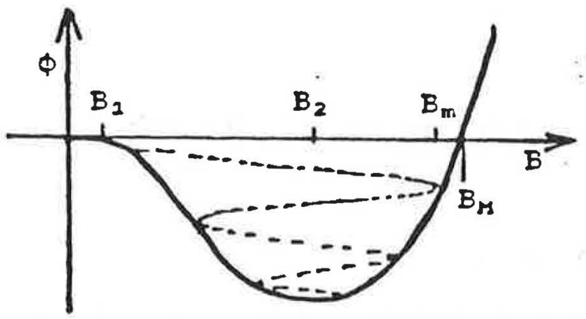
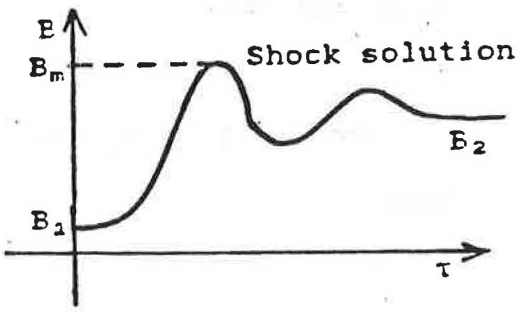


Fig.15



The dotted line shows the schematic variation of B as the "particle" moves in the "well" whilst Fig.15 shows the shape of B in space, corresponding to the dotted line in Fig.14.

Now ,provided

$$B(u_x=0) > B_m$$

where B_m is the maximum value of B in the solution to (B) in the range (B_1, B_M) a shock-like solution exists. (i.e. the transformation must not break down before the "particle" is reflected at the "wall" of the "well")

We have above

$$B(u_x=0) = (2M^2+1)^{1/2}B_1$$

$$\text{and} \qquad B_M = (2M-1)B_1$$

It is easily shown that

$$\frac{B(u_x=0)}{B_M} > 1 \qquad \forall\ M < 2$$

Hence

$$B(u_x=0) > B_m \qquad \forall\ M < 2$$

Altogether we conclude that shock solutions exist for any finite value of ν if

$$1 < M < 2 \tag{5.2.6}$$

Now the minimum of Φ occurs at

$$B_2 = (\lambda u_1 + B_1^2/4)^{1/2} - B_1/2 \tag{5.2.7}$$

so we see that

$$B_2 < B(u_x=0).$$

From Fig.14 we can see that $B_m \to B_2$ as the damping $\nu \to \infty$. Thus,

$$B(u_x=0) > B_m\ \forall\ M \text{ as } \nu \to \infty$$

so there is no longer an upper bound for M in order for shocks to exist.

We have shown how the inclusion of dissipation turns solitary pulse solutions into shock-like solutions. It is important to note that we neglected the pressure term in the fluid equations and therefore did not allow the plasma to heat up even though we included a dissipation term, so energy is not conserved in our model. However, it can be shown [12] that such a model is valid for weak shocks (i.e. $M=1+\delta, \delta$ small). Also, our assumption that the microturbulence is small may also be invalid. This could provide a starting point for further research.

5.3 Phase Plane Analysis of the Profile Equations

Defining

$$y = \frac{dB}{dx}$$

we have from $\frac{d}{dx}$(5.1.13) the autonomous system

$$\frac{dy}{dx} = (g - f\frac{df}{dB}y^2)/f^2 \equiv l(B,y), \text{ say.}$$

and $$\frac{dB}{dx} = y \equiv m(B,y), \text{ say.}$$

We can now apply the standard linearisation technique [17] to determine the nature of the equilibrium points given by

$$m(B,y) = 0 = l(B,y)$$

The equilibrium points are found to be

$y = 0,\ B = B_1$

$y = 0,\ B = B_L$ or B_R

where $B_L = (-B_1 - [B_1^2 + 4\lambda U_1]^{1/2})/2$

and $B_R = (-B_1 + [B_1^2 + 4\lambda U_1]^{1/2})/2$

The eigenvalues, μ, in the linearisation method are found to be

$$\mu = \pm\left(\frac{\partial l}{\partial B}\bigg|_{B_0,y_0}\right)^{1/2}$$

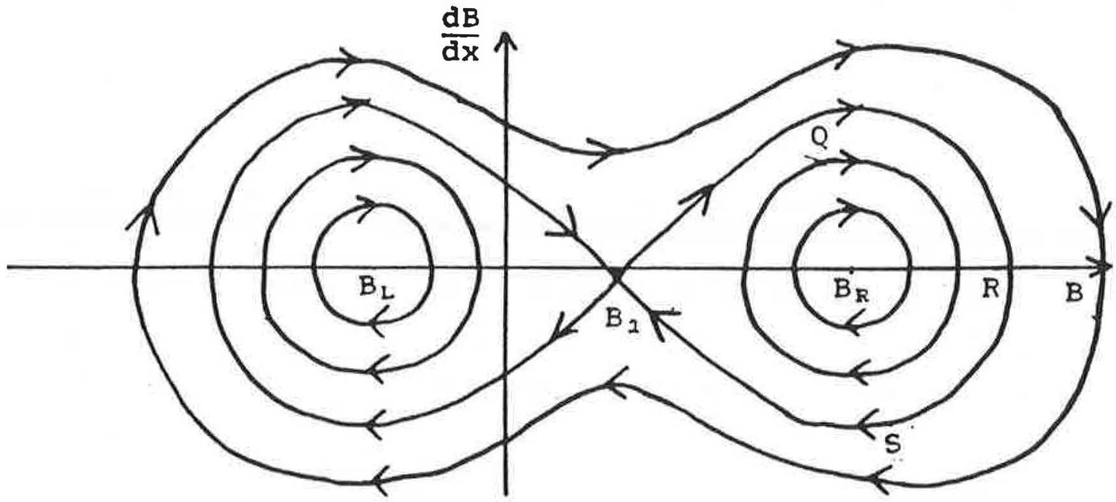
where (B_0, y_0) are the equilibrium points.

