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Sparse numerical linear algebra and interpolation spaces

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Overview of talk

- Norms and duality in finite dimensional Hilbert spaces

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- Collaborators Drosos Kourounis , Rodrigue Kammogne

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$$c||v||_1 \leq ||v||_2 \leq C||v||_1 \quad \text{i.e.} \quad ||\cdot||_1 \sim ||\cdot||_2$$

Finite dimensional Hilbert spaces and R^N

- $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathsf{R}$ scalar product and
  $\|u\|_{\mathcal{H}} = \sqrt{(u, u)} \qquad \forall u \in \mathcal{H}$ norm.

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  $\|u\|_{\mathcal{H}} = \sqrt{(u,u)} \qquad \forall u \in \mathcal{H}$ norm.
- $\exists \{\psi_i\}_{i=1,\ldots,N}$ a basis for $\mathcal{H}$
  $\forall u \in \mathcal{H} \qquad u = \sum_{i=1}^{N} u_i \psi_i \qquad u_i \in \mathrm{R} \quad i = 1, \ldots, N$

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  $\forall u \in \mathcal{H} \qquad u = \sum_{i=1}^{N} u_i \psi_i \qquad u_i \in R \quad i = 1,\ldots,N$
- Representation of scalar product in R^N.
  Let $u = \sum_{i=1}^{N} u_i \psi_i$ and $v = \sum_{i=1}^{N} v_i \psi_i$.
  Then
  $$(u,v) = \sum_{i=1}^{N} \sum_{j=1}^{N} u_i v_j (\psi_i, \psi_j) = \mathbf{v}^T \mathbf{H} \mathbf{u}$$
  where $\mathbf{H}_{ij} = \mathbf{H}_{ji} = (\psi_i, \psi_j)$ and $\mathbf{u}, \mathbf{v} \in R^N$.
  Moreover, $\mathbf{u}^T \mathbf{H} \mathbf{u} > 0 \qquad \text{iff} \qquad \mathbf{u} \neq 0$ and, thus $\mathbf{H}$ SPD.

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- Dual vector
  Let $u \in \mathcal{H},\ u \neq 0$, then $\exists f_u \in \mathcal{H}^\star$ such that

$$f_u(u) = \|u\|_{\mathcal{H}}$$

(Hahn-Banach).

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  The dual vector of $\mathbf{u}$ has the following representation:

  $$\mathbf{f} = \frac{\mathbf{H}\mathbf{u}}{\|\mathbf{u}\|_\mathbf{H}}$$

  and

  $$\|f_u\|^2_{\mathcal{H}^\star} = \mathbf{u}^T\mathbf{H}\mathbf{u} = \mathbf{f}^T\mathbf{H}^{-1}\mathbf{f}$$

Linear operator

- $\mathscr{A} : \mathcal{H} \longrightarrow \mathcal{V}$ where $\mathcal{H}$ and $\mathcal{V}$ finite dimensional Hilbert spaces. $\mathbf{H}$ and $\mathbf{V}$ are the SPD matrices of the scalar products

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$$\mathbf{A}^T\mathbf{V}\mathbf{A}\mathbf{u} = \lambda \mathbf{H}\mathbf{u}$$
- $$\kappa_{\mathbf{H}}(\mathbf{M}) = \|\mathbf{M}\|_{\mathbf{H},\mathbf{H}^{-1}}\|\mathbf{M}^{-1}\|_{\mathbf{H}^{-1},\mathbf{H}}.$$

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The interesting case is $\kappa_{\mathbf{H}}(\mathbf{M})$ independent of N

Interpolation spaces

$$\begin{aligned}\mathcal{H} &= (\mathrm{R}^N, (u,v)_{\mathcal{H}} = \mathbf{u}^T\mathbf{H}\mathbf{v})\\ \mathcal{M} &= (\mathrm{R}^N, (u,v)_{\mathcal{M}} = \mathbf{u}^T\mathbf{M}\mathbf{v})\end{aligned}$$

Then $\exists \mathscr{S}$ self-adjoint such that

$$(u,v)_{\mathcal{H}} = (u, \mathscr{S}v)_{\mathcal{M}} = (\mathscr{S}u, v)_{\mathcal{M}}$$

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i.e.

$$(\mathbf{u},\mathbf{v})_{\mathbf{H}} = (\mathbf{u},\mathbf{S}\mathbf{v})_{\mathbf{M}} = (\mathbf{S}\mathbf{u},\mathbf{v})_{\mathbf{M}}$$

where $\mathbf{S} = \mathbf{M}^{-1}\mathbf{H}$
S (self-adjoint in the good scalar product!)

Interpolation spaces

$$\mathcal{H} = (\mathrm{R}^N, (u,v)_{\mathcal{H}} = \mathbf{u}^T\mathbf{H}\mathbf{v})$$
$$\mathcal{M} = (\mathrm{R}^N, (u,v)_{\mathcal{M}} = \mathbf{u}^T\mathbf{M}\mathbf{v})$$

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$\mathbf{S}$ (self-adjoint in the good scalar product!)

$$\left\{\mathbf{S}\mathbf{x} = \mu\mathbf{x} \quad \Leftrightarrow \quad \mathbf{H}\mathbf{x} = \mu\mathbf{M}\mathbf{x}\right\} \Rightarrow \mu = \delta^2 > 0$$

$\exists\mathbf{W}$ s.t. $\mathbf{M} = \mathbf{W}^T\mathbf{W}$, $\mathbf{H} = \mathbf{W}^T\boldsymbol{\Delta}^2\mathbf{W}$, $\boldsymbol{\Delta}$ diagonal $\boldsymbol{\Delta} \geq 0$

Interpolation spaces

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$$\mathbf{M}\mathbf{\Lambda} = \mathbf{W}^{T}\mathbf{W}\mathbf{W}^{-1}\mathbf{\Delta}\mathbf{W}^{-T}\mathbf{W}^{T}\mathbf{W} = \mathbf{\Lambda}^{T}\mathbf{M} \Longrightarrow (\mathbf{u}, \mathbf{\Lambda}\mathbf{v})_{\mathbf{M}} = (\mathbf{\Lambda}\mathbf{u}, \mathbf{v})_{\mathbf{M}}$$

