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Covariant Extremisation of Flavour-Symmetric Jarlskog Invariants and the Neutrino Mixing Matrix

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Abstract

We examine the possibility that the form of the lepton mixing matrix can be determined by extremising the Jarlskog flavour invariants associated, eg. with the commutator (C) of the lepton mass matrices. Introducing a strictly covariant approach, keeping masses fixed and extremising the determinant $(\mathrm{Tr}\, C^3/3)$ leads to maximal CP violation, while extremising the sum of the 2×2 principal minors $(-\mathrm{Tr}\, C^2/2)$, leads to a non-trivial mixing with zero CP violation. Extremising, by way of example, a general linear combination of two CP-symmetric invariants together, we show that our procedures can lead to acceptable mixings and to non-trivial predictions, eg. $|U_{e3}| \simeq \sqrt{2}/3 \; \sqrt{\Delta m_{12}^2/\Delta m_{23}^2}(1 - m_\mu/m_\tau)^2 \simeq 0.07$.

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1. Introduction

It is now almost twenty years since Jarlskog [1] first alerted us to the inherent U(3) invariance representing the freedom in any weak basis [2] to transform fermion mass matrices in the generation space, whilst keeping the charged-current weak-interaction diagonal and universal, ie. whilst always staying in a weak basis. While all observables (eg. masses and mixing angles) must of course be Jarlskog-invariant, we focus here on flavour-symmetric invariants, ie. those invariants without flavour indices of any kind (ie. involving no flavour projection operators in their definition).

While it has long been clear [3] that any fundamental laws underlying the masses and mixings should be Jarlskog-covariant (as defined below), this principle may not always be easy to respect in practice. In this paper we will show that extremisation of flavour-symmetric Jarlskog invariants [4] [5] leads naturally to Jarlskog-covariant constraints, which (as we shall see) can even have viable phenomenology.

The archetypal example of a flavour-symmetric Jarlskog invariant is the famous determinant of the commutator [6], which for the charged-lepton (L) and neutrino (N) mass matrices (taken here to be hermitian) may be written:

$$\text{Det } C = \text{Tr } C^3/3 = \text{Tr } i[L,N]^3/3 = -2 \text{ Det diag}(\Delta_l) \text{ Det diag}(\Delta_\nu) J \qquad (1)$$

$$\Delta_l^T = (m_\mu - m_\tau, \, m_\tau - m_e, \, m_e - m_\mu), \quad \Delta_\nu^T = (m_2 - m_3, \, m_3 - m_1, \, m_1 - m_2) \qquad (2)$$

where $C := -i[L,N]$, with m_e, m_μ, m_τ the charged-lepton masses and m_1, m_2, m_3 the neutrino masses, and with the invariant measure of CP violation [6] given by:

$$J = c_{12} \, s_{12} \, c_{23} \, s_{23} \, c_{13}^2 \, s_{13} \, s_\delta = (1 - s_{12}^2)^{\frac{1}{2}} \, s_{12} \, (1 - s_{23}^2)^{\frac{1}{2}} \, s_{23}(1 - s_{13}^2) \, s_{13} \, s_\delta. \qquad (3)$$

Extremising Eq. 3, with respect to the standard PDG [7] variables s_{12}, s_{23}, s_{13} and δ is known [8] to lead to $s_{12} = 1/\sqrt{2}$, $s_{23} = 1/\sqrt{2}$, $s_{13} = 1/\sqrt{3}$ and $\delta = \pi/2$, corresponding to maximal CP violation, ie. to trimaximal mixing [9].

It should be remarked that the assumption that the mass matrices are hermitian (see above) may in fact be realised in two distinct ways: one may simply imagine applying a suitable transformation to the right-handed fields (which are anyway inert to the charged-current weak-interaction) bringing L and N into hermitian form, or one may instead everywhere re-interpret L and N as hermitian squares of masss matrices $L \to LL^\dagger$, $N \to NN^\dagger$, operating between left-handed fields. In the latter case, masses need to be replaced by masses-squared throughout ($m_{l/\nu} \to m_{l/\nu}^2$, eg. in Eqs. 1-2). In this paper, in order to circumvent this as yet unresolved ambiguity, we shall give results, where appropriate, for both the linear and quadratic cases.

Of course we now know that lepton mixing is not trimaximal, it being actually much closer to the so-called tri-bimaximal form [10] [11]. The question then arises, as

to whether extremising some other invariant (or combination of invariants) aside from $\mathrm{Tr}\, C^3$, might perhaps yield a more realistic mixing prediction.

In this paper we seek to kindle interest in these kinds of questions, and we start out (Section 2) by re-deriving the results for $\mathrm{Tr}\, C^3$ above, introducing and bringing to bear a strictly covariant (basis-independent) approach. We focus here on the leptonic case: in the context of extremisation, quasi-maximal mixing in the lepton sector looks a priori more comprehensible than anything involving the quarks (which are anyway subject to larger radiative corrections, eg. dependent on the top mass).

We proceed (Section 3) with a similar covariant extremisation of the only other independent invariant depending on C alone, ie. $\mathrm{Tr}\, C^2$. The mixing prediction is now CP-conserving and, whilst not phenomenologically viable as it stands, is non-trivial, with even a suggestive resemblance to some previously proposed lepton-mixing ansatze.

Finally, taking a particular linear combination of two CP-symmetric invariants as an illustrative example, we show (Section 4) how covariant extremisation of slightly more general combinations of invariants can readily lead to realisic mixing predictions.

Our "Perspective" section (Section 5) simply consolidates what we have learned, also briefly pointing forward to what one might hope to learn in future studies, eg. in terms of incorporating CP violation, dropping mass constraints etc.

2. Extremising the Jarlskog Determinant, $\mathrm{Tr}\, C^3/3$

One might argue that the PDG variables are arbitrary, eg. it matters (Eq. 3) that we chose to extremise with respect to δ and not $s_\delta = \sin\delta$. We choose to extremise with respect to the Yukawa couplings themselves (ie. with respect to the mass matrices) using some established results [13] on differentiation of matrix traces with respect to matrix variables (see Appendix A). In terms of matrix derivatives $\partial_L := \partial/\partial L$ etc., the extremisation conditions for, eg. $\mathrm{Tr}\, C^3$ may then be written (see Appendix A) in a manifestly Jarlskog-covariant (ie. basis-independent) way:

$$\partial_L \,\mathrm{Tr}\, C^3/3 \;=\; -i[N, C^2]^T = 0 \tag{4}$$

$$\partial_N \,\mathrm{Tr}\, C^3/3 \;=\; +i[L, C^2]^T = 0, \tag{5}$$

in that both sides of the equation are evidently form-invariant under an arbitrary $U(3)$ transformation $L \to U(3)L\,U(3)^\dagger$, $N \to U(3)N\,U(3)^\dagger$, ie. we would obtain the same equations performing the extremisation in any basis. When the masses are to be held fixed, the zeros on the RHS of Eqs. 4-5 must be replaced by arbitrary matrix polynomials in L^T and N^T respectively, representing the dependence on unknown Lagrange multipliers (see below), with the manifest covariance clearly maintained.

The covariance property allows us to solve our equations in any basis, so we choose a basis convenient to us where the charged-lepton mass matrix is diagonal and where all imaginary parts of the off-diagonal elements of the neutrino mass matrix are equal (the 'epsilon' basis [12]). The neutrino mass matrix may then be written:

$$
N = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}
\begin{pmatrix}
\begin{array}{ccc}
e & \mu & \tau
\end{array} \\
\begin{pmatrix}
a & z+id & y-id \\
z-id & b & x+id \\
y+id & x-id & c
\end{pmatrix}
\end{pmatrix}
\tag{6}
$$

where the seven variables a, b, c, x, y, z and d determine the three neutrino masses and the four mixing parameters (the charged-lepton masses being directly the diagonal elements of L). In terms of these variables the Jarlskog determinant is given by:

$$
\text{Det } C = \text{Tr } C^3/3 = 2d(m_e - m_\mu)(m_\mu - m_\tau)(m_\tau - m_e)(d^2 - xy - yz - zx) \tag{7}
$$

so that CP violation vanishes in the case $d = 0$ and also in the case $d^2 = xy + yz + zx$.