The sign of $\left.\frac{\partial l}{\partial B}\right|_{B_o, y_o}$ determines whether the eigenvalues are real and of opposite sign (saddle point) or purely imaginary (centre). [See 17] It is worthwhile noting that this analysis assumes that the phase paths of the original equation and those of the linearised equation near the equilibrium point are of the same character. This is true for spirals, nodes and saddle points but not always for a centre. Symmetry about the y = 0 axis has to be checked to confirm the equilibrium point is indeed a centre and not a spiral. This is easily verified by showing

$$\left.\frac{dy}{dB}\right|_{y} = \frac{l(B,y)}{m(B,y)} = -\frac{l(B,-y)}{m(B,-y)} = \left.\frac{dy}{dB}\right|_{-y}$$

For the equilibrium point $y = 0$, $B = B_1$ it transpires that if $M > 1$ then it is a saddle point or if $M < 1$ then it is a centre. For the equilibrium point at $y = 0$, $B = B_R$ the reverse is true because for $M > 1$ it is a centre whilst for $M < 1$ it is a saddle point. The point $y = 0$, $B = B_L$ is always a centre but we are not concerned with negative values of B because they are not applicable to the shock problem. In fact, as already explained in section 5.2 we only consider the case $M > 1$. An analysis of the behaviour of $\frac{dy}{dB}$ as $B \to \pm\infty$, $y \to \pm\infty$ enables the phase plane to be completed as in Fig.16 The phase plane path B_1QRS corresponds to the solitary wave solution discussed in section 5.2. We now outline the approach to find the phase plane for equation (B) where we have also included dissipation via anomalous resistivity.

Fig.16



We have the autonomous first order system

$$\frac{dy}{d\tau} = -\frac{\partial\Phi}{\partial B} - \frac{\nu}{U_1}y$$

$$\frac{dB}{d\tau} = y$$

The equilibrium points are found to be the same as before, namely

$$B = B_1,\ y = 0$$

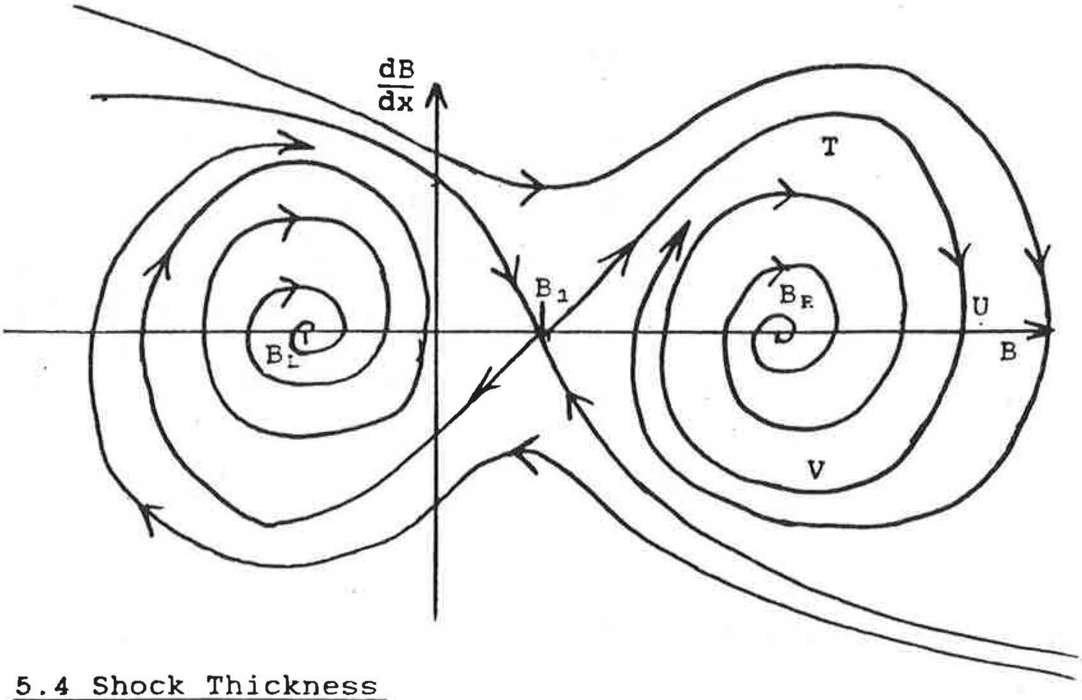
$$B = B_L,\ y = 0$$

$$B = B_R,\ y = 0$$

Notice that $B = B_R$ is actually $B = B_2$ in 5.2.7 which corresponds to the bottom of the "well". The equilibrium point at $B = B_1$ is a saddle if $M > 1$ and the equilibrium points that were previously centres at B_L, B_R have now been deformed into stable spirals by the addition of the damping ν_s. The phase plane is sketched in Fig.17.

The phase path B_1TUVB_R corresponds to the shock-like solution discussed in section 5.2 so the phase plane clearly shows how the addition of dissipation allows B to change from one value, B_1, to another at B_R via an oscillatory behaviour.

Fig.17



5.4 Shock Thickness

The damping length scale $L \sim U_1/\nu_s$, associated with the shock, enables a knowledge of ν_s from quasilinear theory to compare theoretical predictions with satellite data. The actual calculations are performed in Chapter 6.

Chapter 6

6.1 Calculation of the Effective Collision Frequency

The theory of Chapter 4 is now used to provide an estimation of the effective collision frequency for both the ion acoustic instability and the modified two stream instability. This will result in an estimate of the shock thickness as outlined in Chapter 5, enabling a comparison with satellite data to be made. This will determine whether the anomalous resistivity produced by either of the instabilities provides enough dissipation to support the earth's shock.