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$$\mathbf{M\Lambda} = \mathbf{W}^{T}\mathbf{W}\mathbf{W}^{-1}\mathbf{\Delta}\mathbf{W}^{-T}\mathbf{W}^{T}\mathbf{W} = \mathbf{\Lambda}^{T}\mathbf{M} \Longrightarrow (\mathbf{u}, \mathbf{\Lambda v})_{\mathbf{M}} = (\mathbf{\Lambda u}, \mathbf{v})_{\mathbf{M}}$$

and

$$(\mathbf{\Lambda}^{1/2}\mathbf{u}, \mathbf{\Lambda}^{1/2}\mathbf{u})_{\mathbf{M}} = (\mathbf{u}, \mathbf{\Lambda u})_{\mathbf{M}}$$

Interpolation spaces

$$\left[\mathcal{H},\mathcal{M}\right]_{\vartheta} = \left\{\mathbf{u} \in \mathsf{R}^N; \left((\mathbf{u},\mathbf{u})_{\mathcal{M}} + (\mathbf{u},\mathbf{S}^{1-\vartheta}\mathbf{u})_{\mathcal{M}}\right)^{1/2}\right\}$$

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Interpolation spaces

$$\left[\mathcal{H}, \mathcal{M}\right]_{\vartheta} = \left\{\mathbf{u} \in \mathrm{R}^N; \left((\mathbf{u}, \mathbf{u})_{\mathcal{M}} + (\mathbf{u}, \mathbf{S}^{1-\vartheta}\mathbf{u})_{\mathcal{M}}\right)^{1/2}\right\}$$

$$\left[\mathcal{H}, \mathcal{M}\right]_{1/2} = \left\{\mathbf{u} \in \mathrm{R}^N; \left((\mathbf{u}, \mathbf{u})_{\mathcal{M}} + (\mathbf{u}, \mathbf{\Lambda}\mathbf{u})_{\mathcal{M}}\right)^{1/2}\right\}$$

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$$\left[\mathcal{H}, \mathcal{M}\right]_{\vartheta} = \left\{\mathbf{u} \in \mathrm{R}^{N}; \left((\mathbf{u}, \mathbf{u})_{\mathcal{M}} + (\mathbf{u}, \mathbf{S}^{1-\vartheta}\mathbf{u})_{\mathcal{M}}\right)^{1/2}\right\}$$

$$||\mathbf{v}||^2_{\vartheta,h} = ||\mathbf{v}||^2_{\mathbf{H}_{\vartheta,h}} = \mathbf{v}^T\left(\mathbf{M} + \mathbf{M}\mathbf{S}^{1-\vartheta}\right)\mathbf{v}$$

$$\mathbf{H}_{\vartheta,h} = \mathbf{M}\left(\mathbf{I} + \mathbf{S}^{1-\vartheta}\right) = \mathbf{W}^T\left(\mathbf{I} + \mathbf{\Delta}^{2(1-\vartheta)}\right)\mathbf{W} \quad \text{(Bessel)}$$

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Let us drop one of the **M**

$$\left\{ \mathbf{u} \in \mathrm{R}^N; (\mathbf{u}, \mathbf{S}^{1-\vartheta}\mathbf{u})^{1/2}_{\mathcal{M}} \right\}$$

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$$\mathbf{H}_{\vartheta} = \mathbf{M}\left(\mathbf{S}^{1-\vartheta} \right) = \mathbf{W}^T \left(\boldsymbol{\Delta}^{2(1-\vartheta)} \right) \mathbf{W} \quad \text{(Riesz)}$$

$$\mathbf{H}_{\vartheta} \sim \mathbf{H}_{\vartheta,h}$$

Interpolation spaces (duality)

$\mathcal{M}^\star$ and $\mathcal{H}^\star$ dual spaces of $\mathcal{M}$ and $\mathcal{H}$

$$\left[\mathcal{H}, \mathcal{M}\right]_\vartheta^\star = \left[\mathcal{M}^\star, \mathcal{H}^\star\right]_{1-\vartheta}$$

$$\mathbf{H}_{\vartheta,h}^{-1} \sim \mathbf{H}_{1-\vartheta,h}^\star \sim \mathbf{H}_{1-\vartheta}^\star$$

where

Interpolation spaces (duality)

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$$\Big[\mathcal{H},\mathcal{M}\Big]^\star_\vartheta = \Big[\mathcal{M}^\star,\mathcal{H}^\star\Big]_{1-\vartheta}$$

$$\mathbf{H}^{-1}_{\vartheta,h} \sim \mathbf{H}^\star_{1-\vartheta,h} \sim \mathbf{H}^\star_{1-\vartheta}$$

where

$$\mathbf{H}^\star_{1-\vartheta} = \mathbf{H}^{-1}(\mathbf{H}\mathbf{M}^{-1})^\vartheta = \mathbf{W}^{-1}\boldsymbol{\Delta}^{2(\vartheta-1)}\mathbf{W}^{-T} = \mathbf{H}^{-1}_\vartheta$$

Interpolation spaces (∞ dimensional case)

- X, Y two Hilbert spaces with $X \subset Y$, X **dense and continuously embedded** in Y. $\langle \cdot, \cdot \rangle_X, \langle \cdot, \cdot \rangle_Y$ scalar product and $\| \cdot \|_X, \| \cdot \|_Y$ the respective norms.
- (Riesz representation theory) $\exists \mathscr{S} : X \to Y$ positive and self-adjoint with respect to $\langle \cdot, \cdot \rangle_Y$ such that $\langle u, v \rangle_X = \langle u, \mathscr{S} v \rangle_Y$. $\mathscr{E} = \mathscr{S}^{1/2} : X \to Y$,
- $X = D(\mathscr{E})$ with $\|u\|_X \sim \|u\|_{\mathscr{E}} := \left(\|u\|_Y^2 + \|\mathscr{E} u\|_Y^2\right)^{1/2}$.
- $\|u\|_\theta := \left(\|u\|_Y^2 + \|\mathscr{E}^{1-\theta} u\|_Y^2\right)^{1/2}$.
- The *interpolation space of index* θ $[X, Y]_\theta := D(\mathscr{E}^{1-\theta}), \quad 0 \le \theta \le 1$, with the inner-product $\langle u, v \rangle_\theta = \langle u, v \rangle_Y + \left\langle u, \mathscr{E}^{1-\theta} v \right\rangle_Y$ is a Hilbert space (Lions Magenes 1968).
- $[X, Y]_0 = X$ and $[X, Y]_1 = Y$. If $0 < \theta_1 < \theta_2 < 1$ then

$$X \subset [X, Y]_{\theta_1} \subset [X, Y]_{\theta_2} \subset Y.$$

Interpolation Theorem

Let $\mathfrak{X}, \mathfrak{Y}$ Hilbert spaces $\mathfrak{X} \subset \mathfrak{Y}$ with $\mathfrak{X}$ dense in $\mathfrak{Y}$, and with inclusion compact and continuous. Let $\mathcal{X}, \mathcal{Y}$ satisfy similar properties. Let $\pi \in \mathcal{L}(\mathfrak{X}; \mathcal{X}) \cap \mathcal{L}(\mathfrak{Y}; \mathcal{Y})$. Then for all $\theta \in (0, 1)$,

$$\pi \in \mathcal{L}([\mathfrak{X}, \mathfrak{Y}]_\theta; [\mathcal{X}, \mathcal{Y}]_\theta).$$