The off-diagonal elements of $-i[N, C^2]$ must be set to zero for an extremum (Eq. 4), leading to the following cyclically symmetric constraints, from the real (Re) parts:

$$
\text{Re} \quad
\begin{aligned}
-d\,(m_e - m_\mu)\,(m_\tau - m_e)\,\big((y-z) - (b-c)\big)\,(y+z) &= 0 \\
-d\,(m_\mu - m_\tau)\,(m_e - m_\mu)\,\big((z-x) - (c-a)\big)\,(z+x) &= 0 \\
-d\,(m_\tau - m_e)\,(m_\mu - m_\tau)\,\big((x-y) - (a-b)\big)\,(x+y) &= 0
\end{aligned}
\tag{8}
$$

and from the imaginary (Im) parts:

$$
\text{Im} \quad
\begin{aligned}
-(m_e - m_\mu)\,(m_\tau - m_e)\,\big((b-c)\,(d^2 - yz) - (y^2 - z^2)\,x\big) &= 0 \\
-(m_\mu - m_\tau)\,(m_e - m_\mu)\,\big((c-a)\,(d^2 - zx) - (z^2 - x^2)\,y\big) &= 0 \\
-(m_\tau - m_e)\,(m_\mu - m_\tau)\,\big((a-b)\,(d^2 - xy) - (x^2 - y^2)\,z\big) &= 0.
\end{aligned}
\tag{9}
$$

It should be remarked that the cyclic symmetry $(e \to \mu \to \tau,\ a \to b \to c,\ x \to y \to z)$ will be seen to be a useful and important generic feature of our approach, resulting from starting with flavour-symmetric invariants and working in the epsilon basis.

A non-trivial solution to Eq. 8, ie. that corresponding to the real parts above (putting aside the $d = 0$ possibility for one moment), may be written:

$$
\begin{array}{lll}
(x - y) = (a - b) & & a = \sigma + x \\
(y - z) = (b - c) & \text{i.e.} & b = \sigma + y \\
(z - x) = (c - a) & & c = \sigma + z
\end{array}
\tag{10}
$$

where σ is an undetermined overall constant offset. Eq. 10 is precisely the 'S3 invariant constraint' [14] (or 'magic square constraint' [12]) whereby the neutrino mass matrix

is determined to be an 'S3 group matrix' [14], ie. a matrix having all row/column sums equal and hence (at least) one trimaximal eigenvector.

Turning to Eq. 9 (corresponding to the imaginary parts) a non-trivial solution is:

$$
\begin{aligned}
a = b &\qquad \text{and} \qquad & x = y \\
b = c &\qquad \text{and} \qquad & y = z \\
c = a &\qquad \text{and} \qquad & z = x
\end{aligned}
\tag{11}
$$

whereby the mass matrix is circulant [4] and the mixing takes the trimaximal form [9] (Eq. 10 being also satisfied). More trivially, coming to the $d = 0$ case (above), we have that twofold-maximal mixing, eg. $b = c$, $y = z = 0$ also satisfies both sets of equations.

Of course, these solutions do not solve *all* the extremisation equations, Eqs. 4-5. It transpires however, that in the case that the masses are held fixed, all the remaining equations serve only to determine suitable Lagrange multipliers λ_{Li}, λ_{Ni} ($i = 0, 1, 2$):

$$
(\partial_L \operatorname{Tr} C^3/3)^T = -i[N, C^2] = \lambda_{L0} + \lambda_{L1} L + \lambda_{L2} L^2
\tag{12}
$$
$$
(\partial_N \operatorname{Tr} C^3/3)^T = +i[L, C^2] = \lambda_{N0} + \lambda_{N1} N + \lambda_{N2} N^2
\tag{13}
$$

with the apparently excess constraints turning out to be redundant (and furthermore with this circumstance occuring for all the extremisations studied in this paper).

In the epsilon basis, L is diagonal and any polynomial in L is diagonal also, whereby Eqs. 8-9 (and their solutions, eg. Eqs. 10-11) remain valid, even with the Lagrange multipliers (Eq. 12). In addition, the *on*-diagonal elements of $+i[L, C^2]$ vanish (as in Eq. 5) due to L being diagonal, and in the case of non-zero Lagrange multipliers (Eq. 13) the consistency requirement on the coefficients of the λ_{Ni} takes the form:

$$
\begin{vmatrix}
1 & a & a^2 + y^2 + z^2 \\
1 & b & x^2 + b^2 + z^2 \\
1 & c & x^2 + y^2 + c^2
\end{vmatrix} = 0.
\tag{14}
$$

All our solutions above do indeed turn out to satisfy this determinant condition. (In fact *any* $S3$ group matrix, Eq. 10, automatically satisfies this condition, Eq. 14).

Covariance implies that the $\lambda_{Li/Ni}$ themselves will be Jarlskog scalars, expressible in terms of, eg. traces of powers of mass matrices:

$$
\begin{aligned}
L_1 &:= \operatorname{Tr} L = m_e + m_\mu + m_\tau &\qquad N_1 &:= \operatorname{Tr} N = m_1 + m_2 + m_3 \\
L_2 &:= \operatorname{Tr} L^2 = m_e^2 + m_\mu^2 + m_\tau^2 &\qquad N_2 &:= \operatorname{Tr} N^2 = m_1^2 + m_2^2 + m_3^2 \\
L_3 &:= \operatorname{Tr} L^3 = m_e^3 + m_\mu^3 + m_\tau^3 &\qquad N_3 &:= \operatorname{Tr} N^3 = m_1^3 + m_2^3 + m_3^3
\end{aligned}
\tag{15}
$$

or traces of commutators of mass matrices etc. Note that the definitions (Eq. 15) are our 'constraint equations', eg. $\operatorname{Tr} L^2 = L_2$ for fixed L_2 etc., which when differentiated, $\partial_L (\operatorname{Tr} L^2 - L_2) = 2L^T$ etc., lead to the polynomial forms on the RHS of Eqs. 12-13.

The on-diagonal elements of Eq. 12 (which are purely real) are given by:

$$2d\,(m_\mu + m_\tau - 2m_e)\,(m_\mu - m_\tau)\,(d^2 - x^2 - y^2 - z^2) \;=\; \lambda_{L0} \;+\; \lambda_{L1}m_e \;+\; \lambda_{L2}m_e^2$$
$$2d\,(m_\tau + m_e - 2m_\mu)\,(m_\tau - m_e)\,(d^2 - x^2 - y^2 - z^2) \;=\; \lambda_{L0} \;+\; \lambda_{L1}m_\mu \;+\; \lambda_{L2}m_\mu^2 \quad (16)$$
$$2d\,(m_e + m_\mu - 2m_\tau)\,(m_e - m_\mu)\,(d^2 - x^2 - y^2 - z^2) \;=\; \lambda_{L0} \;+\; \lambda_{L1}m_\tau \;+\; \lambda_{L2}m_\tau^2.$$