The saturated energy level 4.5.2 is substituted into 4.1.31 so that our definition 4.2.1 of the effective collision frequency becomes

$$\nu_s = \frac{\pi\epsilon_0 m_i^2 \lambda_D \omega_{ps}^2}{8e^2 V_d n_s m_s} \int_0^{1/\lambda_D} k(V_d - \omega_{rm}/k)^4 \left.\frac{\partial F_{so}}{\partial u}\right|_{u=v_p} dk$$

where we have used $\epsilon_{si}(k,\omega) = \pi\frac{\omega_{ps}^2}{k^2}\left.\frac{\partial F_{so}}{\partial u}\right|_{u=v_p}$ from 4.1.27a and 2.3.3a. The variable ω_{rm} is the value of ω_r when ω_i is maximum since this was our model for obtaining the saturated energy level. (See section 4.5) The integration limits are form 0 to $1/\lambda_D$ because the waves are severely damped for $k\lambda_D \gtrsim 1$. Note that we have assumed the one dimensional case when $\underline{k}$ is parallel to $\underline{V}_d$ (the drift velocity between the ions and electrons in the instabilities) which will suffice for our purposes as we are considering instabilities that grow along the shock.

We can now substitute a Maxwellian for F_{so} to obtain

$$\nu_s = a_s \int_0^{1/\lambda_D} k\left(V_d - \frac{\omega_{rm}}{k}\right)^4 \frac{\omega_r}{k} \exp\left(\frac{-\omega_r^2}{2v_{ths}^2 k^2}\right) dk \qquad 6.1.1$$

where $a_s = \dfrac{\pi\epsilon_0 m_i^2 \lambda_D \omega_{ps}^2}{8\surd(2\pi) e^2 V_d v_{ths}^3 n_s m_s}$ is a constant.

6.2 Calculation of ν_s for the Ion Acoustic Instability

To calculate the integral in 6.1.1 exactly involves finding the value of k for which 3.2.3 is maximum and then substituting into 3.2.2a to find ω_{rm}. As an order of magnitude estimate we see from the ion acoustic dispersion relation that we may approximate ω_r/k to be c_s for all k within the integration limits of 6.1.1. Recall that the theory of Chapter 5 requires only a knowledge of the effective collision frequency for electrons so we now have

$$\nu_e = \frac{a_e}{2\lambda_D^2}(V_d - c_s)^4 c_s \exp(-c_s^2/2v_{ths}^2) \qquad 6.2.1$$

where the constant $a_e/2\lambda_D^2$ simplifies to

$$\frac{a_e}{2\lambda_D^2} = \frac{\surd\pi m_i^2 \omega_{pe}}{8\surd 2 V_d \kappa^2 T_e^2} \qquad 6.2.2$$

which shows how ν_e varies with the different parameters more easily. Notice the sensitivity of ν_e to the parameter $V_d - c_s$.

It is worthwhile remembering at this stage that we require

$$\left|\frac{\omega_r}{k}\right| >> \left(\frac{\kappa T_i}{m_i}\right)^{1/2}, \quad \left|\frac{\omega_r}{k} - V_d\right| << \left(\frac{\kappa T_e}{m_e}\right)^{1/2}$$

for ion waves to exist and that we require

$$T_e >> T_i, \; V_d > \left(\frac{\kappa T_i}{m_i}\right)^{1/2}$$

for instability to occur.

Papadopolous [19] has shown that electron-ion temperature ratios of $T_e/T_i \approx 8$ are required for the ion-acoustic instability to operate. Such ratios are not observed in the earth's bow shock so we shall simply leave the expression 6.2.1 as a means of calculating anomalous resistivity for some other application. Instead we move onto the process whereby the modified two-stream instability provides the anomalous resistivity.

6.3 Calculation of ν_e for the Modified Two Stream Instability

This instability is a more realistic one when applied to the earth's shock because the currents flowing along the shock do actually travel approximately perpendicular to a magnetic field. From section 3.4 we have that $\omega_r/k = V_d/2$ for all values of k at the angle given by $\cos\theta = \left(\frac{m_e}{m_i}\right)^{1/2}$ which simplifies the integration in 6.1.1 immensely.

Thus 6.2.1 becomes

$$\nu_e = \frac{1}{256}\sqrt{\left(\frac{\pi}{2}\right)}\frac{m_i^2}{m_e^2}\left(\frac{V_d}{v_{the}}\right)^4 \omega_{pe}\exp(-V_d^2/8v_{ths}^2)$$

Notice that this is independent of ion temperature but is very sensitive to the relative drift speed, V_d, between the ions and electrons and also the electron thermal velocity. Recall that the modified two-stream instability only occurs provided that

$$V_d \gtrsim v_{thi},\ V_d \gtrsim \left(\frac{KT_e}{m_i}\right)^{1/2}$$

We have also assumed that $\omega_{ce} >> \omega >> \omega_{ci}$ and we have only considered the case $\cos\theta = \left(\frac{m_e}{m_i}\right)^{1/2}$. This instability, unlike the ion acoustic instability, is insensitive to the ratio T_e/T_i and may therefore operate in a wider regime. The data satisfies these conditions and so the instability may indeed operate. Substituting in typical values from [21] listed in

Appendix 6 yields

$$\nu_e = 258s^{-1}$$

giving a value for the shock thickness as

$$L = u_1/\nu_e = 2321m.$$

The actual observed value for L, which is difficult to measure in practice, was in this case found to be 50km.

The data given by Scudder et al [20] listed in Appendix 6 gives a value for the shock thickness as L = 90m. The observed value for L at this time was approximately 15km.

Chapter 7

Discussion and Conclusions

The philosophy in this dissertation has been to highlight the fundamental processes involved with the concept of anomalous resistance in as simple manner as possible and to incorporate them into an overall model for the dissipation mechanism in the earth's bow shock. Mathematical rigor and more realistic models have sometimes been sacrificed to prevent the physical understanding from being obscured. For example, the three dimensional nature of the earth's bow shock has been ignored and the turbulence level in Chapter 5 was assumed to be small. This latter assumption is consistent with using the weakly nonlinear theory in Chapter 4 for particle trapping. In Chapter 5 we demonstrated that anomalous resistivity is necessary to support the earth's bow shock and Chapter 6 showed that it is more than sufficient.

