Science & Technology Facilities Council

Technical Report

RAL-TR-2009-008

The matrix of unitarity triangle angles for quarks

PF Harrison, S Dallison, WG Scott

October 2009

RAL Library
Science and Technology Facilities Council
Rutherford Appleton Laboratory
Harwell Science and Innovation Campus
Didcot
OX11 0QX

Tel: +44(0)1235 445384
Fax: +44(0)1235 446403
email: library@rl.ac.uk

Science and Technology Facilities Council reports are available online at: http://epubs.cclrc.ac.uk/

ISSN 1358-6254

RAL-TR-2009-008
18 Oct 2009

The Matrix of Unitarity Triangle Angles for Quarks

P. F. Harrison[1]
Department of Physics, University of Warwick,
Coventry, CV4 7AL. UK.

and

S. Dallison[2] and W. G. Scott[3]
Rutherford Appleton Laboratory,
Chilton, Didcot, Oxon OX11 0QX. UK.

Abstract

In the context of quark (as for lepton) mixing, we introduce the concept of the matrix of unitarity triangle angles Φ, emphasising that it carries equivalent information to the complex mixing matrix V itself. The angle matrix Φ has the added advantage, with respect to V, of being both basis- and phase-convention independent and consequently observable (indeed several Φ-matrix entries, eg. $\Phi_{cs} = \alpha$, $\Phi_{us} = \beta$ etc. are already long-studied as directly measurable/measured in B-physics experiments). We give complete translation formulae between the mixing-matrix and angle-matrix representations. In terms of Wolfenstein parameters, the invariant flavour-symmetric condition Det $K = 0$ (consistent with both quark and lepton data) predicts $\cos \Phi_{cs} = \bar{\eta}\lambda^2$. We go on to consider briefly the present state of the experimental data on the full angle matrix and some of the prospects for the future, with reference to both the quark and lepton cases.

[1]E-mail:p.f.harrison@warwick.ac.uk
[2]deceased
[3]E-mail:w.g.scott@rl.ac.uk

1. Introduction, Concept and Motivation

Following the pioneering early papers [1] on CP violation in B-meson decays, and the tremendous successes of the B-factory experiments (see e.g. [2]), understanding of CP violation within the standard three-generation (CKM [3]) scenario has continued to grow. Nonetheless, on the key role of the unitarity triangles in the phenomenology, the specific contribution of Aleksan et al. [4] and others [5] in effectively parameterising the CKM matrix in terms of four unitarity-triangle angles, still merits further emphasis and development. In the present paper, building on the above [4] [5], we introduce "the matrix of unitarity triangle angles for the quarks", Φ, as a useful conceptual (and notational) advance in the study of quark mixing (as for the leptons [6]). Our angle matrix (Φ) closely mirrors, and is in fact entirely equivalent to the complex mixing matrix V itself (the CKM matrix [3]), but with the important advantage of being at once both real and basis- and phase-convention independent. Section 2 gives the explicit proofs of equivalence. Section 3 gives some relevant Wolfenstein expansions[7].

Concerning the complex mixing matrix V, it has long been appreciated [8] that essentially all mixing observables (including the magnitudes of the CP-violating asymmetries) are determined by any four independent modulii - up to, in fact, only an overall sign ambiguity affecting all CP-violating asymmetries together in a correlated way. Indeed, it is with this last proviso in mind, that we say that the matrix of moduli:

$$|V| = \begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} 0.974 & 0.226 & 0.004 \\ 0.226 & 0.973 & 0.042 \\ 0.009 & 0.041 & 0.999 \end{pmatrix} \end{array} \tag{1}$$

is *essentially* equivalent to the complex mixing matrix itself (the numerical values of the moduli [9] are displayed in Eq. 1 without their experimental errors, for simplicity). To facilitate a comparison, we immediately introduce and display, on a similar footing, the matrix of unitarity triangle (UT) angles for the quarks:

$$\Phi = \begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} \Phi_{ud} & \Phi_{us} & \Phi_{ub} \\ \Phi_{cd} & \Phi_{cs} & \Phi_{cb} \\ \Phi_{td} & \Phi_{ts} & \Phi_{tb} \end{pmatrix} \end{array} \simeq \begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} 1^o & 22^o & 157^o \\ 67^o & 90^o & 23^o \\ 112^o & 68^o & \sim 0^o \end{pmatrix} \end{array} \tag{2}$$

where the entries are positive internal angles. The row and column sums of the angle matrix (Eq. 2) are all 180^o, while of course the moduli (squared) in any row or column of the moduli matrix (Eq. 1) sum to unity. Note further that the labelling of the rows and columns is identical between the two matrices (Eq. 1 vs. Eq. 2), and we see that the angle matrix already starts to "mirror" the mixing matrix, as advertised above.

The mixing matrix elements, $V_{\alpha i}$, represent directly the amplitudes for transition between the up-type flavours (mass eigenstates) u, c, t in the rows and the down-type flavours (mass eigenstates) d, s, b in the columns. We define the angle matrix entries (Eq. 2) as the phases of the corresponding plaquette products $\Pi_{\alpha i}$ [10] (see Section 2):

$$\Phi := \begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} \quad d \quad & \quad s \quad & \quad b \quad \end{array} \\ \begin{pmatrix} \mathrm{Arg}\,(-\Pi^*_{ud}) & \mathrm{Arg}\,(-\Pi^*_{us}) & \mathrm{Arg}\,(-\Pi^*_{ub}) \\ \mathrm{Arg}\,(-\Pi^*_{cd}) & \mathrm{Arg}\,(-\Pi^*_{cs}) & \mathrm{Arg}\,(-\Pi^*_{cb}) \\ \mathrm{Arg}\,(-\Pi^*_{td}) & \mathrm{Arg}\,(-\Pi^*_{ts}) & \mathrm{Arg}\,(-\Pi^*_{tb}) \end{pmatrix} \end{array} . \tag{3}$$

The minus sign and the complex conjugation are needed in Eq. 3 to convert from external to internal angles, maintaining consistency with existing conventions.