Solving for the λ_{Li} gives the charged-lepton discriminant L_Δ^2 in the denominator of the solution (from the determinant-of-coefficients, Eq. 16, RHS), where L_Δ is given by:

$$L_\Delta \;:=\; \sqrt{L_2^3/2 + 3L_1^4 L_2/2 + 6L_1 L_2 L_3 - 7L_1^2 L_2^2/2 - 3L_3^2 - 4L_1^3 L_3/3 - L_1^6/6}$$
$$=\; (m_e - m_\mu)(m_\mu - m_\tau)(m_\tau - m_e). \quad (17)$$

After some work to cast the numerators also into invariant form, we find that:

$$\lambda_{L0} \;=\; \frac{-L_1^5/2 + 3L_1^3 L_2 - 7L_1 L_2^2/2 - 2L_1^2 L_3 + 3L_2 L_3}{3L_\Delta^2}\,\mathrm{Tr}\,C^3 \quad (18)$$

$$\lambda_{L1} \;=\; \frac{3L_1^4/2 - 7L_1^2 L_2 + 3L_2^2/2 + 6L_1 L_3}{L_\Delta^2}\,\mathrm{Tr}\,C^3 \quad (19)$$

$$\lambda_{L2} \;=\; \frac{-9L_3 + 9L_1 L_2 - 2L_1^3}{3L_\Delta^2}\,\mathrm{Tr}\,C^3. \quad (20)$$

Of course, these expressions can also be obtained in a fully covariant way simply by multiplying Eq. 12 by successive powers of L ($I = L^0$, $L = L^1$, L^2) and taking traces (powers higher than L^2 must always be reduced using the characteristic equation). Analogous $L \leftrightarrow N$ expressions for the λ_{Ni} solve all the remaining equations.

Recognising the coefficients of $\mathrm{Tr}\,C^3$ on the RHS above as $\partial_{L/N}(L_\Delta N_\Delta)/(L_\Delta N_\Delta)$, we see that we have, in effect, extremised the Jarlskog invariant $J = \mathrm{Tr}C^3/(L_\Delta N_\Delta)$, recovering the known result [8], and validating our procedure. Finally, we note that the above analysis is essentially unchanged, taking L and N as the hermitian squares of mass matrices (see Section 1) substituting masses-squared for masses throughout.

3. The Sum of the 2×2 Principal Minors, $-\mathrm{Tr}\,C^2/2$

The other independent invariant function of C may be taken to be the sum of the 2×2 principal minors, expressible in terms of the K-matrix ($K_{l\nu} := -K_{l'l''}^{\nu'\nu''}$ [15]) as follows:

$$Q_{11} = -\mathrm{Tr}\,C^2/2 = \mathrm{Tr}\,[L, N]^2/2 = \Delta_l^T\,\mathrm{diag}(\Delta_l)\,K\,\mathrm{diag}(\Delta_\nu)\,\Delta_\nu. \quad (21)$$

The K-matrix is a key oscillation observable [15], carrying equivalent information to the moduli-squared of the mixing elements [16]. The K-matrix comprises the real parts of the plaquette products [17]: $\Pi_{l\nu} := -K_{l\nu} + iJ$ (indices to be interpreted mod 3) and may be viewed as the CP-conserving analogue of the Jarlskog invariant J [6].

From Eq. 21, taking for simplicity initially the extreme hierarchical approximation m_e, m_μ, m_1, $m_2 \to 0$ (ie. the 2×2 mixing limit), we have:

$$Q_{11} \to m_\tau^2 m_3^2 \left(K_{e1} + K_{e2} + K_{\mu1} + K_{\mu2} \right) + O(m_\tau^2 m_2^2) + \ldots$$
$$= m_\tau^2 m_3^2 \, c_{13}^2 c_{23}^2 (s_{13}^2 + c_{13}^2 s_{23}^2) + O(m_\tau^2 m_2^2) + \ldots \tag{22}$$

Extremising this expression with respect to s_{23} and s_{13} gives $s_{23} = s_{13} = 0$ or $s_{23} = s_{13} = 1$ or $c_{23} c_{13} = 1/\sqrt{2}$ (where the latter looks at first sight very promising phenomenologically, see below). Extremising rather with respect to θ_{23} and θ_{13} gives in addition $s_{23} = 1$ or $s_{13} = 1$ as solutions, for arbitrary s_{13} and s_{23} respectively. Since mixing matrices which are just permutation matrices give no mixing as such, we have that 2×2 maximal mixing is the only non-zero mixing solution in the 2×2 case.

Just as for Tr C^3 above, however, we will extremise exactly here with respect to the mass matrices (see again Appendix A):

$$(-\partial_L \operatorname{Tr} C^2/2)^T = +[N, [L, N]] = 0, \quad \lambda_{L0} + \lambda_{L1} L + \lambda_{L2} L^2 \tag{23}$$
$$(-\partial_N \operatorname{Tr} C^2/2)^T = -[L, [L, N]] = 0, \quad \lambda_{N0} + \lambda_{N1} N + \lambda_{N2} N^2 \tag{24}$$

where the matrix polynomials replace the RHS zeros, in the case of mass constraints. In the epsilon basis the off-diagonal elements of Eq. 23 are, for the real parts:

$$\text{Re} \quad \begin{aligned} (m_\mu + m_\tau - 2m_e)\,(d^2 - y\,z) \; + \; (m_\mu - m_\tau)\,(b - c)\,x \;&=\; 0 \\ (m_\tau + m_e - 2m_\mu)\,(d^2 - z\,x) \; + \; (m_\tau - m_e)\,(c - a)\,y \;&=\; 0 \\ (m_e + m_\mu - 2m_\tau)\,(d^2 - x\,y) \; + \; (m_e - m_\mu)\,(a - b)\,z \;&=\; 0 \end{aligned} \tag{25}$$

and for the imaginary parts:

$$\text{Im} \quad \begin{aligned} d\,(m_\mu + m_\tau - 2m_e)\,(y + z) \; + \; d\,(m_\mu - m_\tau)\,(b - c) \;&=\; 0 \\ d\,(m_\tau + m_e - 2m_\mu)\,(z + x) \; + \; d\,(m_\tau - m_e)\,(c - a) \;&=\; 0 \\ d\,(m_e + m_\mu - 2m_\tau)\,(x + y) \; + \; d\,(m_e - m_\mu)\,(a - b) \;&=\; 0. \end{aligned} \tag{26}$$

We observe that Eqs. 25-26 imply that either $d = 0$ or $d^2 = xy + yz + zx$, and so from Eq. 7 we see immediately that there can be no CP violation in this case.

It turns out that the $d^2 = xy + yz + zx$ solution is equivalent to the $d = 0$ solution, so that we need to consider only the $d = 0$ case (note that in the case $d^2 = xy + yz + zx$ the epsilon basis is not unique, and the imaginary part can be rephased to zero). Setting $d = 0$ then, solves Eq. 26, corresponding to the imaginary parts. Clearly, setting any pair of x, y, z to zero and the corresponding pair of a, b, c equal, solves also Eq. 25 for the real parts, so that, just as for Tr C^3 above, 2×2 maximal mixing in any sector gives an exact extremum of Tr C^2, independent of the neutrino mass spectrum.