The failure of the ion-acoustic instability to operate in the regime where $T_i \gtrsim T_e$ has been shown from the data [20,21] to mean that this can not account for the anomalous resistivity required to support the shock. The modified two-stream instability was then considered as a mechanism for producing anomalous resistivity and was used in this text for the first time to predict the shock thickness. This is intuitively more realistic as it necessitates passage of a current across a magnetic field which does actually occur in the shock. A further assumption involved here was that only the angle $\theta = \theta_0$ where $\cos\theta_0 = \left(\frac{m_e}{m_i}\right)^{1/2}$ between the direction of propogation of the instability and the background magnetic field was treated.

The results indicated an excess of anomalous resistance since the predicted value of the shock thickness was smaller by a factor of an order of magnitude. This discrepency might be explained by the one dimensional nature of the model or any of the other simplifications made in the model. Further extensions and investigations are required. In particular, the work of Davidson in [4] might be modified to provide a more sophisticated way of incorporating particle trapping.

Appendix 1

Common Plasma Parameters

Listed below are the definitions and notation for parameters used throughout the text.

Plasma frequency $\omega_{ps} = \left(\frac{n_o e^2}{m_s \epsilon_o}\right)^{1/2}$

These are oscillations of species s, assuming a uniform background of the other species, due to just the electric fields produced when the particles are displaced.

Cyclotron frequency $\omega_{cs} = \frac{|q_s| B}{m_s}$

This is the frequency of gyration of particles of species s about a field $\underline{B}$ in the absence of $\underline{E}$.

Debye length $\lambda_D = \left(\frac{\epsilon_o \kappa T_e}{n_o e^2}\right)^{1/2}$

When a test ion is placed in a plasma its field is shielded by the resulting cloud of electrons surrounding the ion. The Debye length is the distance for the ion's field to fall by 1/e times its original value.

Plasma parameter $N_D = 4\pi n_o \lambda_D^3 / 3$

This is the number of particles in a Debye sphere.

Thermal velocity $v_{ths} = \left(\frac{\kappa T_s}{m_s}\right)^{1/2}$

This is a measure of the average K.E. of particles of species s.

Ion acoustic speed $$c_s = \left(\frac{\gamma_e T_e + \gamma_i T}{m_i}\right)^{1/2} \approx \left(\frac{\gamma_e T_e}{m_i}\right)^{1/2}$$

where γ_s is the ratio of specific heats.

This is the speed of propogation of low frequency compression waves in a plasma. It is reminiscent of sound waves in an ideal gas.

Alfven speed $$V_A = \left(\frac{B_o^2}{\mu_o m_s n_s}\right)^{1/2}$$

This is the speed of propogation of waves whose restoring force is magnetic pressure, with the fluid density providing the inertia.

Lower hybrid frequency $$\omega_{LH} = (\omega_{ce}\omega_{ci})^{1/2}$$

This is the lower cut-off frequency for the so called extraordinary mode which is a wave travelling perpendicular to $\underline{B}_o$ that is neither purely transverse or purely longitudinal. For very low frequencies this wave degenerates into the magnetosonic wave of Chap.5 which is mainly a longitudinal disturbance propogating across $\underline{B}_o$. Hence the name magnetosonic.

Note: simple algebra $\Rightarrow$ $$\lambda_D = v_{the}/\omega_{pe}$$

Note: $\nabla_{\underline{v}}$ is used to denote the operator $\left(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3}\right)$ where $\underline{v} = (v_1, v_2, v_3)$ is a position in velocity space, whilst ∇ is the familiar $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ where $\underline{x} = (x_1, x_2, x_3)$ is a position in configuration space.

Appendix 2

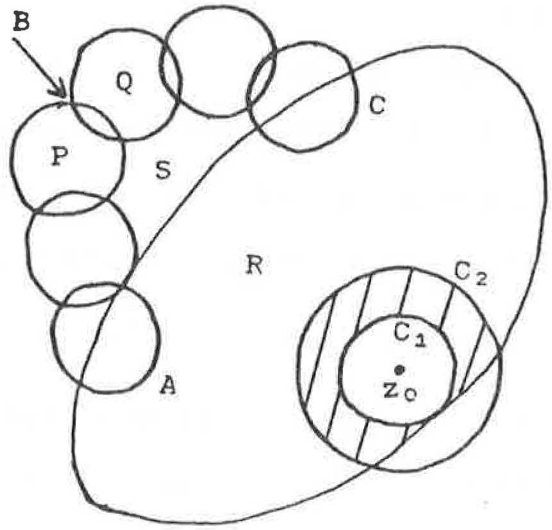
Analytic Continuation

Let $f(z)$ be a function which is analytic within a region R. We wish to invent a function $g(z)$ that coincides with $f(z)$ in R and that is analytic in some extension of this region. Then $g(z)$ is called an analytic continuation of $f(z)$.

e.g. $f(z) = \sum_{n=0}^{\infty} z^n$ is analytic for $|z|<1$ but is divergent for $|z|\geq 1$. Consider $g(z) = \frac{1}{1-z}$. This is analytic everywhere except $z=1$ and agrees with $f(z)$ within $|z|<1$. It is therefore an analytic continuation of $f(z)$. However, it is not always possible to be able to replace an expression by an equivalent one possessing a wider range of validity. In which case we shall use the following method.