To appreciate the labelling of the angle matrix (Eqs. 2-3) we recall that any unitarity triangle is defined by the inner product of two given rows (or columns) of the complex mixing matrix V. So here unitarity triangles are simply indexed by the single (row or column) flavour label *not* featuring in the inner product. Then, with any

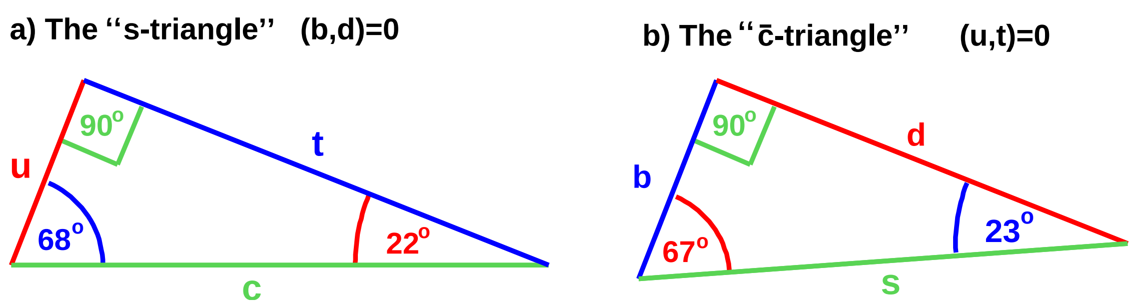

Figure 1: The indexing of unitarity triangles and their angles: a) The familiar row-based $(b,d) =:$ s-triangle and b) the closely similar $(u,t) =:$ $\bar{c}$-triangle. The indexing of the angles follows from the flavour sub-amplitude *opposite* the angle (see text).

inner product comprising the sum of three sub-amplitudes, each corresponding to a particular intermediate quark flavour (mass eigenstate) and to a particular side of the triangle, we have that any angle within a given triangle is simply labelled by the flavour label associated with the side *opposite* to that angle (see Figure 1).

It should now be clear that each row and each column of the angle matrix corresponds to a particular unitarity triangle, whereby there are three row-based triangles (labelled u, c, t, respectively) and three column-based triangles (labelled d, s, b, respectively) giving, as is well-known, six unitarity triangles in all [1]. Each row/column then lists the angles of the corresponding triangle in (mass) order. Clearly we have

[1] The complex conjugate triangles $\bar{d}$, $\bar{s}$ ... etc., formed reversing the order of arguments in the inner products (e.g. the $(u,t) = (t,u)^* =:$ $\bar{c}$-triangle, see Figure 1b), are *not* counted separately here.

that each angle $\Phi_{\alpha i}$ appears in just one row-based triangle and just one column-based triangle, as immediately specified by its row and column indices respectively.

Given the inherent difficulty of visualising six unitarity triangles at once, each sharing each angle with just one of the other triangles, we expect the Φ-matrix, as the experimental focus (see Section 4) moves on after the B-factory era, soon to come to be seen as the natural way to appreciate at a glance all the standard-model (SM) weak-phases and their inter-relations, i.e. angles in common, relevant 180^o sums etc. Importantly, if incidentally, it also offers a simple and definitive naming convention for the (SM) quark UT angles, to parallel that for the leptons [6], free of the arbtrariness of prior nomenclatures (see Section 4) arising in the case of the quarks.

2. The Equivalence of the Mixing Matrix and the Angle Matrix

The definition of the Φ-matrix given in the previous section (Eq. 3) relies on that of the plaquette products. Any given plaquette product $\Pi_{\alpha i}$ is obtained from the mixing matrix V, by deleting the row and column containing the element $V_{\alpha i}$ to leave the complementary 2×2 sub-matrix, or "plaquette" [10]. The plaquette product $\Pi_{\alpha i}$ is then formed by multiplying the four elements of the plaquette together, with the appropriate pair of diagonally-related elements complex conjugated:

$$\Pi_{\alpha i} := V_{\beta j} V^*_{\beta k} V_{\gamma k} V^*_{\gamma j} \tag{4}$$

where the indices are cyclically defined, i.e. in the columns, i, j, k $(i \neq j \neq k)$ retain the cyclic order of d, s, b and similarly for the up-type quarks in the rows (equivalently, in terms of "generation number" we may write: $j = i+1 \bmod 3$, $k = i+2 \bmod 3$, etc.).

Such plaquette products $\Pi_{\alpha i}$ constitute the minimal non-trivial loop amplitudes possible in flavour space [11], and are ubiquitous as interference terms in e.g. squared penguin amplitudes, after summing over the intermediate flavour in the loop (particularly in the leptonic case, neutrino oscillations etc. [12], plaquette products have also been referred to as "boxes"[13]). Thus Φ is seen as a "loop-space variable" [14], relating to the complex mixing matrix V much in the way that, e.g., the observable field-strength tensor F relates to the vector potential A in electromagnetism.