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Let $\mathfrak{X} \supset \mathcal{X}_h$ and $\mathfrak{Y} \supset \mathcal{Y}_h$ ($\mathcal{X}_h$ and $\mathcal{Y}_h$ finite-dimensional spaces) $i_h : \mathscr{L}(\mathcal{X}_h; \mathfrak{X}) \cap \mathcal{L}(\mathcal{Y}_h; \mathfrak{Y})$ the continuous injection operator

$$i_h \in \mathcal{L}([\mathcal{X}_h, \mathcal{Y}_h]_\theta; [\mathfrak{X}, \mathfrak{Y}]_\theta).$$

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Let $\mathfrak{X}, \mathfrak{Y}$ Hilbert spaces $\mathfrak{X} \subset \mathfrak{Y}$ with $\mathfrak{X}$ dense in $\mathfrak{Y}$, and with inclusion compact and continuous. Let $\mathcal{X}, \mathcal{Y}$ satisfy similar properties. Let $\pi \in \mathcal{L}(\mathfrak{X}; \mathcal{X}) \cap \mathcal{L}(\mathfrak{Y}; \mathcal{Y})$. Then for all $\theta \in (0, 1)$,

$$\pi \in \mathcal{L}([\mathfrak{X}, \mathfrak{Y}]_\theta; [\mathcal{X}, \mathcal{Y}]_\theta).$$

$$\forall u_h \in [\mathcal{X}_h, \mathcal{Y}_h]_\theta, \ \|i_h u_h\|_\theta = \|u_h\|_\theta \leq C_1 \|u_h\|_{\theta,h}.$$

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Assume now that there exists an interpolation operator $\exists I_h$ such that $I_h : \mathcal{L}(\mathfrak{X}; \mathcal{X}_h) \cap \mathcal{L}(\mathfrak{Y}; \mathcal{Y}_h)$ and $I_h u = u_h$ for all $u_h \in \mathcal{X}_h$.

$$I_h \in \mathcal{L}([\mathfrak{X}, \mathfrak{Y}]_\theta; [\mathcal{X}_h, \mathcal{Y}_h]_\theta)$$

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$$\forall u \in [\mathfrak{X}, \mathfrak{Y}]_\theta, \ \|I_h u\|_{\theta,h} \leq C_2 \|u\|_\theta.$$

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$$\pi \in \mathcal{L}([\mathfrak{X}, \mathfrak{Y}]_\theta; [\mathcal{X}, \mathcal{Y}]_\theta).$$

$$\forall u_h \in [\mathcal{X}_h, \mathcal{Y}_h]_\theta, \; \|i_h u_h\|_\theta = \|u_h\|_\theta \leq C_1 \|u_h\|_{\theta,h}.$$

$$\forall u \in [\mathfrak{X}, \mathfrak{Y}]_\theta, \; \|I_h u\|_{\theta,h} \leq C_2 \|u\|_\theta.$$

Since $[\mathcal{X}_h, \mathcal{Y}_h]_\theta \subset [\mathfrak{X}, \mathfrak{Y}]_\theta$ then $\frac{1}{C_1}\|u_h\|_\theta \leq \|u_h\|_{\theta,h} \leq C_2 \|u_h\|_\theta$.

i.e. $\|u_h\|_\theta \sim \|u_h\|_{\theta,h}$

Interpolation spaces (∞ dimensional case)

$\Omega \subset \mathrm{R}^n$ open bounded with smooth boundary Γ and let $\boldsymbol{\alpha}$ denote a multi-index of order m where m is a positive integer

$$H^m(\Omega) = \left\{ u : D^{\boldsymbol{\alpha}} u \in L^2(\Omega), \quad |\boldsymbol{\alpha}| \leq m \right\} \qquad (H^0(\Omega) = L^2(\Omega))$$

$$H^s(\Omega) := [H^m(\Omega), H^0(\Omega)]_{1-s/m}$$

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$H_0^s(\Omega)$ completion of $C_0^\infty(\Omega)$ in $H^m(\Omega)$, where $s > 0$.

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For $0 \leq s_2 < s_1$ and k integer

$$\begin{aligned} [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta &= H_0^{(1-\theta)s_1+\theta s_2}(\Omega) \\ &\text{if} \quad (1-\theta)s_1 + \theta s_2 \neq k + 1/2 \end{aligned}$$

$$\begin{aligned} [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta &= H_{00}^{k+1/2}(\Omega) \subset H_0^{k+1/2} \\ &\text{if} \quad (1-\theta)s_1 + \theta s_2 = k + 1/2 \end{aligned}$$

$$H^{-s}(\Omega) = (H_0^s(\Omega))^\star \quad s > 0$$

If $(1-\theta)s_1 + \theta s_2 = 1/2$

$$[H^{-s_1}(\Omega), H^{-s_2}(\Omega)]_\theta = \left(H_{00}^{1/2}(\Omega) \right)^\star.$$

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Finite-element example

$$H_{00}^{1/2}(\Omega) = [H_0^1(\Omega), L^2(\Omega)]_{1/2}.$$

Let $\mathcal{X}_h \subset H_0^1(\Omega), \mathcal{Y}_h \subset L^2(\Omega)$. Let $\{\phi_i\}_{1\leq i\leq n} \in \mathcal{X}_h$ be a spanning set for $\mathcal{Y}_h$ and let $\mathbf{L}_k \in \mathrm{R}^{n\times n}$ denote the Grammian matrices corresponding to the $\langle \cdot, \cdot \rangle_{H_0^k(\Omega)}$-inner product $(H^0(\Omega) = L^2(\Omega))$:

$$(\mathbf{L}_k)_{ij} = \langle \phi_i, \phi_j \rangle_{H_0^k(\Omega)}.$$

$\mathbf{H} = \mathbf{L}_1, \;\; \mathbf{M} = \mathbf{L}_0$ and $\mathbf{H}_{1/2,h} = \mathbf{L}_0\left(\mathbf{I} + (\mathbf{L}_0^{-1}\mathbf{L}_1)^{1/2}\right)$ (Bessel)