Let $f(z)$ be analytic in some region R. Since $f(z)$ is analytic there exists a Taylor series expansion that converges to $f(z)$ within some circle, C_1, about any point z_0 in R. In fact, for any given $z_0 \in R$, we know that there exists a Taylor series within the largest circle, centre z_0, that lies entirely within R. However, it is possible that this series converges within some larger circle C_2. The series must agree with $f(z)$ within C_1 but the question arises as to whether it agrees with $f(z)$ in $(R \cap C_2) \setminus C_1$. Moreover, for a point z_1 outside R there may exist two chains of circles from R continuing $f(z)$ into a region containing z_1 (See Fig.18) so the question also arises as to whether the two values for $f(z_1)$ agree. If the answers to these questions are in the affirmative then we have a way of analytically continuing $f(z)$ to some larger region.

Fig.18



To demonstrate this is indeed true we shall require the following theorem.

Identity Theorem for Single-Valued Analytic Functions

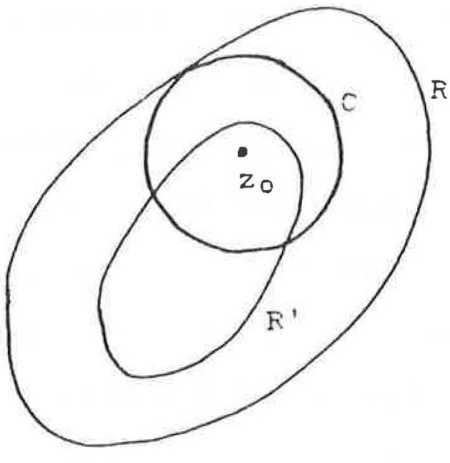
Let $f_1(z), f_2(z)$ be analytic in a region R. Let $f_1(z), f_2(z)$ coincide in some portion R' of R.

[N.B. R' may be a subregion, a segment of a curve or even an infinite set of points having a limit point in R.]

Then $f_1(z), f_2(z)$ coincide in the whole of R.

Proof

Fig.19



Choose any point z_0 in R'.

Draw the largest circle, C, of centre z_0, that fits entirely in R. Each of f_1, f_2 can be represented by a Taylor series about z_0, convergent inside this circle, since they are analytic within R. But f_1, f_2 coincide within R' and the identity theorem for power series then implies that the two power series are identical. So f_1, f_2 are the same throughout the interior of C. We can then choose another point z, within $R \cup C$ until the whole of R is used - i.e. f_1, f_2 identical throughout the region R.

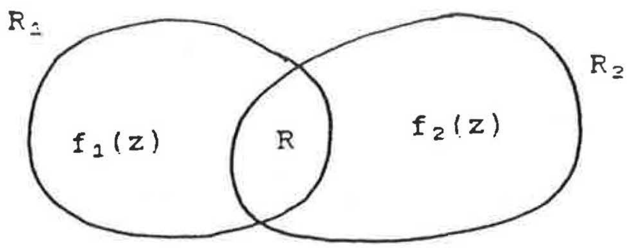
Returning to our original question then we see that this theorem implies the series does agree with f(z) within $(R \cap C_2) \setminus C_1$. To answer our second query concerning uniqueness of our continuation at some point z_1 we require the following corollary.

Corollary Analytic Continuation is Locally Unique

Proof

Consider a function $f_1(z)$ analytic in a region R_1. Let R_2 be some other region which has a region R in common with R_1. Let $f_2(z)$ be some analytic function in R_2 which coincides with $f_1(z)$ in R.

Fig.20



Suppose g(z) is some other function, analytic in R_2, that coincides with $f_1(z)$ in R. Thus $g(z) = f_2(z)$ in R. But f_2, g are analytic in R_2 with a subregion R of R_2 where they coincide. The above theorem implies $f_2(z) \equiv g(z)$ throughout R_2. This means that $f_2(z)$ is the unique local analytic continuation of $f_1(z)$.

[N.B. Continuing f_2 into some region R_3 with an analytic function f_3 we will only have $f_1=f_3$ if there exists a common region between all three of R_1, R_2, R_3.]

Returning to Fig.18 we know that the function in B is the unique function that is both analytic in B and agrees with f in P and we know the function is the unique analytic function that agrees with f in Q. However, we may still have two different functions but provided that there is no singularity within region S we can cover S with overlapping regions and apply the previous theorem to see f(z) is the same in B irrespective of the path taken.

The total structure of an analytic function is then deducible from only a small sample of the function - a quite remarkable feature. Once all possible continuations are performed we arrive at the so called complete analytic function. It may happen in the process of continuation that we discover a curve across which it is impossible to continue the function. Such a curve is called a natural boundary of the complete analytic function.

In order to illustrate the idea of analytic continuation as used in Chapter 2 consider, as an example, the function

$$f(\zeta) = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x-\zeta)} dx \qquad (a > 0)$$

which is only defined for $\mathrm{Im}\zeta > 0$, say.

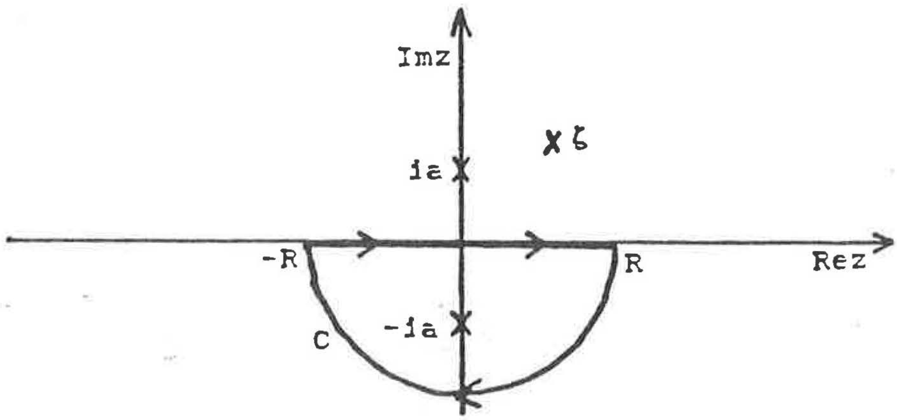
Employing Cauchy's residue theorem

$$f(\zeta) = \int_C \frac{1}{(z^2+a^2)(z-\zeta)}\, dz$$

$$= -2\pi i \sum_{\alpha} \text{res}_{\alpha}\left(\frac{1}{(z-ia)(z+ia)(z-\zeta)}\right)$$

where C is the contour in Fig.21, in the limit of $R \to \infty$ and where $\text{res}_{\alpha} g(z)$ is the coeffecient of $\frac{1}{z-\alpha}$ in the Laurent expansion of $g(z)$ about $z = \alpha$.