The matrix of plaquette products then takes the explicit form:

$$\Pi = \begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} \quad d \quad\quad\quad & \quad\quad s \quad\quad\quad & \quad\quad b \quad \end{array} \\ \begin{pmatrix} V_{tb}V^*_{ts}V_{cs}V^*_{cb} & V_{td}V^*_{tb}V_{cb}V^*_{cd} & V_{ts}V^*_{td}V_{cd}V^*_{cs} \\ V_{ub}V^*_{us}V_{ts}V^*_{tb} & V_{ud}V^*_{ub}V_{tb}V^*_{td} & V_{us}V^*_{ud}V_{td}V^*_{ts} \\ V_{cb}V^*_{cs}V_{us}V^*_{ub} & V_{cd}V^*_{cb}V_{ub}V^*_{ud} & V_{cs}V^*_{cd}V_{ud}V^*_{us} \end{pmatrix} \end{array} \tag{5}$$

where the pattern of complex conjugation is seen to follow in straightforward analogy to the usual pattern of computation of a 3×3 determinant in terms of cofactors. Likewise the labelling of the Π-matrix entries is analogous to the labelling of the entries in "the matrix of cofactors", as encountered, e.g. in calculating a matrix inverse.

The plaquette products are basis- and phase-convention independent complex numbers. As is well known, all the imaginary parts are equal [15]:

$$-\Pi^* =: \begin{pmatrix} K_{ud} & K_{us} & K_{ub} \\ K_{cd} & K_{cs} & K_{cb} \\ K_{td} & K_{ts} & K_{tb} \end{pmatrix} \; + \; i \begin{pmatrix} J & J & J \\ J & J & J \\ J & J & J \end{pmatrix} \tag{6}$$

and furthermore define J, the Jarlskog CP invariant [15]. We remark that alternating signs ($\pm J$) often encountered for the imaginary parts of the plaquette products do not enter here, with the cyclic definitions specified above.

To establish an equivalence between the mixing matrix and the angle matrix we still have to show that we can re-obtain the mixing matrix starting from the angle matrix. We shall take it as given [8] that the matrix of mixing moduli is (essentially) equivalent to the complex mixing matrix, and content ourselves in the first instance with showing how to obtain the mixing-matrix moduli $|V_{\alpha i}|$, starting from the angles.

We begin by defining, in terms of the Φ-matrix, a $\mathrm{Sin}\,\Phi$ matrix:

$$\mathrm{Sin}\,\Phi := \begin{pmatrix} \sin\Phi_{ud} & \sin\Phi_{us} & \sin\Phi_{ub} \\ \sin\Phi_{cd} & \sin\Phi_{cs} & \sin\Phi_{cb} \\ \sin\Phi_{td} & \sin\Phi_{ts} & \sin\Phi_{tb} \end{pmatrix} = J \times \begin{pmatrix} \frac{1}{|\Pi_{ud}|} & \frac{1}{|\Pi_{us}|} & \frac{1}{|\Pi_{ub}|} \\ \frac{1}{|\Pi_{cd}|} & \frac{1}{|\Pi_{cs}|} & \frac{1}{|\Pi_{cb}|} \\ \frac{1}{|\Pi_{td}|} & \frac{1}{|\Pi_{ts}|} & \frac{1}{|\Pi_{tb}|} \end{pmatrix} \tag{7}$$

where the trigonometric function Sin must be understood to act independently on the individual matrix entries as shown. From Eq. 7 the entries in the $\mathrm{Sin}\,\Phi$ matrix are clearly inversely proportional to the moduli of the plaquette products, $|\Pi_{\alpha i}|$, which are themselves each expressible as a product of four mixing-matrix moduli via Eq. 4.

Starting from the $\mathrm{Sin}\,\Phi$ matrix (Eq. 7) and keeping the same (cyclic) definitions of the flavour indices as in Eq, 4, we may now define certain products of sines, $\Xi_{\alpha i}$:

$$\Xi_{\alpha_i} := \sin\Phi_{\alpha j}\sin\Phi_{\alpha k}\sin\Phi_{\beta i}\sin\Phi_{\gamma i} \tag{8}$$

mutiplying together the (four) $\mathrm{Sin}\,\Phi$ entries in the same row and column as $\sin\Phi_{\alpha i}$, excluding $\sin\Phi_{\alpha i}$ itself. Clearly every mixing modulus-squared *except* $|V_{\alpha i}|^2$ enters in the denominator of the product Eq. 8, whereby the $\Xi_{\alpha i}$ must be proportional to $|V_{\alpha i}|^2$. The relevant normalising factor may be obtained by summing over any row or column (or indeed over both rows and columns). It should now be clear that:

$$|V_{\alpha i}|^2 = \Xi_{\alpha i} \,/\, (\sum_{\beta} \Xi_{\beta i}) = \Xi_{\alpha i} \,/\, (\sum_{j} \Xi_{\alpha j}) \tag{9}$$

$$= 3\ \Xi_{\alpha i} \,/\, (\sum_{\beta j} \Xi_{\beta j}). \tag{10}$$

The equivalence of the Φ-matrix and the $(|V|)$-matrix is clearly established, taking the positive square-root (In Eq. 9-10, both numerator and denominator are positive).

In a similar vein we may obtain the magnitude of the CP-invariant J:

$$|J| = 9(\prod_{\alpha i} \Xi_{\alpha i})^{1/4} \,/\, (\sum_{\alpha i} \Xi_{\alpha i})^2. \tag{11}$$

Of course in the case of the quarks, the "sense" of the unitarity triangles has already been determined experimentally, so that J is anyway already known to be positive for the quarks. While clearly the Ξ-matrix above carries no sign information, more generally the Φ-matrix itself (and the SinΦ matrix) carries explicitly the sign of J:

$$J = 9 \prod_{\alpha i} \sin \Phi_{\alpha i} \;/\; (\sum_{\alpha i} \Xi_{\alpha i})^2 \tag{12}$$

and the equivalence of Φ to the complex mixing matrix V is seen to be complete. We will return to this point again later in connection with mixing in the lepton sector.

We may further remark that the normalisation factor in Eq. 9-10 is itself a non-trivial flavour-symmetric observable [16] [17]:

$$\sum_{\alpha i} \Xi_{\alpha i} = 3 \;/\; (\prod_{\alpha i} |V_{\alpha i}|^2) \tag{13}$$

recognisable as (the reciprocal of) the product of all the mixing-moduli squared [17].