Moreover, we have

$\mathbf{H}_{1/2,h} \sim \mathbf{H}_{1/2} = \mathbf{L}_0\left(\mathbf{L}_0^{-1}\mathbf{L}_1\right)^{1/2}$ (Riesz)

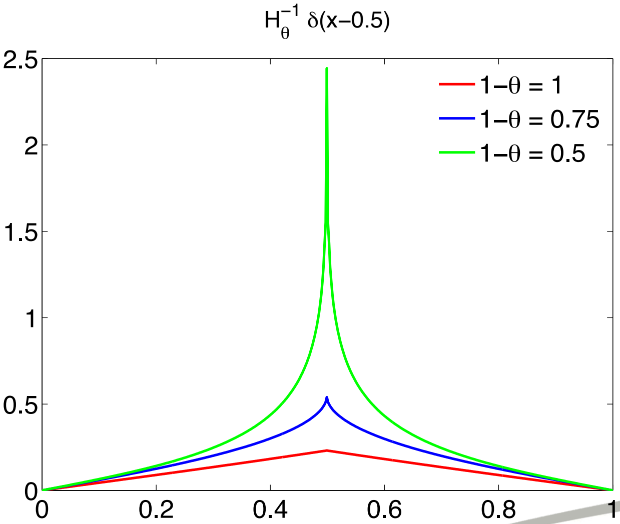
Interpolation theorem for FEM

Let the assumptions of Interpolation Theorem hold with $(\mathcal{X}, \mathcal{Y})$ replaced by $(\mathcal{X}_h, \mathcal{Y}_h)$ defined above. Let $\mathbf{H}_{\theta,h} = \mathbf{L}_0\left(\mathbf{I} + (\mathbf{L}_0^{-1}\mathbf{L}_1)^{1-\theta}\right)$, $\mathbf{H}_\theta = \mathbf{L}_0(\mathbf{L}_0^{-1}\mathbf{L}_1)^{1-\theta}$. Then there exist constants c, C independent of n such that

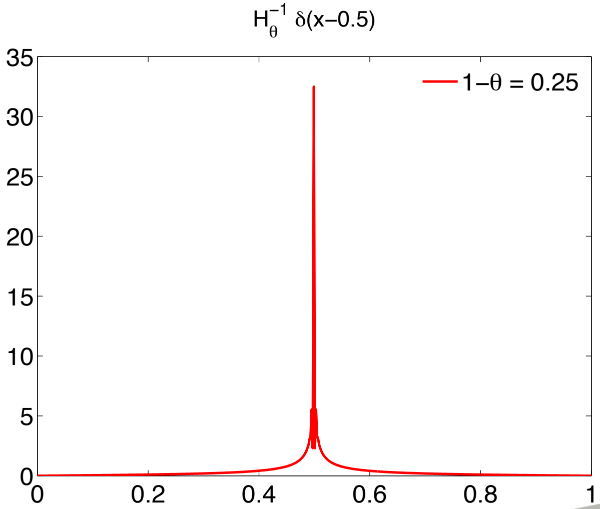
$$\begin{array}{rcl} c\|u_h\|_{[\mathfrak{X},\mathfrak{Y}]_\theta} \leq & \|\mathbf{u}\|_{\mathbf{H}_{\theta,h}} & \leq C\|u_h\|_{[\mathfrak{X},\mathfrak{Y}]_\theta}, \\ c\|u_h\|_{[\mathfrak{X},\mathfrak{Y}]_\theta} \leq & \|\mathbf{u}\|_{\mathbf{H}_\theta} & \leq C\|u_h\|_{[\mathfrak{X},\mathfrak{Y}]_\theta}, \end{array}$$

for all $u_h \in [\mathcal{X}_h, \mathcal{Y}_h]_\theta$ and with $\theta \in (0,1)$, $\mathfrak{X} = L^2(\Omega)$, and $\mathfrak{Y} = H_0^1(\Omega)$

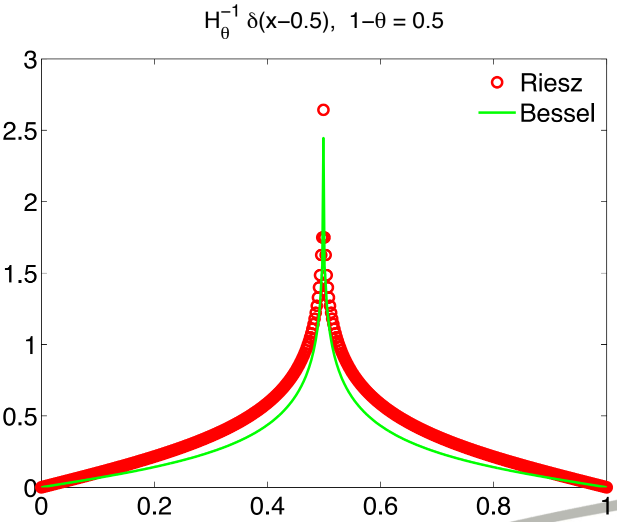
Few examples



Few examples



Few examples



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Evaluation of $\mathbf{H}_\theta \mathbf{z}$

- Generalised Lanczos
  $\mathbf{H}\mathbf{V}_k = \mathbf{M}\mathbf{V}_k\mathbf{T}_k + \beta_{k+1}\mathbf{M}\mathbf{v}_{k+1}\mathbf{e}_k^T, \quad \mathbf{V}_k^T\mathbf{M}\mathbf{V}_k = \mathbf{I}_k$
  ($\mathbf{T}_k$ tridiagonal).

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  ($\mathbf{T}_k$ tridiagonal).
- $\mathbf{v}_0 = \mathbf{z}$
  $\mathbf{H}_\theta \mathbf{z} \approx \mathbf{M}\mathbf{V}_k \mathbf{T}_k^{1-\theta} \mathbf{e}_1 \|\mathbf{z}\|_{\mathbf{M}}$ and
  $\mathbf{H}_{\theta,h} \mathbf{z} \approx \mathbf{M}\mathbf{V}_k (\mathbf{I}_k + \mathbf{T}_k^{1-\theta}) \mathbf{e}_1 \|\mathbf{z}\|_{\mathbf{M}}$.