Fig.21



Only $z = -ia$ is within the contour and

$$\text{res}_{z=-ia} = \frac{1}{2ia(\zeta+ia)}$$

So $$f(\zeta) = \frac{-\pi}{a(\zeta+ia)} \qquad (\zeta_i > 0) \qquad \text{A2.1}$$

This is only valid for $\zeta_i > 0$ since the singularity at $z = \zeta$ is then outside the contour of integration.

If $\zeta_i < 0$ we have to add the residue at $z = \zeta$ which is

$$\left.\frac{1}{(z+ia)(z-ia)}\right|_{z=\zeta} = \frac{1}{\zeta^2+a^2}$$

So $$f(\zeta) = \frac{-\pi}{a(\zeta+ia)} - \frac{2\pi i}{\zeta^2+a^2} \qquad (\zeta_i < 0) \qquad \text{A2.2}$$

We can not use A2.2 as the analytic continuation of A2.1 since the function is discontinous at $\zeta_i = 0$ and is therefore not analytic. We subtract the extra term in A2.2 and write

$$f(\zeta) = \int_{-\infty}^{+\infty} \frac{1}{(x^2+a^2)(x-\zeta)}\, dx + \frac{2\pi i}{\zeta^2+a^2} \qquad (\zeta_i < 0)$$

$$f(\zeta) = \int_{-\infty}^{+\infty} \frac{1}{(x^2+a^2)(x-\zeta)}\, dx \qquad (\zeta_i > 0)$$

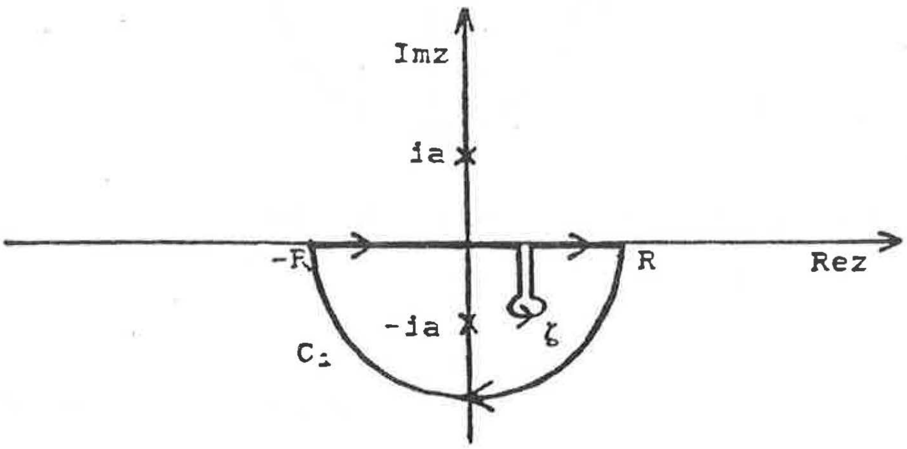
as the analytic continuation of A2.1. Alternatively we can use

$$f(\zeta) = \int_{C_1} \frac{1}{(z^2+a^2)(z-\zeta)}\, dz \qquad (\zeta_i < 0)$$

$$= \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x-\zeta)}\, dx \qquad (\zeta_i > 0)$$

where C_1 is the contour in Fig.22 in the limit of $R \to \infty$ since the singularity at $z = \zeta$ is then in the exterior of C_1.

Fig.22



Appendix 3

With reference to Chapter 4 we now prove

$$\underline{E}_1(\underline{k},t) = \underline{E}_1{}^*(-\underline{k},t) \quad \text{and} \quad \omega(\underline{k},t) = -\omega^*(-\underline{k},t)$$

where * denotes complex conjugate.

According to linear theory we may express $\underline{E}_1(\underline{x},t)$ in terms of its Fourier components

$$\underline{E}_1(\underline{x},t) = \frac{1}{(2\pi)^3}\int \overline{\underline{E}}(\underline{k})\exp i(\underline{k}.\underline{x}+\omega t)d\underline{k}$$

where we have assumed

$$\underline{E}_1(\underline{k},t) = \overline{\underline{E}}(\underline{k})\exp(i\omega t)$$

So $$\overline{\underline{E}}(k)\exp i\omega(\underline{k})t = \frac{1}{(2\pi)^3}\int \underline{E}_1(\underline{x},t)\exp(-i\underline{k}.\underline{x})d\underline{k} \qquad \text{A3.1}$$

by inverting the Fourier transform.

Thus

$$\overline{\underline{E}}(-\underline{k})\exp i\omega(-\underline{k})t = \frac{1}{(2\pi)^3}\int \underline{E}_1(\underline{x},t)\exp(i\underline{k}.\underline{x})d\underline{k}$$

which $\Rightarrow$

$$\overline{\underline{E}}^*(-\underline{k})\exp -i\omega^*(-\underline{k})t = \frac{1}{(2\pi)^3}\int \underline{E}_1(\underline{x},t)\exp(-i\underline{k}.\underline{x})d\underline{k} \qquad \text{A3.2}$$

since $\underline{E}_1$ is real.