3. The Sin Φ and Cos Φ Matrices in the Wolfenstein Parameterisation.

Underlying the famous Wolfenstein parametrisation [9] [7] [18] of the CKM matrix, is the apparent power hierarchy of quark mixing angles first remarked upon by Wolfenstein [7], by which $\theta_{12} : \theta_{23} : \theta_{13} \simeq \lambda : \lambda^2 : \lambda^3$, where $\lambda = \sin\theta_C \sim 0.22$, with $\theta_C \simeq \theta_{12}$ the Cabbibo angle [3]. The corresponding hierarchy of Φ-matrix elements then follows immediately, recalling our "complementary labelling" of the triangles and angles, and taking the inner (dot) products in pairs of the rows or columns of V:

$$|V| \sim \begin{array}{c} \\ u \\ c \\ t \end{array} \!\!\begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \end{array} \quad \Longrightarrow \quad \Phi \sim \begin{array}{c} \\ u \\ c \\ t \end{array} \!\!\begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} \lambda^2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \lambda^4 \end{pmatrix} \end{array} \tag{14}$$

where we have worked initially here only with the bare powers of λ, for simplicity. Notice that the centermost row and column of the Φ-matrix, corresponding to the $(t, u) := c$ and $(b, d) := s$ triangles respectively, have all angles "large" as a result of $\theta_{12}\theta_{23}/\theta_{13} \sim 1$ (hence Figure 1). Corresponding to the first row and column of

Eq. 14, the u and d triangles have one small angle $\Phi_{ud} \sim \theta_{12}\theta_{13}/\theta_{23} \sim \lambda\lambda^3/\lambda^2 \sim \lambda^2$, while from the last row and column, the t and b triangles have one (very) small angle $\Phi_{tb} \sim \theta_{23}\theta_{13}/\theta_{12} \sim \lambda^5/\lambda \sim \lambda^4$ (these various "long and thin" triangles are not shown).

Now using the full Wolfenstein parameterisation [7] in terms of improved paremeters $\bar{\rho}$ and $\bar{\eta}$ [9] [18], we have, to lowest order (element-wise) in small quantities:

$$\mathrm{Sin}\,\Phi \simeq \bar{\eta} \begin{pmatrix} \lambda^2 & b & b \\ g & bg & b \\ g & g & A^2\lambda^4 \end{pmatrix} \tag{15}$$

where

$$b := \frac{1}{[(1-\bar{\rho})^2 + \bar{\eta}^2]^{\frac{1}{2}}}, \quad g := \frac{1}{(\bar{\rho}^2 + \bar{\eta}^2)^{\frac{1}{2}}}. \tag{16}$$

Notice that the $\mathrm{Sin}\,\Phi$ matrix is proportional to the (signed) CP-violating quantity $\bar{\eta}$.

We may now use Eq. 15 together with Eqs. 8 and Eq. 12 above, to recover the usual Wolfenstein approximation for J:

$$J = \frac{\prod_{\alpha i} \sin\Phi_{\alpha i}}{(\sum_\alpha \Xi_{\alpha i})^2} \simeq \frac{\bar{\eta}^9 A^2 \lambda^6 b^4 g^4}{(\bar{\eta}^4 b^2 g^2)^2} \simeq \bar{\eta} A^2 \lambda^6. \tag{17}$$

The $\mathrm{Cos}\,\Phi$ matrix similarly takes the form:

$$\mathrm{Cos}\,\Phi \simeq \begin{pmatrix} 1 & b(1-\bar{\rho}) & -b(1-\bar{\rho}) \\ g\bar{\rho} & bg[\bar{\eta}^2 - \bar{\rho}(1-\bar{\rho})] & b(1-\bar{\rho}) \\ -g\bar{\rho} & g\bar{\rho} & 1 \end{pmatrix}. \tag{18}$$

A vanishing $\mathrm{Cos}\,\Phi$ element implies a right angle in the Φ matrix, e.g. $\cos\Phi_{cs} = 0 \Rightarrow \Phi_{cs} = 90^o$ [17] [19] (see Eq. 2), corresponding to the exact constraint:

$$\bar{\eta}^2 - \bar{\rho}(1-\bar{\rho}) = 0 \qquad \text{ie.} \;\; \bar{\rho} = \bar{\rho}^2 + \bar{\eta}^2. \tag{19}$$

We may remark that, while the $\mathrm{Sin}\,\Phi$ matrix is clearly independent of a possible choice to work with external rather than internal angles, the $\mathrm{Cos}\,\Phi$ matrix would change sign.

The K-matrix is just J times the $\mathrm{Cot}\,\Phi$ matrix:

$$K = J\,\mathrm{Cot}\,\Phi \simeq \frac{J}{\bar{\eta}} \begin{pmatrix} 1/\lambda^2 & (1-\bar{\rho}) & -(1-\bar{\rho}) \\ \bar{\rho} & \bar{\eta}^2 - \bar{\rho}(1-\bar{\rho}) & (1-\bar{\rho}) \\ -\bar{\rho} & \bar{\rho} & 1/A^2\lambda^4 \end{pmatrix}. \tag{20}$$

In a previous publication [16] (and see [17]) we have considered the possibility that it is $\mathrm{Det}\,K$ which vanishes exactly (rather than $\cos\Phi_{cs}$ as above). Eq. 20 readily gives:

$$\mathrm{Det}K \simeq (J/\bar{\eta})^3 (1/A^2\lambda^6)\{[\bar{\eta}^2 - \bar{\rho}(1-\bar{\rho})] - \lambda^2\bar{\rho}(1-\bar{\rho})\} \tag{21}$$

which is valid for $|\bar{\eta}^2 - \bar{\rho}(1-\bar{\rho})| \lesssim \mathcal{O}(\lambda)$. Setting $\mathrm{Det}\, K$ exactly to zero then predicts:

$$\cos\Phi_{cs} = bg[\bar{\eta}^2 - \bar{\rho}(1-\bar{\rho})] \simeq \lambda^2 bg\bar{\rho}(1-\bar{\rho}) \simeq \bar{\eta}\lambda^2, \tag{22}$$

i.e. we predict a small correction to $\cos\Phi_{cs} = 0$ above, such that $\cos\Phi_{cs} \simeq \bar{\eta}\lambda^2$ [17][16].