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  $\mathbf{H}_{\theta,h}\mathbf{z} \approx \mathbf{M}\mathbf{V}_k(\mathbf{I}_k + \mathbf{T}_k^{1-\theta})\mathbf{e}_1\|\mathbf{z}\|_\mathbf{M}$.
- $\mathbf{v}_0 = \mathbf{M}^{-1}\mathbf{z}$
  $\mathbf{H}_\theta^{-1}\mathbf{z} \approx \mathbf{V}_k\mathbf{T}_k^{\theta-1}\mathbf{e}_1\|\mathbf{z}\|_{\mathbf{M}^{-1}}$ and
  $\mathbf{H}_{\theta,h}^{-1}\mathbf{z} \approx \mathbf{V}_k(\mathbf{I}_k + \mathbf{T}_k^{1-\theta})^{-1}\mathbf{e}_1\|\mathbf{z}\|_{\mathbf{M}^{-1}}$.
- Alternative: N. Hale, and N. J. Higham and L. N. Trefethen,

  SIAM J. Numer. Anal.

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Preconditioners for the Steklov-Poincaré operator

Let Ω be an open subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ and consider the model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Given a partition of Ω into two subdomains $\Omega \equiv \Omega_1 \cup \Omega_2$ with common boundary Γ this problem can be equivalently written as

$$\begin{cases} -\Delta u_1 = f & \text{in } \Omega_1, \\ u_1 = 0 & \text{on } \partial\Omega_1 \setminus \Gamma, \end{cases} \qquad \begin{cases} -\Delta u_2 = f & \text{in } \Omega_2, \\ u_2 = 0 & \text{on } \partial\Omega_2 \setminus \Gamma, \end{cases}$$

with the 'interface conditions'

$$\begin{cases} u_1 = u_2 \\ \frac{\partial u_1}{\partial n_1} = -\frac{\partial u_2}{\partial n_2} \end{cases} \quad \text{on } \Gamma$$

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Preconditioners for the Steklov-Poincaré operator

Given $\lambda_1, \lambda_2 \in H^{1/2}_{00}(\Gamma)$, ψ_1, ψ_2 denote the harmonic extensions of λ_1, λ_2 respectively into Ω_1, Ω_2, i.e., for $i = 1, 2$, ψ_i satisfy

$$\begin{cases} -\Delta\psi_i &= 0 \qquad \text{in } \Omega_i, \\ \psi_i &= \lambda_i \qquad \text{on } \Gamma, \\ \psi_i &= 0 \qquad \text{on } \partial\Omega_i \setminus \Gamma. \end{cases}$$

The Steklov-Poincaré operator $\mathscr{S} : H^{1/2}_{00}(\Gamma) \to H^{-1/2}(\Gamma)$

$$\langle \mathscr{S}\lambda_1, \lambda_2 \rangle_{H^{1/2}(\Gamma)} = \langle \nabla\psi_1, \nabla\psi_2 \rangle_{L^2(\Omega)} =: s(\lambda_1, \lambda_2).$$

$$c_1 \|\lambda\|^2_{H^{1/2}(\Gamma)} \le s(\lambda, \lambda) \le c_2 \|\lambda\|^2_{H^{1/2}(\Gamma)}.$$

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Preconditioners for the Steklov-Poincaré operator

$$
\text{(i)} \quad \begin{cases} -\Delta u_i^{\{1\}} & = f \ \text{ in } \Omega_i, \\ \quad\ \ u_i^{\{1\}} & = 0 \ \text{ on } \partial\Omega_i, \end{cases}
$$

$$
\text{(ii)} \quad \left\{ \ \mathscr{S}\lambda = -\frac{\partial u_1^{\{1\}}}{\partial n_1} - \frac{\partial u_2^{\{1\}}}{\partial n_2} \ \text{ on } \Gamma, \right.
$$

$$
\text{(iii)} \quad \begin{cases} -\Delta u_i^{\{2\}} & = 0 \ \text{ in } \Omega_i, \\ \quad\ \ u_i^{\{2\}} & = \lambda \ \text{ on } \partial\Omega_i. \end{cases}
$$

The resulting solution is

$$
u|_{\Omega_i} = u_i^{\{1\}} + u_i^{\{2\}}.
$$

An other problem

$$\begin{cases} -\nu\Delta u + \vec{b}\cdot\nabla u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Discrete Formulation

$$\mathcal{V}^h = \mathcal{V}^{h,r} := \left\{ w \in C^0(\Omega) : w|_{\mathfrak{t}} \in P_k \quad \forall \mathfrak{t} \in \mathfrak{T}_h \right\} \subset H^1(\Omega)$$

be a finite-dimensional space of piecewise polynomial functions defined on some subdivision $\mathfrak{T}_h$ of Ω into simplices $\mathfrak{t}$ of maximum diameter h. Let further $\mathcal{V}_I^h, \mathcal{V}_B^h \subset \mathcal{V}^h$ satisfy $\mathcal{V}_I^h \oplus \mathcal{V}_B^H \equiv \mathcal{V}^h$ where $\mathcal{V}_I^h = \left\{ w \in \mathcal{V}^h : w|_{\partial\Omega} = 0 \right\}$. Let $\mathcal{X}_h \subset H_0^1(\Gamma)$ denote the space spanned by the restriction of the basis functions of $\mathcal{V}_I^h$ to the internal boundary Γ.