A somewhat more interesting non-trivial solution to Eq. 25 (still with $d = 0$ from Eq. 26) clearly reflects the inherent cyclic symmetry:

$$x = \pm\sqrt{(a-b)(c-a)E} \qquad E = \frac{(m_e - m_\mu)(m_\tau - m_e)}{(m_e + m_\mu - 2m_\tau)(m_\tau + m_e - 2m_\mu)}$$

$$y = \pm\sqrt{(b-c)(a-b)M} \qquad M = \frac{(m_\mu - m_\tau)(m_e - m_\mu)}{(m_\mu + m_\tau - 2m_e)(m_e + m_\mu - 2m_\tau)} \qquad (27)$$

$$z = \pm\sqrt{(c-a)(b-c)T} \qquad T = \frac{(m_\tau - m_e)(m_\mu - m_\tau)}{(m_\tau + m_e - 2m_\mu)(m_\mu + m_\tau - 2m_e)}.$$

With the charged-lepton masses positive, we have $E > 0 > M, T$, whereby either $b < a < c$ or $c < a < b$. We may take all upper (positive) signs in the case $b < a < c$, corresponding to an inverted hierarchy, and all lower (negative) signs in the case $c < a < b$, corresponding to a normal hierarchy (as considered below) with other sign combinations being just trivial rephasings of these two possibilities. For these solutions the determinant condition is again automatically satisfied, since $E + M + T + 1 = 0$ appears as a factor in the determinant Eq. 14.

As in the case of $\mathrm{Tr}\, C^3$ above, the remaining constraints (from Eqs. 23-24) will serve only to fix the Lagrange multipliers. While we again have the option to determine the Lagrange multiplers in a general and fully covariant way, it will prove useful as before to consider eg. the on-diagonal ∂_L equations (which are again purely real) explicitly:

$$
\begin{aligned}
2(m_\tau - m_e)y^2 - 2(m_e - m_\mu)z^2 &= \lambda_{L0} + \lambda_{L1}\, m_e + \lambda_{L2}\, m_e^2 \\
2(m_e - m_\mu)z^2 - 2(m_\mu - m_\tau)x^2 &= \lambda_{L0} + \lambda_{L1}\, m_\mu + \lambda_{L2}\, m_\mu^2 \qquad (28)\\
2(m_\mu - m_\tau)x^2 - 2(m_\tau - m_e)y^2 &= \lambda_{L0} + \lambda_{L1}\, m_\tau + \lambda_{L2}\, m_\tau^2.
\end{aligned}
$$

Since Lagrange multipliers need in general only be evaluated locally at the extremum, we now have the possibility to achieve some simplification by substituting (on the LHS of Eq. 28) the specific solutions Eq. 27, before solving for the λ_{Li} and again casting the resulting expressions (curly brackets denote anticommutators) in invariant form:

$$\lambda_{L0} = \frac{(\mathrm{Tr}\,\{L^2, N\}\, L_1 - \mathrm{Tr}\,\{L, N\}\, L_2)(3\mathrm{Tr}\,\{L, N\} - 2L_1 N_1)}{2(9L_3 - 9L_1 L_2 + 2L_1^3)} \qquad (29)$$

$$\lambda_{L1} = \frac{-(3\mathrm{Tr}\,\{L^2, N\} - 2L_2 N_1)(3\mathrm{Tr}\,\{L, N\} - 2L_1 N_1)}{2(9L_3 - 9L_1 L_2 + 2L_1^3)} \qquad (30)$$

$$\lambda_{L2} = \frac{(3\mathrm{Tr}\,\{L, N\} - 2L_1 N_1)^2}{2(9L_3 - 9L_1 L_2 + 2L_1^3)}. \qquad (31)$$

(Similar expressions also result for $L \leftrightarrow N$, as before). While this means that our Lagrange multipliers are now solution-specific (ie. they fail for the mass-independent

2×2 maximal-mixing solution above), one at least has that Eqs. 23-24 and Eqs. 29-31 (with their $L \leftrightarrow N$ counterparts) together constitute a covariant statement of specifically the non-trivial (cyclically symmetric) solution given by Eq. 27.

From Eq. 27 the mixing is uniquely predicted, in essence by fitting a, b and c (Eq. 6) to give the correct neutrino mass spectrum. Of course absolute neutrino masses (as distinct from mass-squared differences) are not yet known directly, but we may still obtain a specific prediction if we are prepared to assume a normal (ie. non-inverted) 'classic' fermion mass hierarchy $m_1 << m_2 << m_3$ (ie. with no offset, $m_1 \simeq 0$).

We then find that an acceptable neutrino mass-squared difference hierarchy ($h_\nu^2 := \Delta m_{12}^2/\Delta m_{23}^2 \simeq m_2^2/m_3^2 \simeq 0.03$) results, simply by setting the ratio $(b-a)/(a-c) = 1$, corresponding, it turns out, to the ν_e being exactly trimaximally mixed (this ratio is in fact the only operative parameter here, since rescaling and off-setting of a, b and c clearly does not influence the mixing matrix). The resulting $|U|^2$-matrix (ie. the matrix of moduli-squared of the resulting mixing-matrix elements) is given numerically by:

$$(|U_{l\nu}|^2) = \begin{matrix} & \nu_1 & \nu_2 & \nu_3 \\ e \\ \mu \\ \tau \end{matrix} \begin{pmatrix} .33333 & .33333 & .33333 \\ .17079 & .16257 & .66663 \\ .49587 & .50409 & .00003 \end{pmatrix} \simeq \begin{matrix} & \nu_1 & \nu_2 & \nu_3 \\ e \\ \mu \\ \tau \end{matrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix}. \quad (32)$$

While being closely a transpose/permutation of the tri-bimaximal form, and bearing a clear relation to some previously proposed ansatze (see in particular Acker et al. [18] and Karl et al. [19] as well as the Fritzsch-Xing ansatz [20]), this mixing (Eq. 32) clearly cannot at this point be considered acceptable phenomenologically.

We find that the situation is not improved substituting masses-squared in place of masses throughout. Indeed the large discrepancy in $|U_{\tau 3}|$ is considerably worsened, with $|U_{\tau 3}|^2 \sim 3 \times 10^{-9}$ (maintaining the mass hierarchy unchanged requires setting $(b-a)/(a-c) \sim 0.36$). Having also explored less minimalist assumptions regarding the neutrino mass-spectrum, ie. inverted mass hierarchy, large mass offsets etc. we can report no phenomenlogically acceptable solutions to the problem of extremising Tr C^2. For $m_1 << m_2 << m_3$, we see that all our exact solutions are consistent with the (approximate) constraints derived differentiating Eq. 22 with respect to PDG angles (although we learn that the initially promising $U_{\tau 3} = c_{23}c_{13} = 1/\sqrt{2}$ condition in fact corresponds only to phenomenologically uninteresting 2×2 maximal mixing solutions).

4. Extremising More General Mixing Invariants

More general commutator invariants are readily constructed, involving higher powers of mass matrices [21]. In this section we shall extremise a very simple composite

function, comprising an arbitrary linear combination of two such invariants:

$$A := Q_{11} + qQ_{21} \tag{33}$$

with q a scalar constant of suitable dimensionality (assumed Jarlskog-invariant) and:

$$Q_{11} := \text{Tr}\,[L, N]\,[L, N]\,/\,2 \qquad Q_{21} := \text{Tr}\,[L, N]\,[L^2, N]\,/\,2, \tag{34}$$

our Q_{mn} notation being essentially self-explanatory [1] at this point.

Of course, Q_{11} has already been individually extremised in Section 3 (Eqs. 23-24), so we only need in addition here the relevant derivatives of Q_{21} (see Appendix A):

$$(\partial Q_{21}/\partial L)^T = [N, [L^2, N]]/2 + \{L, [N, [L, N]]\}/2 \tag{38}$$
$$(\partial Q_{21}/\partial N)^T = -[L^2, [L, N]] \tag{39}$$

where again curly brackets signify the anti-commutator.