Concerning RGE evolution, it should be remarked that several authors [20] have noted that of the four Wolfenstein parameters (λ, A, $\bar{\rho}$, $\bar{\eta}$) fixing the CKM matrix, it is only the parameter A which evolves to leading order in the SM and MSSM [20]. We then have from Eq. 15 that, while mixing moduli $|V_{\alpha i}|$ and UT sides etc. definitely should evolve, by contrast in the angle matrix Φ only the very small bottom-right corner element $\Phi_{tb} \sim A^2\lambda^4$ should evolve significantly, so that the angle matrix Φ, at least as regards its general appearance (Eq. 2) may be said to be largely invariant. We do not consider RGE evolution any further here, only remarking that the angle matrix Φ (Eq. 2) may already be revealing its basic form at the very highest energy scales.

4. Prior Nomenclatures, Current Results and some Future Prospects

As a prelude to discussing the experimental measurements, errors etc. on the angle-matrix entries, it will be necessary to establish the correspondence with some of the historical namings of UT angles, at least in those cases for which prior namings exist. In particular, UT angles β, α, γ, also known as ϕ_1, ϕ_2, ϕ_3, have of course already been intensively studied theoretically and experimentally (the "switch" in ordering between the Babar [21] α, β, γ and Belle [22] ϕ_1, ϕ_2, ϕ_3 nomenclatures is well-known and perhaps a little unfortunate, but can hardly be a cause of any major confusion). From our definitions [2] (Eqs. 3-5) we have $\Phi_{us} = \beta = \phi_1$, $\Phi_{cs} = \alpha = \phi_2$ and $\Phi_{ts} = \gamma = \phi_3$, comprising the central column of our full Φ-matrix:

$$\Phi = \begin{array}{c} \\ u \\ c \\ t \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} \quad d \quad & \quad s \quad & \quad b \quad \end{array} \\ \begin{pmatrix} \Phi_{ud} = \beta_s = \chi & \Phi_{us} = \beta = \phi_1 & \Phi_{ub} \\ \Phi_{cd} = \gamma' = \gamma - \delta\gamma & \Phi_{cs} = \alpha = \phi_2 & \Phi_{cb} = \beta + \delta\gamma \\ \Phi_{td} & \Phi_{ts} = \gamma = \phi_3 & \Phi_{tb} = \beta_K = \chi' \end{pmatrix} \end{array}. \tag{23}$$

In the B_s-sector, the angle Φ_{ud} (top-left entry in Eq. 23) is quite naturally seen as the analogous angle to $\Phi_{us} = \beta$ in the B_d-sector, whereby one has $\Phi_{ud} = \beta_s$ [24]. Note however that since the direct measurements in the B_s-sector are potentially very sensitive to possible new physics contributions, the more specific designation

[2] We note that UT angles in the literature [9] e.g. α, β, γ, are often defined in terms of *ratios* of mixing matrix elements, e.g. $\alpha = \arg(-V_{td}V_{tb}^*/V_{ud}V_{ub}^*)$, $\beta = \arg(-V_{cd}V_{cb}^*/V_{td}V_{tb}^*)$ etc. [9]. The definition of Section 1-2 (Eq. 3-5) with all four relevant mixing elements multiplied systematically on an essentially equal footing, is of course entirely equivalent to those definitions in terms of ratios [9].

β_s^{SM} is also sometimes used [23] to indicate explicitly the SM contribution alone, which ultimately defines our Φ-matrix here. Since Φ_{ud} is often denoted χ in the theory literature [25] we take $\Phi_{ud} = \beta_s = \beta_s^{SM} = \chi$. Other notations have sometimes been used (e.g. $-\phi_s/2$ [24]) especially to denote the directly measured empirical angle inclusive of any new physics (this distinction is far from academic [23] [24] as discussed below).

To complete a set of four independent angles (clearly α, β, γ above are *not* mutually independent) one might easily take Φ_{cd}, to complete a "Φ-plaquette" [12] (with Φ_{us}, Φ_{cs} and Φ_{ud}) enabling all other angles to be readily calculated, summing rows and columns to 180^o. We note that Φ_{cd} has been denoted $\gamma' := \gamma - \delta\gamma$ [26]. This latter notation exploits the fact that the row-based c-triangle and the column-based s-triangle, being otherwise independent, have the angle α in common (see again Figure 1). We may thus write $\Phi_{cd} = \gamma - \delta\gamma$ and (equivalently) also $\Phi_{cb} = \beta + \delta\gamma$ (since $\alpha + \beta + \gamma = 180^o$). As we will see, however, neither Φ_{cd}, Φ_{cb} nor $\delta\gamma$ is necessarily an optimal choice to complete the set in practice. Indeed, we will rather take Φ_{tb} (bottom-right entry in Eq. 23) as our fourth parameter here [5], as will be discussed in more detail below.