Discrete Formulation

$$
\begin{array}{ll}
\text{(i)} & \mathbf{A}_{II,i}\mathbf{u}_i^{\{1\}} = \mathbf{f}_{I,i}, \\
\text{(ii)} & \mathbf{S}\mathbf{u}_B = \mathbf{f}_B - \mathbf{A}_{IB,1}^T\mathbf{u}_1^{\{1\}} - \mathbf{A}_{IB,2}^T\mathbf{u}_2^{\{2\}}, \\
\text{(iii)} & \mathbf{A}_{II,i}\mathbf{u}_i^{\{2\}} = -\mathbf{A}_{IB,1}^T\mathbf{u}_B - \mathbf{A}_{IB,2}^T\mathbf{u}_B,
\end{array}
$$

where $\mathbf{S}$ is the Schur complement corresponding to the boundary nodes

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2, \quad \mathbf{S}_i = \mathbf{A}_{BB,i} - \mathbf{A}_{IB,i}^T\mathbf{A}_{II,i}^{-1}\mathbf{A}_{IB,i}.$$

The resulting solution is $(\mathbf{u}_{I,1},\ \mathbf{u}_{I,2},\ \mathbf{u}_B)$ where

$$\mathbf{u}_{I,i} = \mathbf{u}_i^{\{1\}} + \mathbf{u}_i^{\{2\}}.$$

$H_{00}^{1/2}$-preconditioners

Let $\mathcal{X}_h = \mathrm{span}\,\{\phi_i, 1 \le i \le m\}$ *be defined as above and let* $(\mathbf{L}_k)_{ij} = \langle \phi_i, \phi_j \rangle_{H_0^k(\Gamma)}$ *for* $k = 0, 1$. *Let*

$$\mathbf{H}_{1/2} := \mathbf{L}_0 (\mathbf{L}_0^{-1} \mathbf{L}_1)^{1/2}.$$

Then for all $\boldsymbol{\lambda} \in R^m \setminus \{\mathbf{0}\}$

$$\kappa_1 \le \frac{\boldsymbol{\lambda}^T \mathbf{S} \boldsymbol{\lambda}}{\boldsymbol{\lambda}^T \mathbf{H}_{1/2} \boldsymbol{\lambda}} \le \kappa_2$$

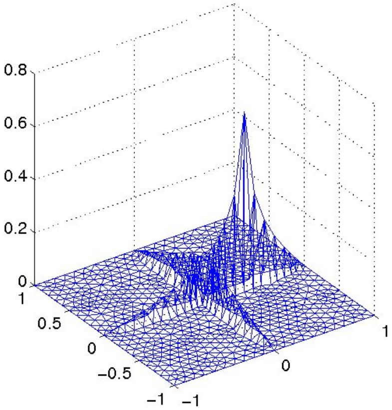
with κ_1, κ_2 *independent of* h.

Discrete DD and Preconditioning

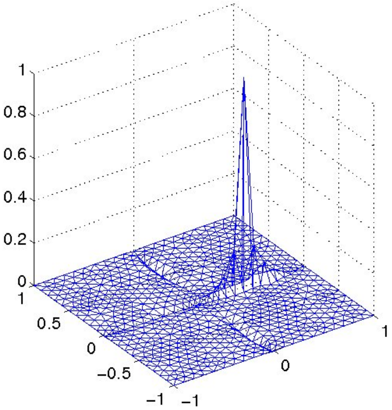
$$\mathbf{P} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ 0 & \mathbf{P}_S \end{pmatrix}$$

with $\mathbf{A}_{II} = \nu \mathbf{L}_{II} + \mathbf{N}_{II}$ where $\mathbf{L}_{II}$ is the direct sum of Laplacians assembled on each subdomain and $\mathbf{N}_{II}$ is the direct sum of the convection operator $\vec{b} \cdot \nabla$ assembled also on each subdomain. With P_S we denote the approximation of $\mathbf{H}_{00}^{1/2}$ by a vector or of $\mathbf{H}^{-1/2}$ by a vector. Then we use FGMRES.

Green functions on wirebasket

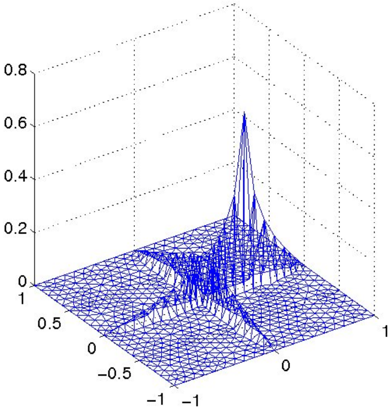


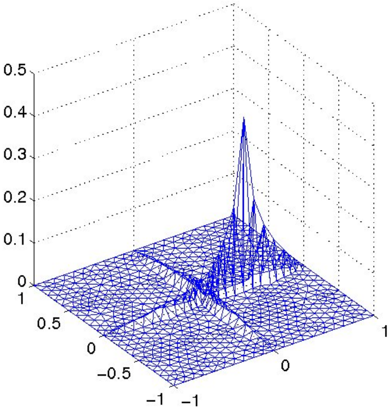
Steklov-Poincaré



Neumann-Neumann

Green functions on wirebasket





Steklov-Poincaré $H^{1/2}$

Numerical results: Poisson equation

			Linear			Quadratic		
#*dom*	*n*	*m*	$H_{1/2,h}$	$H_{1/2}$	$\widehat{H}_{1/2}$	$H_{1/2,h}$	$H_{1/2}$	$\widehat{H}_{1/2}$
4	45,377	449	10	9	9	11	11	11
	180,865	897	10	10	10	11	11	11
	722,177	1793	11	11	11	11	11	11
16	45,953	1149	13	12	12	13	13	13
	183,041	2301	13	13	13	13	13	13
	730,625	4605	13	13	13	13	13	13
64	66,049	3549	16	14	14	16	15	15
	263,169	7133	16	15	15	16	15	15
	1,050,625	14,301	17	16	15	17	15	15

FGMRES iterations for model problem .

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Numerical results: an other problem

			Linear			Quadratic		
#*dom*	*n*	*m*	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$
4	45,377	449	10	12	21	12	13	20
	180,865	897	11	11	20	12	13	19
	722,177	1793	11	11	19	12	12	18
16	45,953	1149	12	17	37	13	17	35
	183,041	2301	13	17	35	13	16	32
	730,625	4605	12	15	32	12	15	30
64	66,049	3549	16	22	55	17	21	51
	263,169	7133	17	22	52	16	20	46
	1,050,625	14,301	15	19	47	16	19	43