The R.H.S.'s of A3.1, A3.2, are equal so

$$\overline{\underline{E}}(k)\exp i\omega(\underline{k})t = \overline{\underline{E}}^*(-\underline{k})\exp -i\omega^*(-\underline{k})t \quad \forall\ \underline{k},\ \forall\ t \qquad \text{A3.3}$$

Since this is true $\forall$ t, at t = 0, we have

$$\overline{\underline{E}}(\underline{k}) = \overline{\underline{E}}(-\underline{k})$$

but $\overline{\underline{E}}$ is independent of t so

$$\overline{\underline{E}}(\underline{k}) = \overline{\underline{E}}(-\underline{k}) \quad \forall\ \underline{k},\ \forall\ t$$

Thus $$\omega(\underline{k}) = -\omega^*(-\underline{k}) \qquad \text{from A3.3}$$

Hence

$$\underline{E}_1(\underline{k},t) = \overline{\underline{E}}(\underline{k})\exp i\omega(\underline{k})t = \overline{\underline{E}}^*(-\underline{k})\exp -i\omega^*(-\underline{k})t = \underline{E}_1{}^*(-\underline{k},t)$$

as required.

Appendix 4

We shall prove the identity

$$\nabla.(\underline{a}\underline{b}) = \underline{a}.(\nabla\underline{b}) + \underline{b}\nabla.\underline{a} \quad \forall \text{ vectors } \underline{a},\underline{b}$$

where the diad $\underline{a}\underline{b}$ means the tensor with components $a_i b_j$ for $i,j = 1,2,3$ with $\underline{a} = (a_1,a_2,a_3)$, $\underline{b} = (b_1,b_2,b_3)$.

This diadic notation has been used in Chapter 4 since it is so common in the plasma literature. However, the suffix notation is more lucid when proving this result.

Write $$\nabla.(\underline{a}\underline{b}) = \underline{e}_j \frac{\partial(a_i b_j)}{\partial x_i}$$

where $\underline{e}_j$ is the unit vector in the x_j direction and the summation convention is employed.

So $$\nabla.(\underline{a}\underline{b}) = \underline{e}_j \left(a_i \frac{\partial b_j}{\partial x_i} + b_j \frac{\partial a_i}{\partial x_i} \right)$$

$$= a_i \underline{e}_j \frac{\partial b_j}{\partial x_i} + b_j \underline{e}_j \frac{\partial a_i}{\partial x_i}$$

which, in diadic notation, is simply

$$\nabla.(\underline{a}\underline{b}) = \underline{a}.(\nabla\underline{b}) + \underline{b}(\nabla.\underline{a})$$

Appendix 5

Cauchy Integrals and Plemelj Formulae

The function h(p) on page 9 is a Cauchy integral. [17] i.e. it takes the form

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt$$

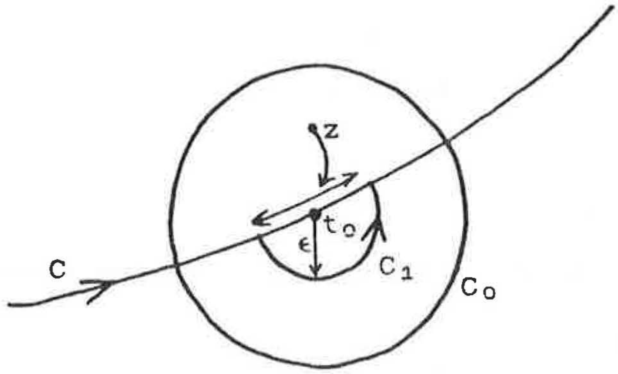
where C is some curve, or collection of curves, in the complex t plane, where f(t) is a complex valued function prescribed on C and where z is a point not on C.

For suitable curves C and functions f(t), F(z) will be analytic. Denote by t_0 a point on C, other than an end point. We wish to examine the behaviour of F(z) as $z \to t_0$ from the left and right (as viewed facing the direction of integration) by $F_+(t_0)$ and $F_-(t_0)$ respectively.
Define the principal value integral by

$$F_P(t_0) = \frac{1}{2\pi i}(P)\int_C \frac{f(t)}{t-t_0}dt = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{C-C_2} \frac{f(t)}{t-t_0}dt$$

where C_2 is the portion of the curve C contained within a small circle of radius ϵ, centred on t_0. We wish to obtain relations between $F_+(t_0)$, $F_-(t_0)$ and $F_P(t_0)$. We assume f(t) is analytic at t_0 (and continuous elsewhere). Then there is a small circle, C_0, centre t_0 such that f(t) is analytic within this circle. See Fig.23.

Fig.23



By Cauchy's theorem, C may be indented around t_o by use of a semi-circle of radius ϵ, centre t_o, lying within C_o. This allows us to carry out the limiting process in which z approaches t_o.

$$F_+(t_o) \equiv \lim_{z\to t_o} \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt$$

$$= \lim_{z\to t_o} \frac{1}{2\pi i}\int_{C+C_1-C_2} \frac{f(t)}{t-z}dt$$

$$= \frac{1}{2\pi i}\int_{C-C_2} \frac{f(t)}{t-t_o}dt + \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{t-t_o}dt$$

As $\epsilon\to 0$ we obtain

$$F_+(t_o) = F_P(t_o) + f(t_o)/2 \qquad \text{V.1}$$

If z was outside C we would need a small semi-circle whose integration sense is clockwise giving rise to a minus sign. Thus,

$$F_-(t_o) = F_P(t_o) - f(t_o)/2 \qquad \text{V.2}$$

V.1,V.2 are called the Plemelj formulae. Our derivation has assumed f(t) to be analytic at t_o, which is sufficient for our purposes, but weaker conditions on f(t) are possible.[18]

Appendix 6

Typical Parameter Sizes

Typical values for the following parameters for the calculations performed in Chapter 6 are