Turning now to the experimental results themselves, as is well-known, a number of direct measurements of the angle $\Phi_{us} = \beta$ in the $B_d \to J/\psi K_s$ mode have been carried out over many years, especially at the B-factories [27] [28] yielding a world average $\sin 2\beta = 0.684 \pm 0.022$ [30]. Ambiguities can be resolved [29] to give finally: $\Phi_{us} = \beta = 21.58^o \pm 0.86^o$ [30]. The combined decay modes $B_d \to \pi\pi$, $B_d \to \rho\rho$ and $B_d \to \rho\pi$ yield: $\Phi_{cs} = \alpha \simeq 90.6^o \pm 4.0^o$ [31] [32] [30] (consistent with $\alpha = 90^o$ as earlier noted [17]). Direct measurements of the angle γ yield $\Phi_{ts} = \gamma \sim 70^o \pm 30^o$ [33] [34] [30] consistent with SM unitarity ($\alpha + \beta + \gamma = 180^o$). LHCb [26], followed by Super Flavour Factories [35], can ultimately reduce the errors on the angles $\Phi_{cs} = \alpha$ and $\Phi_{us} = \beta$ by as much as a factor of four or so, with both projects vastly improving the measurement of $\Phi_{ts} = \gamma$ to reach an error of perhaps $\pm 2^o$ or better.

The angle $\Phi_{ud} = \beta_s = \chi$ is in fact already highly constrained by existing indirect measurements through SM unitarity constraints, giving $\Phi_{ud} \simeq \bar{\eta}\lambda^2(1 + (1 - \bar{\rho})\lambda^2) \sim 1.04^o \pm 0.05^o$ using latest fits [30]. Direct measurements of $\Phi_{ud} = \beta_s$ in the $B_s^0 \to J/\psi\phi$ channel are potentially sensitive to new physics contributions and indeed, combining recent CDF-II [23] and D0 [24] results gives $\Phi_{ud} = \beta_s \simeq (19 \pm 7)^o$, approximately 2.6σ from the expected SM value above. LHCb [26] is expected to make the definitive measurement of $\Phi_{ud} = \beta_s$, ultimately to better than $\pm 1^o$ accuracy. Such measurements, if anomalies persist, may eventually need to be re-interpreted, possibly with reference to a new/different angle matrix (or even several such matrices) relevant to the new physics, after subtraction of the SM contribution above.

Similarly, our best knowledge of $\Phi_{tb} \simeq \bar{\eta} A^2 \lambda^4 (1+\lambda^2) \sim 0.035^o \pm^{0.003^o}_{0.002^o}$ [30] comes indirectly through SM unitarity constraints. Φ_{tb} is sometimes denoted β_K [36] being at least in principle related to CP violation in K^0 mixing (and similarly in D^0 mixing). In the theory literature Φ_{tb} is sometimes denoted χ' [25]. One chooses $\Phi_{tb} = \beta_K = \chi'$ as the fourth primary parameter here, rather than $\Phi_{cd} = \gamma - \delta\gamma$ above, since $\delta\gamma$ and β_s are anyway very nearly equal [3], due to the smallness of Φ_{tb} (note that $\Phi_{tb} = \beta_s - \delta\gamma$).

Given Φ_{us}, Φ_{cs}, Φ_{ud} and Φ_{tb} above, all the remaining entries in the Φ-matrix can now be determined in an obvious way, summing rows and columns to 180^o. We find:

$$\Phi \simeq \begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} \quad d \quad & \quad\quad s \quad\quad & \quad\quad b \quad \end{array} \\ \begin{pmatrix} 1.04^o \pm 0.05^o & 21.58^o \pm 0.86^o & 157.38^o \mp 0.89^o \\ 66.82^o \mp 4.20^o & 90.60^o \pm 4.00^o & 22.58^o \pm 0.89^o \\ 112.14^o \pm 4.21^o & 67.82^o \mp 4.22^o & 0.035^o \pm 0.003^o \end{pmatrix} \end{array}, \tag{24}$$

where the main correlations are indicated by the signs on the errors. In Eq. 24 we have padded the value of Φ_{cs} given above by an extra decimal digit, $\Phi_{cs} \rightarrow 90.60^o \pm 4.00^o$, for uniformity with the higher precision of the other input angles. Also, the asymmetric error on Φ_{tb} quoted above has been (conservatively) symmetrised, $\Phi_{tb} \rightarrow 0.035^o \pm 0.003^o$, for simplicity of presentation in the matrix (Eq. 24).

In the leptonic case the angles may be determined in neutrino oscillation experiments. The measurement of at least one non-zero CP-violating asymmetry in the leptonic case will be needed to determine the "sense" of the leptonic triangles (as for the quarks, *triangles* are defined with the inner-product arguments in ascending (cyclic) mass order, and the "sense" similarly from the mass-ordering of the sides). The measurement will determine the sign of the leptonic J and hence the (common) sign of all the entries in the angle matrix, so that row and column sums in the leptonic case may sum to $+180^o$ as for the quarks, or possibly to -180^o, as remains to be seen.

5. Concluding Remarks

In this paper we have introduced and developed the concept of a matrix of unitarity triangle angles for the quarks, as for the leptons, showing that it carries equivalent information to the complex mixing matrix itself, with the added advantage of being basis- and phase-convention independent and hence fully observable. Each row and each column of the angle matrix lists directly the angles of a specific unitarity triangle (three row-based and three column-based triangles, making six unitarity triangles in

[3] Indeed one occasionally sees the $\bar{c}$-triangle of Figue 1b with its two base angles labelled $\gamma - \chi$ and $\beta + \chi$ [37] or (equivalently) $\gamma - \beta_s$ and $\beta + \beta_s$ [38] respectively. This is a good working approximation in the SM since $\Phi_{ud} = \beta_s = \chi \simeq 1.04^o$ (Eq. 24) while $\delta\gamma = \Phi_{ud} - \Phi_{tb} \simeq 1.00^o$ (Eq. 24) i.e. currently indistinguishable from each other within the errors on the direct measurement of $\Phi_{ud} = \beta_s = \chi$.

all, of which four have largely RGE-invariant shape in the SM and MSSM). Individual angles are labelled in a systematic, well-defined, physically-motivated way. The marked hierarchy of the CKM elements, as reflected in the Wofenstein parameterisation, translates in the angle matrix into just two small angles: $\Phi_{ud} \sim \lambda^2$ and $\Phi_{tb} \sim \lambda^4$. The centremost row and column of the angle matrix each comprise all "large" angles and correspond to the two most familiar unitarity triangles, which have been long studied theoretically and experimentally already in the context of $B \leftrightarrow \bar{B}$ oscillations.