The off-diagonal terms derived from Q_{21}, analogously to Eq. 25, are (Re parts):

$$(m_\mu^2 + m_\tau^2 + m_\mu m_\tau - m_\tau m_e - m_e m_\mu - m_e^2)\,(d^2 - y\,z)\ +\ (m_\mu^2 - m_\tau^2)\,(b - c)\,x$$
$$(m_\tau^2 + m_e^2 + m_\tau m_e - m_e m_\mu - m_\mu m_\tau - m_\mu^2)\,(d^2 - z\,x)\ +\ (m_\tau^2 - m_e^2)\,(c - a)\,y \tag{40}$$
$$(m_e^2 + m_\mu^2 + m_e m_\mu - m_\mu m_\tau - m_\tau m_e - m_\tau^2)\,(d^2 - x\,y)\ +\ (m_e^2 - m_\mu^2)\,(a - b)\,z$$

and (Im parts):

$$d\,(m_\mu^2 + m_\tau^2 + m_\mu m_\tau - m_\tau m_e - m_e m_\mu - m_e^2)\,(y + z)\ +\ d\,(m_\mu^2 - m_\tau^2)\,(b - c)$$
$$d\,(m_\tau^2 + m_e^2 + m_\tau m_e - m_e m_\mu - m_\mu m_\tau - m_\mu^2)\,(z + x)\ +\ d\,(m_\tau^2 - m_e^2)\,(c - a) \tag{41}$$
$$d\,(m_e^2 + m_\mu^2 + m_e m_\mu - m_\mu m_\tau - m_\tau m_e - m_\tau^2)\,(x + y)\ +\ d\,(m_e^2 - m_\mu^2)\,(a - b)$$

and the combined constraint equations are obtained simply by adding these terms (weighted by q) into Eqs. 25-26.

---

[1]We are considering here flavour-symmetric quadratic commutator invariants (vanishing in the case of zero mixing) which may be usefully arranged as a 3×3 matrix:

$$Q = \frac{1}{2}\begin{pmatrix} \text{Tr}\,[L, N]^2 & \text{Tr}\,[L, N][L, N^2] & \text{Tr}\,[L, N^2]^2 \\ \text{Tr}\,[L, N][L^2, N] & \text{Tr}\,[L, N][L^2, N^2] & \text{Tr}\,[L, N^2][L^2, N^2] \\ \text{Tr}\,[L^2, N]^2 & \text{Tr}\,[L^2, N][L^2, N^2] & \text{Tr}\,[L^2, N^2]^2 \end{pmatrix}. \tag{35}$$

As usual, powers of mass matrices higher than L^2, N^2 are not considered, by virtue of the characteristic equation (and $\text{Tr}\,[L^2, N]\,[L, N^2] \equiv \text{Tr}\,[L, N]\,[L^2, N^2]$, so there are indeed only nine such invariants). Generalising Eq. 21, the Q_{mn} ($m,n = 1, 2, 3$) are the double moments of the K-matrix:

$$Q_{mn} = \Delta_l^T\,\text{diag}(\Delta_l)\,(\text{diag}\,\Sigma_l)^{m-1}\,K\,(\text{diag}\,\Sigma_\nu)^{n-1}\,\text{diag}(\Delta_\nu)\,\Delta_\nu \tag{36}$$
$$\Sigma_l = (m_\mu + m_\tau,\ m_\tau + m_e,\ m_e + m_\mu) \qquad \Sigma_\nu = (m_2 + m_3,\ m_3 + m_1,\ m_1 + m_2). \tag{37}$$

For known masses, the Q-matrix and the K-matrix are therefore equivalent, and both are equivalent to the $|U_{l\nu}|^2$-matrix (see [15] [16]), with only four elements functionally independent in each case.

10

As in Section 3 we may again set $d = 0$, and the solutions to these combined equations are then readily written down from Eqs. 27, by modifying the expressions for E, M and T in Eq. 27, making the (cyclically-symmetric) substitutions:

$$
\begin{aligned}
(m_e - m_\mu) &\rightarrow (m_e - m_\mu) + q(m_e^2 - m_\mu^2) \\
(m_\mu - m_\tau) &\rightarrow (m_\mu - m_\tau) + q(m_\mu^2 - m_\tau^2) \\
(m_\tau - m_e) &\rightarrow (m_\tau - m_e) + q(m_\tau^2 - m_e^2)
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
(m_e + m_\mu - 2m_\tau) &\rightarrow (m_e + m_\mu - 2m_\tau) + q(m_e^2 + m_\mu^2 + m_e m_\mu - m_\tau L_1) \\
(m_\mu + m_\tau - 2m_e) &\rightarrow (m_\mu + m_\tau - 2m_e) + q(m_\mu^2 + m_\tau^2 + m_\mu m_\tau - m_e L_1) \\
(m_\tau + m_e - 2m_\mu) &\rightarrow (m_\tau + m_e - 2m_\mu) + q(m_\tau^2 + m_e^2 + m_\tau m_e - m_\mu L_1).
\end{aligned}
\tag{43}
$$

The determinant condition Eq. 14 remains valid since the equation $E + M + T + 1 = 0$ (see text just after Eq. 27) itself remains valid under these substitutions.

Fitting to the 'mass' spectrum as in Section 3, we now have (replacing the unique mixing prediction Eq. 32) a trajectory of mixings, each mixing given by a different value of the parameter q. To locate the phenomenologically viable solution along this trajectory (supposing even that there is a viable solution), we will simply impose the "$S3$ constraint" (Eq. 10), which although not forced on us here by the extremisation, is well-known [12] [14] to be consistent with the phenomenology. We shall see below how the $S3$-constraint (by its nature) can be imposed in a fully covariant way.

In practice, we shall substitute the $S3$ constraint Eq. 10 into the solution Eq. 27 (as modified by the substitutions Eqs. 42-43), thereby fixing all the parameters in the neutrino mass matrix, as functions of q, as follows:

$$
\begin{aligned}
a &= x + \sigma & x &= \frac{k^3}{(m_\mu - m_\tau)^2(1 + q(m_\mu + m_\tau))} \\
b &= y + \sigma & y &= \frac{k^3}{(m_\tau - m_e)^2(1 + q(m_\tau + m_e))} \\
c &= z + \sigma & z &= \frac{k^3}{(m_e - m_\mu)^2(1 + q(m_e + m_\mu))}
\end{aligned}
\tag{44}
$$

where the mixing clearly cannot depend on the scale factor k or the offset σ.

Assuming a normal classic neutrino mass hierarchy as before, the operative parameter is now q, fixing both the hierarchy ratio $h_\nu^2 := \Delta m_{12}^2 / \Delta m_{23}^2 \simeq m_2^2 / m_3^2$ and the mixing simultaneously. We see immediately from Eq. 44 that, eg. as $q \rightarrow -1/(m_\mu + m_\tau)$ we have a pole where $x \rightarrow -\infty$ while the hierarchy factor $h_\nu \rightarrow 0$, with the mixing approaching the exact tri-bimaximal form [10] [11] in the limit.

Parametrising deviations from this pole in the form $q = -(1 + \epsilon)/(m_\mu + m_\tau)$, we find: $\epsilon \simeq 4/3\, h_l^2 h_\nu (1 + h_\nu/3 + 8h_\nu^2/9 + \ldots)$. For $h_\nu^2 \simeq 0.03$ (with $h_l := m_\mu/m_\tau \simeq 0.06$)

we have $\epsilon \simeq (0.03)^2$, and taking $k \simeq 0.38$ meV$^{1/3}$GeV$^{2/3}$ and $\sigma \simeq 25$ meV, gives a neutrino mass spectrum $(m_1, m_2, m_3) \simeq (0, 8.7, 50)$ meV, and a mixing matrix:

$$
(|U_{l\nu}|^2) = \begin{matrix} & \nu_1 & \nu_2 & \nu_3 \\ e \\ \mu \\ \tau \end{matrix}\!\!\left(\begin{matrix} 0.662 & 0.333 & 0.005 \\ 0.219 & 0.333 & 0.448 \\ 0.120 & 0.333 & 0.547 \end{matrix}\right) \simeq \begin{matrix} & \nu_1 & \nu_2 & \nu_3 \\ e \\ \mu \\ \tau \end{matrix}\!\!\left(\begin{matrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{matrix}\right). \quad (45)
$$

The mixing Eq. 45, being very close to the tribimaximal form, is certainly acceptable phenomenologically. There is the non-trivial prediction $|U_{e3}| \simeq \sqrt{2}/3\, h_\nu(1-h_l)^2 \simeq 0.07$ (cf. Ref. [12]) and violations of $\mu - \tau$ symmetry [22] also, eg. $\theta_{23} \simeq \pi/4 - |U_{e3}|/\sqrt{2}$. Related poles in y and z, for which the mixing matrix would have rows correspondingly permuted with respect to Eq. 45, would seem not to be phenomenologically relevant.