	Scudder et al [20]	D.Lepine [21]
n_o	$20\times10^6 m^{-3}$	$25\times10^6 m^{-3}$
u_1	$300\times10^3 ms^{-1}$	$600\times10^3 ms^{-1}$
V_d	$75\times10^3 ms^{-1}$	$50\times10^3 ms^{-1}$
κT_e	$5.5\times10^{-18} J$	$9.6\times10^{-18} J$
κT_i	$1.4\times10^{-18} J$	$6.4\times10^{-18} J$
ω_{pe}	$2.5\times10^5 rad/s$	$2.8\times10^5 rad/s$
v_{the}	$2.5\times10^6 ms^{-1}$	$3.3\times10^6 ms^{-1}$
L	15km	100km

We also have the absolute constants

$m_e = 9.1\times10^{-31} Kg$ $\quad$ $m_i = 1.67\times10^{-27} Kg$

$\kappa = 1.38\times10^{-23} J/K$ $\quad$ $e = 1.6\times10^{-19} C$

$\epsilon_o = 8.85\times10^{-12} C/Nm^2$

N.B. $1eV \equiv 1.6\times10^{-19} J$

Notice that the two data sets indicate that in reality these physical variables are by no means constant making it difficult in some instances to obtain reliable data.

Appendix 7

To evaluate $\displaystyle \mathop{\mathrm{C}\!\!\int}_{-\infty}^{\infty} \frac{\frac{\partial F_{eo}}{\partial u}}{u-v_p}\,du$ we split the integration into four regions

$$\int_{-\infty}^{-v_p} \, , \quad \int_{-v_p}^{v_p-\delta} \, , \quad \mathrm{C}\!\!\int_{v_p-\delta}^{v_p+\delta} \, , \quad \int_{v_p+\delta}^{\infty} \qquad \text{A7.1}$$

$$\text{(a)} \qquad \text{(b)} \qquad \text{(c)} \qquad \text{(d)}$$

where δ is a small but fixed constant. We shall aim to show that term (b) is the most significant.

Since $F_{eo} = \left(\frac{m_e}{2\pi\kappa T_e}\right)^{1/2} \exp\left(\frac{-m_e u^2}{2\kappa T_e}\right)$ term (a) is bounded above by

$$\mathrm{C}\!\!\int_{-\infty}^{-v_p} \frac{\frac{\partial F_{eo}}{\partial u}}{-v_p}\,du = \frac{1}{\surd(2\pi) v_p v_{the}} \exp(-v_p^2/2v_{the}^2)$$

The important fact to note is that this varies as $\exp(-v_p^2/v_{the}^2)$ and because we are dealing with the case $|v_p| >> |v_{the}|$, this term is very small. A similar argument applies to term (d). A more subtle approach is required for the Cauchy Principle Value in term (c).

Define $\quad g(u) = \frac{-u}{\surd(2\pi) v_{the}^3} \exp(-u^2/2v_{the}^2)$

Then term (c) is

$$\mathrm{C}\!\!\int_{v_p-\delta}^{v_p+\delta} \frac{g(u)}{u-v_p}du = \mathrm{C}\!\!\int_{v_p-\delta}^{v_p+\delta} \frac{g(v_p)}{u-v_p}du + \mathrm{C}\!\!\int_{v_p-\delta}^{v_p+\delta} \frac{g(u)-g(v_p)}{u-v_p}du \qquad \text{A7.2}$$

The first integral vanishes whilst for the second integral we can write

$$g(u) - g(v_p) = \frac{h(v_p)-h(u)}{\surd(2\pi) v_{the}^3} \exp(-v_p^2/2v_{the}^2)$$

where we have defined $\quad h(u) = u\exp\left(\frac{v_p^2-u^2}{2v_{the}^2}\right)$.

We are thus required to evaluate the Cauchy integral

$$\mathcal{C}\!\!\int_{v_p-\delta}^{v_p+\delta} \frac{h(u)-h(v_p)}{u-v_p}du$$

Now, h(u) certainly satisfies $|h(u)-h(v_p)| \le K|u-v_p|$ where K is a constant since h(u) is analytic. The second integral in A7.2 is bounded above by

$$\exp(-v_p^2/2v_{the}^2)\,\mathcal{C}\!\!\int_{v_p-\delta}^{v_p+\delta} L(u-v_p)du$$

where L is a constant dependentent on K. Note that this again varies as $\exp(-v_p^2/2v_{the}^2)$ so it is a small term.

For term (b) a Binomial expansion may be used to give

$$\frac{1}{\surd(2\pi)v_{the}^3}\int_{-v_p}^{v_p-\delta} \frac{u}{v_p}\left(1 + \frac{u}{v_p} + \frac{u^2}{v_p^2} + \frac{u^3}{v_p^3} + \ldots\right)\exp(-u^2/2v_{the}^2)du \qquad \text{A7.3}$$

Consider,

$$\int_{-v_p}^{v_p-\delta} \frac{u}{v_p}\exp(-u^2/2v_{the}^2)du = \left[-v_{the}^2\exp(-u^2/2v_{the}^2)\right]_{-v_p}^{v_p-\delta}$$

This again varies as $\exp(-v_p^2/2v_{the}^2)$ which is exponentially small.

However, the second term in A7.3 involves an integral

$$I = \int_{-v_p}^{v_p} u^2\exp(-u^2/2v_{the}^2)du = \left[-v_{the}^2 u\exp(-u^2/2v_{the}^2)\right]_{-v_p}^{v_p-\delta} + \int_{-v_p}^{v_p} v_{the}^2\exp(-u^2/2v_{the}^2)du$$

The integral on the R.H.S. dominates and may be evaluated using the substitution $w = u^2/2v_{the}^2$ to obtain

$$I \approx \sqrt{2v_{the}^2}\int_0^\infty w^{-1/2}\exp(-w)\,dw$$

where we have used the fact that v_p is large to obtain the integration limits.

So,

$$I = \sqrt{2v_{the}^2}\,\Gamma(1/2)$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\exp(-x)\,dx$.

This yields a result $1/v_p^2$ for term (b). A similar argument can be applied to the fourth order term in A7.3 so that 2.2.1 has been proven.

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