FGMRES iterations for 2nd model problem

Laplace-Beltrami

We can extend everything to an interface that is the union of manifolds $\mathfrak{m}_k$ by using the Lapace-Beltrami operator and interpolating between $L^2(\Gamma)$ and $H^1_{\partial\Omega}(\Gamma)$ with the norm

$$\|u\|_{H^1_{\partial\Omega}(\Gamma)} = \left(\sum_{k=1}^{K} \|u_k\|^2_{H^1_{\partial\Omega}(\mathfrak{m}_k)}\right)^{1/2}.$$

using $H^1_0(\mathfrak{m}_k)$ with

$$|v|^2_{H^1_0(\mathfrak{m}_k)} = \int_{\mathfrak{m}_k} \left|\nabla^k_\Gamma v\right|^2 \mathrm{d}s(\mathfrak{m}_k)$$

where ∇^k_Γ denote the tangential gradient of v with respect to $\mathfrak{m}_k$

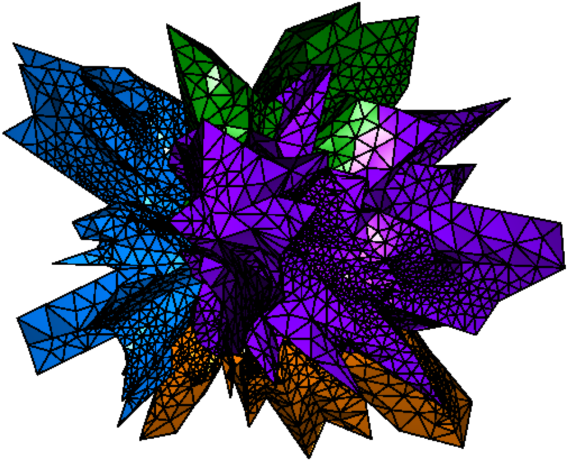
$$\nabla^k_\Gamma v(\mathbf{x}) := \nabla v(\mathbf{x}) - \mathbf{n}_k(\mathbf{x})(\mathbf{n}_k(\mathbf{x}) \cdot \nabla v(\mathbf{x})),$$

where $\mathbf{n}_k(\mathbf{x})$ is the normal to $\mathfrak{m}_k$ at $\mathbf{x}$.

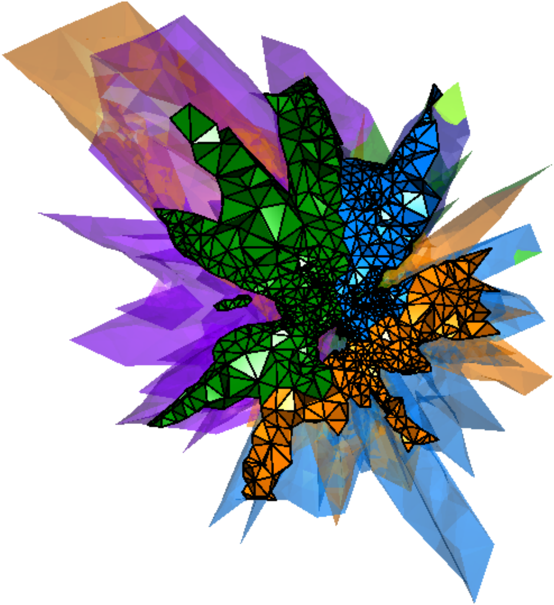
A. , Kourounis, Loghin IMA J. Num. Anal. 2012

Other Domains: CRYSTAL

A. , Kourounis, Loghin IMA J. Num. Anal. 2012



Other Domains: CRYSTAL



Other Domains: CRYSTAL

	$(L, M)_{1/2}$			$(L, I)_{1/2}$		
n	$N = 16$	$N = 64$	$N = 256$	$N = 16$	$N = 64$	$N = 256$
240,832	17	24	31	110	104	141
2,521,753	21	25	29	28	71	65

Iterations for the crystal problem with and without the mass matrix.

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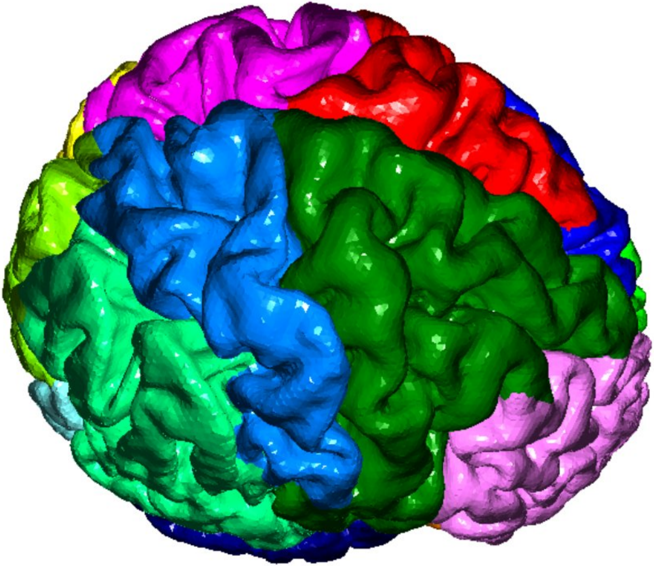
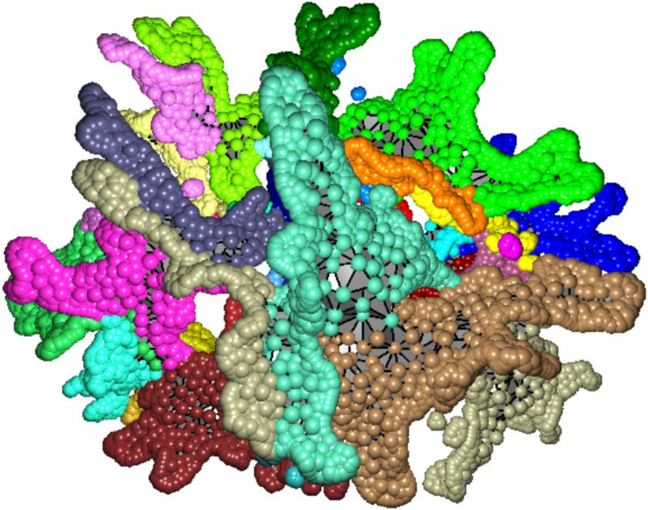
Other Domains: "BRAIN"



Other Domains: "BRAIN"



Other Domains: "BRAIN"

	$n = 5120357$	$n = 25973106$
$N = 1024$	$n_b = 679160$ it $= 22$	$n_b = 2067967$ it $= 22$
$N = 2048$	$n_b = 895170$ it $= 22$	$n_b = 2737064$ it $= 23$
$N = 4096$	$n_b = 1172815$ it $= 24$	$n_b = 3602083$ it $= 23$

Results for reaction-diffusion PDE on Brain (N number of subdomains, n_b number of nodes in interface, it FGMRES iteration number, and $\theta = 0.7$).