Repeating the analysis with masses-squared, in effect takes us closer to the pole (now $\epsilon \sim 5 \times 10^{-7}$) yielding a mixing correspondingly closer to the tri-bimaximal form:

$$
(|U_{l\nu}|^2) = \begin{matrix} & \nu_1 & \nu_2 & \nu_3 \\ e \\ \mu \\ \tau \end{matrix}\!\!\left(\begin{matrix} 0.6665 & 0.3333 & 0.0002 \\ 0.1768 & 0.3333 & 0.4898 \\ 0.1567 & 0.3333 & 0.5100 \end{matrix}\right) \simeq \begin{matrix} & \nu_1 & \nu_2 & \nu_3 \\ e \\ \mu \\ \tau \end{matrix}\!\!\left(\begin{matrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{matrix}\right). \quad (46)
$$

The corresponding prediction for $|U_{e3}|$ (now $|U_{e3}| \sim \sqrt{2}/3\, h_\nu^2 \sim 0.014$) is clearly of less immediate phenomenological interest than that in the unsquared case above.

Lagrange multipliers for Eq. 33 are found in the usual way. For the ∂_L equations, solving the analogue of Eq. 28 (after the substitutions Eq. 42), we find:

$$
\lambda_{L0} = N_1^2 \frac{\lambda_{L00} + \lambda_{L01}q + \lambda_{L02}q^2 + \lambda_{L03}q^3 + \lambda_{L04}q^4 + \lambda_{L05}q^5}{2(q^2 L_{P6} + 2q L_{P2} L_{P3} + L_{P2}^2)^2} \tag{47}
$$

$$
\lambda_{L1} = N_1^2 \frac{\lambda_{L10} + \lambda_{L11}q + \lambda_{L12}q^2 + \lambda_{L13}q^3 + \lambda_{L14}q^4 + \lambda_{L15}q^5}{2(q^2 L_{P6} + 2q L_{P2} L_{P3} + L_{P2}^2)^2} \tag{48}
$$

$$
\lambda_{L2} = N_1^2 \frac{\lambda_{L20} + \lambda_{L21}q + \lambda_{L22}q^2 + \lambda_{L23}q^3 + \lambda_{L24}q^4 + \lambda_{L25}q^5}{2(q^2 L_{P6} + 2q L_{P2} L_{P3} + L_{P2}^2)^2} \tag{49}
$$

where we have explicitly assumed the '$S3$ constrained' solution Eq. 44. Fully invariant and flavour-symmetric, the expressions for $\lambda_{L00}, \lambda_{L01} \ldots \lambda_{L10}$ etc., as functions of the L_i, $i = 1 \ldots 3$, are given in Appendix B.

Similarly, for the ∂_N equations we find:

$$
\lambda_{N0} = -N_2 \frac{L_{P2}^2 + 2q L_{P2} L_{P3} + q^2 L_{P6}}{6 N_1 (L_{P2} + q L_{P3})} \tag{50}
$$

$$
\lambda_{N1} = 0 \tag{51}
$$

$$
\lambda_{N2} = \frac{L_{P2}^2 + 2q L_{P2} L_{P3} + q^2 L_{P6}}{2 N_1 (L_{P2} + q L_{P3})} \tag{52}
$$

with no clear relation to the ∂_L case, since our "action" (Eq. 33) is not $L \leftrightarrow N$ symmetric. The supplementary polynomials L_{P2}, L_{P3}, L_{P6} are defined in Appendix B.

It should be emphasised that (up to the apparently spontaneous choice of pole) the extremisation conditions on the composite action Eq. 33, together with the particular Lagrange multipliers defined by Eqs. 47-52 and Appendix B, constitute an entirely covariant and flavour-symmetric specification of the realistic, non-trivial mixing Eq. 45.

5. Perspective

Motivated by the notion (Section 1) that the fundamental laws underlying the fermion masses and mixings might come from a variational principle, applied to some as yet unspecified function of flavour-symmetric Jarlskog invariants, we have presented a manifestly covariant machinary for extremising such invariants as functions of the mass matrices. (Building on the analogy [4] with the field-strength tensor in terms of covariant derivatives, cf. $F_{\mu\nu} = -i[\mathcal{D}_\mu, \mathcal{D}_\nu]$, we see our extremisation equations, eg. Eqs. 23-24 as analogous to the Yang-Mills [23] equations, cf. $[\mathcal{D}_\mu, F^{\mu\nu}] = 0$, which are themselves derivable from a quadratic Lagrangian $\mathcal{L} = -\text{Tr } F^2/2$, whereby we see the Yukawa couplings as "dynamical variables", somewhat analogous to the gauge fields).

Having tested-out our methods on the familiar case of Tr C^3 (Section 2) we have been able to show (Section 3) that extremising Tr C^2 does not lead to physically realistic lepton mixings. Focussing on quadratic "actions", we see Eq. 33 (Section 4) as just an example of an (approximate) "effective" action, whose main particular merit is that we know how to solve its resulting extremisation equations analytically, and that it leads to at least one mixing (Eq. 45) which is compatable with observation. Clearly the complete absence of CP violation is a significant deficiency of Eq. 33 (given the sucesss of the standard explanation of CP violation in the quark sector) and incorporating CP violation is an obvious challenge for the future.

Most of all, we would like to think that this paper will stimulate the search for a natural and beautiful action function, determining not only the mixing but also the (relative) mass values $m_1 : m_2 : m_3$ and $m_e : m_\mu : m_\tau$, obviating the need for those ugly Lagrange multipliers (and perhaps also shedding some light on the quark case at the same time). Regarding masses, it is clear that our procedures apply directly as they stand: one will only have to notice that for some 'perfect' action, not only is the mixing correctly predicted, but the Lagrange multiplier functions, on the RHS of the extremisation equations, vanish numerically for the correct masses, effectively as in a free extremisation. (Empirical relations like the Koide relation: $512L_3L_1 - 64L_2^2 - 656L_2L_1^2 + 207L_1^4 = 0$ [24] could be relevant here, cf. Appendix B). In removing a large part of the arbitrariness inevitably associated with the flavour

sector as we know it today, such an "action" (if it exists) would surely be welcomed as an economical adjunct to the present Standard Model of Particle Physics.

Acknowledgements

Appendix A

We are making use of a long-established result from matrix calculus [13], whereby, in terms of the matrix derivative $\partial_X = \partial/\partial X$, for some constant matrix A, we have:

$$\partial_X \, \mathrm{Tr} \, XA = \mathrm{Tr} \, (\partial_X X)A = A^T \tag{53}$$

where the superscript T denotes the matrix transpose. Note that $\partial_X X$ is a matrix of matrices (each sub-matrix with a single unit entry), and that in particular $\partial_X X \neq I$, where I is the identity matrix (while $\partial_X \, \mathrm{Tr} \, X = \mathrm{Tr} \, \partial_X \, X = I$).