Acknowledgement

This work was supported by the UK Science and Technology Facilities Council (STFC). PFH acknowledges the hospitality of the Centre for Fundamental Physics (CfFP) Rutherford Appleton Laboratory. We also acknowledge useful contributions at various points from T. Gershon, F. Long, D. Roythorne, S. Sun, M. Williams and F. Wilson.

References

[1] A. B. Carter and A.I. Sanda, Phys. Rev. D23 (1981) 1567.
I.I. Bigi and A.I. Sanda, Nucl. Phys. B193 (1981) 85.

[2] "Experimental Review of CP Violation and CKM Angles at B Factories",
K. F. Chen at DISCRETE'08, Dec. 11-16, 2008, IFIC, Valencia, Spain.
(Also avaiable at: http://belle.kek.jp/belle/talks/DIS08/Chen.pdf).

[3] N. Cabibbo, Phys. Rev. Lett. 10 (1963) 531.
M. Kobayashi and T. Maskawa, Prog. Thor. Phys. 49 (1973) 652.

[4] R. Aleksan, B. Kayser and D. London,
Phys. Rev. Lett. 73 (1994) 18. hep-ph/9403341

[5] R. F. Lebed, Phys. Rev. D 55 (1997) 348. arXiv:hep-ph/9607305
G. C. Branco, L. Lavoura and J. P. Silva, "CP Violation", OUP (1999).

[6] J. D. Bjorken, P. F. Harrison and W. G. Scott,
Phys. Rev. D 74 (2006) 073012. hep-ph/0511201

[7] L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945.

[8] F. J. Botella and Ling-Lie Chau, Phys. Lett. B168 (1986) 97.
C. Jarlskog, Proc. IVth LEAR Workshop, Villar-sur-Ollon, Switzerland (1987).
G. C. Branco and L. Lavoura, Phys. Lett. B 208 (1988) 123. C. Hamzaoui, Phys. Rev. Lett. 61 (1988) 35. C. Jarlskog and R. Stora, Phys. Lett. B 208 (1988) 268.

G. Auberson, Phys. Lett. B 216 (1989) 167. L. Lavoura, Phys. Lett. B 223 (1989) 97. J. D. Bjorken, Phys. Rev. D 39 (1989) 1396.

[9] C. Amsler et al. (Particle Data Group) Phys. Lett. B 667 (2008) 145.
Also available at: http://pdg.lbl.gov/2008/reviews/rpp2008-rev-ckm-matrix.pdf

[10] J. D. Bjorken and I. Dunietz Phys. Rev. D 36 (1987) 2109.
C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039.
I. Dunietz, O. W. Greenberg and D. Wu, Phys. Rev. Lett. 55 (1985) 2935
D. Wu Phys. Rev. D 33 (1986) 860.
D. Du, I. Dunietz and D. Wu Phys. Rev. D 34 (1986) 3414.
J. F. Nieves and P. B. Pal Phys. Rev. D 36 (1987) 315.

[11] P. F. Harrison, W. G. Scott, presented by W. G. Scott at Neutrino Telescopes, Venice, Italy (2009). arXiv:0906.3077 [hep-ph].

[12] P. F. Harrison, W. G. Scott and T. J. Weiler, Phys. Lett. B 641 (2006) 372. hep-ph/0607335.

[13] D. J. Wagner and T. J. Weiler, Phys.Rev. D59 (1999) 113007. hep-ph/9801327.
T. J. Weiler and D. J. Wagner, Proceedings of the 2nd Latin American Symposium on High Energy Physics, San Juan, Puerto Rico (1998) hep-ph/9806490.

[14] B. Durhuus and J. M. Leinaas, Physica Scripta 25 (1982) 504. CERN-TH-3110
H-M. Chan, S. T. Tsou, Int. J. Mod. Phys. A14 (1999) 2139. hep-th/9904102

[15] C. Jarlskog, Z. Phys. C 29 (1985) 491.
C. Jarlskog, Phys. Rev. D35 (1987) 1685; *ibid* D36 (1987) 2128.
C. Jarlskog, "CP violation", Ed. C. Jarlskog (World Scientific 1989).

[16] P. F. Harrison, D. R. J. Roythorne and W. G. Scott, Phys. Lett. B 657 (2007) 210. arXiv:0709.1439 [hep-ph].
P.F. Harrison, D.R.J. Roythorne, W.G. Scott, Proc. 43rd Rencontres de Moriond, EW Interactions, La Thuile, Italy (2008) arXiv:0805.3440 [hep-ph].

[17] P. F. Harrison, D. R. J. Roythorne and W. G. Scott, Proc. 18th Particle and Nuclei Intl. Conf. (PANIC08), Eilat, Israel (2008). arXiv:0904.3014 [hep-ph].

[18] A. J. Buras, Phys. Rev. D 50 (1994) 3433. hep-ph/9403384

[19] H. Fritzsch and Z.-z. Xing, Phys. Lett. B 353 (1995) 114. hep-ph/9502297.
I. Masina and C. A. Savoy, Nucl. Phys. B755 (2006) 1. hep-ph/0603101;
Phys. Lett. B642 (2006) 472. hep-ph/0606097. Z.-z. Xing, arXiv:0904.3172.