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Reaction-Diffusion Systems

Rodrigue Kammogne, D. Loghin Proceed. DD 2012

Rodrigue Kammogne, D. Loghin Tech. Rep. in preparation $\Omega \subset \mathbf{R}^2$

$$\begin{cases} -\mathbf{D}\Delta\mathbf{u} + \mathbf{M}\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

Reaction-Diffusion Systems

Rodrigue Kammogne, D. Loghin Proceed. DD 2012
Rodrigue Kammogne, D. Loghin Tech. Rep. in preparation $\Omega \subset \mathrm{R}^2$

$$\begin{cases} -\mathbf{D}\Delta\mathbf{u} + \mathbf{M}\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \ \mathbf{M} = \begin{bmatrix} \alpha_(x,y) & \beta_1(x,y) \\ \beta_2(x,y) & \alpha_2(x,y) \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \text{ (SPD).}$$

$\mathbf{f} \in L^2(\Omega)$ and $\mathbf{M}$ satisfies

$$0 < \gamma_{min} < \frac{\boldsymbol{\xi}^T \mathbf{M} \boldsymbol{\xi}}{\boldsymbol{\xi}^T \boldsymbol{\xi}} \ \forall \boldsymbol{\xi} \in \mathrm{R}^2 \setminus \{\}\,; \text{ and} ||\mathbf{M}|| < \gamma_{max}.$$

Reaction-Diffusion Systems

Rodrigue Kammogne, D. Loghin Proceed. DD 2012
Rodrigue Kammogne, D. Loghin Tech. Rep. in preparation $\Omega \subset \mathbb{R}^2$

$$\begin{cases} -\mathbf{D}\Delta\mathbf{u} + \mathbf{M}\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

$$\alpha_1 = \begin{cases} 1 & \text{if } x^2 + y^2 < 1/4 \\ 100 & \text{otherwise} \end{cases} ; \alpha_2 = \begin{cases} 100 & \text{if } x^2 + y^2 < 1/4 \\ 1 & \text{otherwise} \end{cases}$$

$$\beta_1 = \begin{cases} 0.1 & \text{if } x^2 + y^2 < 1/4 \\ 1 & \text{otherwise} \end{cases} \quad \beta_2 = \begin{cases} 1 & \text{if } x^2 + y^2 < 1/4 \\ 0.1 & \text{otherwise} \end{cases}$$

Reaction-Diffusion Systems

Rodrigue Kammogne, D. Loghin Proceed. DD 2012

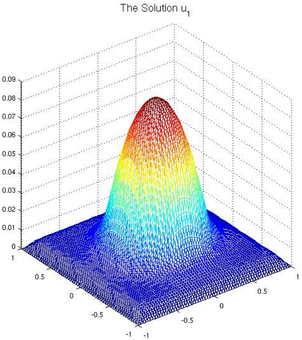
Rodrigue Kammogne, D. Loghin Tech. Rep. in preparation $\Omega \subset \mathbf{R}^2$

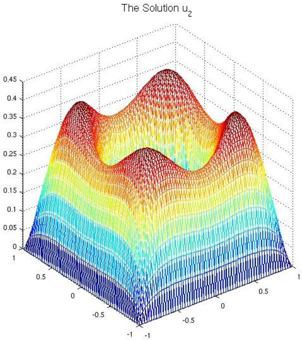
$$\begin{cases} -\mathbf{D}\Delta\mathbf{u} + \mathbf{M}\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

$d_1 = 1$, $d_2 = 0.1$	
domains=	4 16 64
size=8450	18 24 28
33282	19 25 28
132098	20 26 28

Reaction-Diffusion Systems

Rodrigue Kammogne, D. Loghin Proceed. DD 2012
Rodrigue Kammogne, D. Loghin Tech. Rep. in preparation





Other applications: Mathematical Finance

L.Silvestre Communications on Pure and Applied Mathematics 2007

Luis A. Caffarelli, Sandro Salsa, Luis Silvestre Invent. math. 2008

- Let X_t be an α-stable Levy process such that $X_0 = x$ for some point $x \in \mathbb{R}^n$.

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-

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Other applications: Quasi-Geostrophic Equation

L.Silvestre Ann. I. H. Poincare (2010)
L. Caffarelli, L. Silvestre, Comm. Partial Differential Equations (2007)
L. Caffarelli, A. Vasseur, Ann. of Math., (2012)
P. Constantin, J. Wu, Ann. I. H. Poincare Anal. Non Lin. (2008), (2009)
P. Constantin, J. Wu, SIAM J. Math. Anal. (1999)

Other applications: Quasi-Geostrophic Equation

$$\theta : \mathrm{R}^2 \times [0, +\infty) \to \mathrm{R}$$

$$\partial_t \theta(x,t) + w \cdot \nabla\theta(x,t) + (-\Delta)^{\alpha/2}\theta(x,t) = 0, \qquad \theta(x,0) = \theta_0$$

and

$$w = (R_2\theta, R_1\theta)$$

where R_i are the Riesz transforms

$$R_i\theta(x) = cPV \int_{\mathrm{R}^2} \frac{(y_i - x_i)\theta(y)}{|y-x|^3} dy.$$

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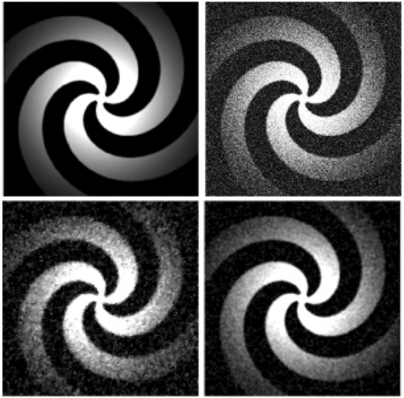
- Interpolation spaces are not only useful in DD
- Link with integro-differential operator such as Riemann-Liouville fractional derivative (M.Riesz 1938,1949).
- In modelling complex phenomena the use of non-local operators is a new promising subject attracting increasing attention.
- Other areas of application that are worth to mention include: BEM and image processing (filtering):

top-left: original top-right:noised



bottom-left: $\min \left\{ \int_\Omega |\nabla u(x)| \mathrm{d}x + 1/50 \int_\Omega (u_0(x) - u(x))^2 \mathrm{d}x \right\}$

Pascal Getreuer (2007)

bottom-right: $\min \left\{ ||u||^2_{1/2} + 1/50 \int_\Omega (u_0(x) - u(x))^2 \mathrm{d}x \right\}$