We first consider $\mathrm{Tr} \, C^3 = \mathrm{Tr} \, i[L,N]^3$ (where $C := -i[L,N]$, see Sections 1-2):

$$
\begin{align}
\partial_L \, \mathrm{Tr} \, C^3 &= \mathrm{Tr} \quad \partial_L \, C^3 \tag{54}\\
&= 3 \, \mathrm{Tr} \quad (\partial_L C) \, C^2 \tag{55}\\
&= -3i \, \mathrm{Tr} \quad [(\partial_L L), N] \, C^2 \tag{56}\\
&= -3i \, \mathrm{Tr} \quad (\partial_L L)[N, C^2] \tag{57}\\
&= -3i \, [N, C^2]^T \tag{58}
\end{align}
$$

making use of the cyclic property of the trace, Eq. 54-55 and Eq. 56-57, and the matrix-calculus theorem (Eq. 53) for the final step Eq. 57-58.

The case of $\mathrm{Tr} \, C^2 = -\mathrm{Tr} \, [L,N]^2$ (see Section 3) follows entirely analogously:

$$
\begin{align}
\partial_L \, \mathrm{Tr} \, C^2 &= -2i \, [N, C]^T \tag{59}\\
&= -2 \, [N, [L, N]]^T \tag{60}
\end{align}
$$

where in the last line we have removed the factor i using $C = -i[L,N]$.

Finally, we consider the case corresponding to $Q_{21} = \mathrm{Tr} \, [L,N][L^2,N]/2$ (Section 4):

$$\partial_L \, \mathrm{Tr} \, [L,N][L^2,N] = \mathrm{Tr} \, \Big([(\partial_L L), N][L^2,N] + [L,N]([(\partial_L L)L, N] + [L(\partial_L L), N]) \Big) \tag{61}$$

Taking each of the above three terms one at a time, we have:

$$
\begin{align}
\mathrm{Tr} \quad [(\partial_L L), N][L^2, N] &= \mathrm{Tr} \quad (\partial_L L)[N, [L^2, N]] \tag{62}\\
\mathrm{Tr} \quad [L, N][(\partial_L L)L, N] &= \mathrm{Tr} \quad (\partial_L L)(LNLN - L^2 N^2 - LN^2 L + LNLN) \\
&= \mathrm{Tr} \quad (\partial_L L)(L[N, [L, N]]) \tag{63}\\
\mathrm{Tr} \quad [L, N][L(\partial_L L), N] &= \mathrm{Tr} \quad (\partial_L L)(NLNL - LN^2 L - N^2 L^2 + NLNL) \\
&= \mathrm{Tr} \quad (\partial_L L)([N, [L, N]]L) \tag{64}
\end{align}
$$

wherby adding the above equations:

$$
\begin{align}
\partial_L \, \mathrm{Tr} \, [L,N][L^2,N] &= \mathrm{Tr} \quad (\partial_L L)([N, [L^2, N]] + \{L, [N, [L, N]]\}) \\
&= [N, [L^2, N]]^T + \{L, [N, [L, N]]\}^T \tag{65}
\end{align}
$$

where the curly brackets denote the anti-commutator (see main text Eq. 38).

Appendix B

The following completes the expressions for the Lagrange multiplers of Section 4:

$$\lambda_{L00} = L_{P2}(12L_3^2L_1 - 6L_3L_2L_1^2 - 18L_3L_2^2$$
$$+4L_3L_1^4/3 + 19L_2^3L_1 - 11L_2^2L_1^3 + 3L_2L_1^5 - L_1^7/3) \tag{66}$$

$$\lambda_{L10} = L_{P2}(-36L_3^2 + 36L_3L_2L1 - 4L_3L_1^3 - 3L_2^3 + 3L_2^2L_1^2 - 5L_2L_1^4 + L_1^6) \tag{67}$$

$$\lambda_{L20} = 4L_{P2}^2(9L_3 - 9L_2L_1 + 2L_1^3) \tag{68}$$

$$\lambda_{L01} = 18L_3^3L_1 - 18L_3^2L_2^2 + 27L_3^2L_2L_1^2 - 13L_3^2L_1^4 - 48L_3L_2^3L_1 - 2L_3L_2^2L_1^3 + 14L_3L_2L_1^5$$
$$-2L_3L_1^7 - 6L_2^5 + 165L_2^4L_1^2/2 - 77L_2^3L_1^4 + 30L_2^2L_1^6 - 6L_2L_1^8 + L_1^{10}/2 \tag{69}$$

$$\lambda_{L11} = -54L_3^3 - 72L_3^2L_2L_1 + 42L_3^2L_1^3 - 18L_3L_2^3 + 144L_3L_2^2L_1^2 - 74L_3L_2L_1^4$$
$$+22L_3L_1^6/3 - 6L_2^4L_1 + 10L_2^3L_1^3 - 20L_2^2L_1^5 + 10L_2L_1^7 - 4L_1^9/3 \tag{70}$$

$$\lambda_{L21} = 81L_3^2L_2 - 27L_3^2L_1^2 + 108L_3L_2^2L_1 - 90L_3L_2L_1^3 + 18L_3L_1^5$$
$$+27L_2^4/2 - 207L_2^3L_1^2 + 186L_2^2L_1^4 - 56L_2L_1^6 + 11L_1^8/2 \tag{71}$$

$$\lambda_{L02} = 36L_3^3L_2 + 36L_3^3L_1^2 - 156L_3^2L_2^2L_1 + 52L_3^2L_2L_1^3 - 16L_3^2L_1^5 + 3L_3L_2^4$$
$$+93L_3L_2^3L_1^2 - 253L_3L_2^2L_1^4/3 + 35L_3L_2L_1^6 - 4L_3L_1^8 - 41L_2^5L_1/2$$
$$+135L_2^4L_1^3/2 - 62L_2^3L_1^5 + 76L_2^2L_1^7/3 - 11L_2L_1^9/2 + L_1^{11}/2 \tag{72}$$

$$\lambda_{L12} = -144L_3^3L_1 + 48L_3^2L_2L_1^2 + 32L_3^2L_1^4 - 30L_3L_2^3L_1 + 126L_3L_2^2L_1^3 - 254L_3L_2L_1^5/3$$
$$+10L_3L_1^7 + 9L_2^5/2 - 43L_2^4L_1^2/2 + 27L_2^3L_1^4 - 25L_2^2L_1^6 + 67L_2L_1^8/6 - 3L_1^{10}/2 \tag{73}$$

$$\lambda_{L22} = 216L_3^2L_2L_1 - 72L_3^2L_1^3 - 27L_3L_2^3 - 45L_3L_2^2L_1^2 - 9L_3L_2L_1^4$$
$$+9L_3L_1^6 + 63L_2^4L_1 - 225L_2^3L_1^3 + 191L_2^2L_1^5 - 59L_2L_1^7 + 6L_1^9 \tag{74}$$

$$\lambda_{L03} = 36L_3^4 - 78L_3^3L_2L_1 + 58L_3^3L_1^3 + 24L_3^2L_2^3 - 63L_3^2L_2^2L_1^2 - 20L_3^2L_2L_1^4/3 - 5L_3^2L_1^6$$
$$-47L_3L_2^4L_1 + 466L_3L_2^3L_1^3/3 - 298L_3L_2^2L_1^5/3 + 100L_3L_2L_1^7/3 - 11L_3L_1^9/3$$
$$+L_2^6 + 3L_2^5L_1^2/2 - 15L_2^4L_1^4/2 - L_2^3L_1^6/3 + 7L_2^2L_1^8/3 - 7L_2L_1^{10}/6 + L_1^{12}/6 \tag{75}$$