[20] S. R. Juarez et al., Phys. Rev. D66 (2002) 116007. hep-ph/0206243
B. A. Kniehl et al. Phys. Rev. D62 (2002) 073010. hep-ph/0005060

[21] B. Aubert et al. (BaBar collab.), Phys. Rev. Lett. 86 (2001) 2515.
J.D. Bjorken SLAC-PUB-5389, 18th SLAC Summer Inst. (1990) 0167-198.

[22] A. Abashian et al. (Belle collab.), Phys. Rev. Lett. 86 (2001) 2509.
I. Bigi et al. in CP Violation, ed C. Jarlskog, World Scientific, Singapore (1989) 175-248. SLAC-PUB-4476, RU-DOE/ER/40352-24 (1987).

[23] T. Aaltonen et al. (CDF collab.) Phys. Rev. Lett. 100 (2008) 161802
arXiv: 0712.2397 [hep-ph]. see also CDF/ANAL/BOTTOM/PUBLIC/9458.

[24] V. M. Abazov et al. (D0 collab.) Phys. Rev. Lett. 101 (2008) 241801
arXiv: 0802.2255 [hep-ph]

[25] F. J. Botella et al. Nucl. Phys. B, 651 (2003) 174.
J.A. Aguilar-Saavedraa et al. Nucl. Phys. B 706 (2005) 204.
M. Artuso, E. Barberio, S. Stone. PMC Physics A (2009) 3:3. arXiv:0902.3743

[26] S. Amato et al. LHCb Technical proposal, CERN LHCC 98-4 LHCC/P4 (1998).

[27] B. Aubert et al. (BaBar collab.), Phys. Rev. Lett. 87 (2001) 091801. Phys. Rev. D 66 (2001) 032003. Phys. Rev. Lett. 89 (2002) 201802. Phys. Rev. Lett. 94 (2005) 161803. Phys. Rev. Lett. 99 (2007) 171803. B. Aubert et al. (Babar collab.) Phys. Rev. D. 79 (2009) 072009, arXiv:0902.1708 [hep-ex].

[28] K. Abe et al. (Belle collab.) Phys. Rev. Lett. 87 (2001) 091802.
K. Abe et al., Phys. Rev. D 66 (2002) 032007; Phys. Rev. D 66 (2002) 071102(R).
K. Abe et al. Phys. Rev. D 71 (2005) 072003.
K-F. Chen et al. (Belle Collab.) Phys. Rev. Lett. 98 (2007) 031802.

[29] I. Dunietz et al. Phys. Rev. D 43 (1991) 2193.
J. Charles et al. Phys. Rev. D 58 (1998) 114021.
B. Aubert et al. (Babar Collab.) Phys. Rev. D 71 (2005) 032005. hep-ex/0411016.
R. Itoh et al. (Belle collab.) Phys. Rev. Lett. 95 (2005) 091601. hep-ex/0504030.
J. Charles et al. Phys. lett. B 425 (1998) 375. hep-ph/9801363.
T. Latham and T. Gershon, J. Phys. G36 (2009) 025006. arXiv:0809.0872

[30] http://ckmfitter.in2p3.fr/plots_Moriond09/ckmEval_results.html
J. Charles et al. (CKMfitter Group) Eur. Phys. J. C 41 (2005) 1. hep-ph/0406184.
http://www.slac.stanford.edu/xorg/hfag/triangle/summer2008/index.shtml

E. Barberio et al. (HFAG collab.) arXiv:0808.1297.
M. Bona et al. (UTfit Collab.) JHEP 0507 (2005) 028. hep-ph/0501199.

[31] B. Aubert et al. (Babar Collab.) Phys. Rev. Lett. 94 (2005) 181802. Phys. Rev. Lett. 94 (2005) 131801. Phys. Rev. Lett. 95 (2005) 151803.
B. Aubert et al. (Babar collab.) SLAC-PUB-13452 BABAR-PUB-08/053 arXiv: 0901.3522 [hep-ex] submitted to Phys. Rev. Lett.

[32] K. Abe et al. (Belle collab) Phys. Rev. Lett. 93 (2004) 021601.
C. C. Wang et al. (Belle collab.) PRL 94 (2005) 121801.
A. Somov, A. J. Schwartz et al. (Belle collab.) Phys. Rev. Lett. 96 (2006) 171801.

[33] B. Aubert et al. (Babar collab.) Phys. Rev. Lett. 95 (2005) 121802.
B. Aubert et al. (Babar collab.) Phys. Rev. D78 (2008) 034023.
T. Gershon, Phys.Rev.D79 (2009) 051301. arXiv:0810.2706.

[34] A. Poluektov (Belle collab.) Phys. Rev. D 70 (2004) 072003.
A. Poluektov (Belle collab.) Phys. Rev. D 73 (2006) 112009.
K. Abe (Belle collab.) BELLE-CONF-0801, arXiv:0803.3375

[35] T. Browder et al. JHEP 0802:110, 2008. arXiv:0710.3799 [hep-ph].
C-h. Cheng, also A. Bevan, at "New Physics with SuperB", Warwick, April 2009: http://www2.warwick.ac.uk/fac/sci/physics/research/epp/meetings/superb2009.
K. Abe et al. SuperKEKB LoI, KEK Report 04-4. see: http://superb.kek.jp/

[36] The Babar Physics Book, eds. P. F. Harrison and H. R. Quinn (1998).

[37] F. Muheim "Flavour in the Era of LHC" HEP Forum, Abingdon, UK (2007). http://hepwww.rl.ac.uk/accel/forum/2007b/forum.html

[38] A. Golutvin "The Flavour Physics Potential of LHCb" *B*-Factory Symposium, QMUL, London, UK (2009). http://pprc.qmul.ac.uk/babarSymposium/