$$\lambda_{L13} = 18L_3^3L_2 - 150L_3^3L_1^2 + 122L_3^2L_2L_1^3 + 22L_3^2L_1^5/3 + 15L_3L_2^4$$
$$-86L_3L_2^3L_1^2 + 92L_3L_2^2L_1^4 - 188L_3L_2L_1^6/3 + 25L_3L_1^8/3$$
$$-3L_2^5L_1 + 14L_2^4L_1^3 - 8L_2^3L_1^5 - 2L_2^2L_1^7 + 11L_2L_1^9/3 - 2L_1^{11}/3 \tag{76}$$

$$\lambda_{L23} = -81L_3^2L_2^2 + 270L_3^2L_2L_1^2 - 81L_3^2L_1^4 + 108L_3L_2^3L_1$$
$$-282L_3L_2^2L_1^3 + 104L_3L_2L_1^5 - 22L_3L_1^7/3 - 9L_2^5/2 + 33L_2^4L_1^2/2$$
$$-53L_2^3L_1^4 + 64L_2^2L_1^6 - 49L_2L_1^8/2 + 17L_1^{10}/6 \tag{77}$$

$$\lambda_{L04} = 48L_3^4L_1 + 24L_3^3L_2^2 - 174L_3^3L_2L_1^2 + 166L_3^3L_1^4/3 - 91L_3^2L_2^3L_1/2 + 1141L_3^2L_2^2L_1^3/6$$
$$-189L_3^2L_2L_1^5/2 + 17L_3^2L_1^7/2 + 5L3L_2^5/4 + 27L_3L_2^4L_1^2/2 - 337L_3L_2^3L_1^4/6$$
$$+89L_3L_2^2L_1^6/3 - 13L_3L_2L_1^8/12 - L_3L_1^{10}/2 + L_2^6L_1/8 - 17L_2^5L_1^3/4$$
$$+101L_2^4L_1^5/8 - 28L_2^3L_1^7/3 + 61L_2^2L_1^9/24 - 5L_2L_1^{11}/12 + L_1^{13}/24 \tag{78}$$

$$\lambda_{L14} = 60L_3^3 L_2 L_1 - 84L_3^3 L_1^3 + 33L_3^2 L_2^3/2 - 249L_3^2 L_2^2 L_1^2/2 + 793L_3^2 L_2 L_1^4/6$$
$$-19L_3^2 L_1^6/2 - 17L_3 L_2^4 L_1/2 + 139L_3 L_2^3 L_1^3/3 - 82L_3 L_2^2 L_1^5/3$$
$$-31L_3 L_2 L_1^7/3 + 5L_3 L_1^9/2 - L_2^6/8 + 9L_2^5 L_1^2/4 - 67L_2^4 L_1^4/8$$
$$+14L_2^3 L_1^6/3 - L_2^2 L_1^8/24 + 5L_2 L_1^{10}/12 - L_1^{12}/8 \tag{79}$$

$$\lambda_{L24} = -54L_3^3 L_2 + 18L_3^3 L_1^2 + 54L_3^2 L_2^2 L_1 + 78L_3^2 L_2 L_1^3 - 32L_3^2 L_1^5$$
$$-27L3 L_2^4/4 + 27L_3 L_2^3 L_1^2 - 293L_3 L_2^2 L_1^4/2 + 199L_3 L_2 L_1^6/3 - 27L_3 L_1^8/4$$
$$+3L_2^5 L_1/4 - 5L_2^4 L_1^3/2 + 33L_2^3 L_1^5/2 - 2L_2^2 L_1^7 - 31L_2 L_1^9/12 + L_1^{11}/2 \tag{80}$$

$$\lambda_{L05} = 6L_3^4 L_2 + 12L_3^4 L_1^2 - 19L_3^3 L_2^2 L_1/2 - 145L_3^3 L_2 L_1^3/3 + 29L_3^3 L_1^5/2 + L_3^2 L_2^4/2$$
$$-5L_3^2 L_2^3 L_1^2/4 + 2467L_3^2 L_2^2 L_1^4/36 - 409L_3^2 L_2 L_1^6/12 + 131L_3^2 L_1^8/36$$
$$-L_3 L_2^5 L_1/6 + 4L_3 L_2^4 L_1^3 - 223L_3 L_2^3 L_1^5/6 + 211L_3 L_2^2 L_1^7/9 - 13L_3 L_2 L_1^9/3$$
$$+2L_3 L_1^{11}/9 + L_2^6 L_1^2/8 - 11L_2^5 L_1^4/6 + 69L_2^4 L_1^6/8$$
$$-79L_2^3 L_1^8/12 + 137L_2^2 L_1^{10}/72 - L_2 L_1^{12}/4 + L_1^{14}/72 \tag{81}$$

$$\lambda_{L15} = L_\Sigma(-36L_3^3 L_1 - 27L_3^2 L_2^2/2 + 57L_3^2 L_2 L_1^2 - 11L_3^2 L_1^4/2 + 6L_3 L_2^3 L_1$$
$$-15L_3 L_2^2 L_1^3 - 8L_3 L_2 L_1^5/3 + L_3 L_1^7 - L_2^4 L_1^2/2 + 3L_2^2 L_1^6/2 - L_2 L_1^8/3) \tag{82}$$

$$\lambda_{L25} = -18L_3^4 + 36L_3^3 L_2 L_1 - 4L_3^3 L_1^3 - 3L_3^2 L_2^3/4 - 141L_3^2 L_2^2 L_1^2/4$$
$$+111L_3^2 L_2 L_1^4/4 - 239L_3^2 L_1^6/36 - 3L_3 L_2^4 L_1 + 193L_3 L_2^3 L_1^3/6$$
$$-87L_3 L_2^2 L_1^5/2 + 33L_3 L_2 L_1^7/2 - 31L_3 L_1^9/18 - L_2^6/8 + 2L_2^5 L_1^2$$
$$-81L_2^4 L_1^4/8 + 163L_2^3 L_1^6/12 - 45L_2^2 L_1^8/8 + 3L_2 L_1^{10}/4 - L_1^{12}/72 \tag{83}$$

where the invariant traces L_1, L_2 and L_3 are defined as in the main text Eq. 15. Some useful supplementary polynomials (used here and in the main text Eqs. 47-52) are:

$$L_\Sigma := (L_1^3 - L_3)/3 = (m_e + m_\mu)(m_\mu + m_\tau)(m_\tau + m_e) \tag{84}$$

$$L_{P2} := (3L_2 - L_1^2)/2 = m_e^2 + m_\mu^2 + m_\tau^2 - m_e m_\mu - m_\mu m_\tau - m_\tau m_e \tag{85}$$

$$L_{P3} := (3L_3 - L_2 L_1)/2 = m_e^3 + m_\mu^3 + m_\tau^3$$
$$-(m_e^2 m_\mu + m_\mu^2 m_e)/2 - (m_\mu^2 m_\tau + m_\tau^2 m_\mu)/2 - (m_\tau^2 m_e + m_e^2 m_\tau)/2 \tag{86}$$

$$L_{P6} := L_{P3}^2 - L_\Delta^2/4 = (m_e^3 + m_\mu^3 + m_\tau^3 - m_e^2 m_\mu - m_\mu^2 m_\tau - m_\tau^2 m_e)$$
$$\times (m_e^3 + m_\mu^3 + m_\tau^3 - m_e m_\mu^2 - m_\mu m_\tau^2 - m_\tau m_e^2) \tag{87}$$

where the charged-lepton discriminant L_Δ^2 is defined in the main text, Eq. 17.

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