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Stitching the Yukawa Quilt

P Ramond R G Roberts and G G Ross

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Stitching the Yukawa Quilt

P. Ramond*
Institute for Fundamental Theory,
Department of Physics, University of Florida,
Gainesville Florida 32611 USA

R. G. Roberts
Rutherford Appleton Laboratory,
Chilton, Didcot, Oxon, OX11 OQX, UK

and

G. G. Ross†
Department of Theoretical Physics,
Oxford University,
Keble Road, Oxford, UK

Abstract

We develop a systematic analysis of quark mass matrices which, starting with the measured values of quark masses and mixing angles, allows for a model independent search for all possible (symmetric or hermitian) mass matrices having texture zeroes at the unification scale. A survey of all six and five texture zero structures yields a total of five possible solutions which may be distinguished by improved measurements of the CKM matrix elements and which may readily be extended to include lepton masses with the Georgi-Jarlskog texture. The solutions naturally suggest a parameterisation for the mass matrices based on a perturbative expansion and we present some speculations concerning the origin of such structure.

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†SERC Senior Research Fellow

1 Introduction

The Yukawa sector of the Standard Model is parameterized in terms of three 3×3 matrices of Yukawa coupling constants. These serve to determine the nine quark and lepton masses as well as the three angles and one phase of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Of the masses only one is presently unknown, the top quark mass although it cannot be heavier than 200 GeV without spoiling the consistency of the Standard Model with experiment.

While the three Yukawa matrices appear as independent parameters at the level of the Standard Model, they can be correlated in various theoretical extensions. In specific GUTs, for example, there are relations between fermion masses, the best known being the $SU(5)$ relation $m_b = m_\tau$ [1] at the unification scale M_X. As with the relation amongst gauge couplings this relation applies at the unification scale and must be radiatively corrected. These corrections offer a further tantalising piece of circumstantial evidence in favour of supersymmetric unification for, starting with the relation $m_b(M_X) = m_\tau(M_X)$, they can bring the prediction for m_b/m_τ into agreement with experiment using the same value for M_X found in the analysis of gauge couplings [2]. This cannot be done in the non-supersymmetric case without adding new interactions.

Further predictions for fermion masses require more sophisticated GUTs because the simple relations $m_s = m_\mu$ and $m_d = m_e$ plus radiative corrections are grossly in disagreement with experiment. One inspired choice was proposed by Georgi and Jarlskog [3] and subsequently developed by Harvey, Ramond and Reiss [4]. Recently it was run in the SUSY case by Dimopoulos, Hall and Raby [5] and by Ramond [6] and Arason, Castaño, Ramond and Piard [7]. In this and other [8] Ansätze the number of parameters needed to specify the mass matrices is limited by the requirement that there be "texture" zeroes [9],[10]. The maximum number of such zeroes in the up and down quark mass matrices consistent with the absence of a zero mass eigenvalue is six. With six zeroes there are left just six real parameters plus one phase to describe the six quark masses, the three mixing angles and the CP-violating phase and so one obtains relations between the masses and mixing angles. It was shown that, including SUSY radiative corrections, the resulting predictions for the low energy parameters are in remarkably good agreement with experiment.

This illustrates how the analysis of fermion masses can lend support to the hypothesis of a stage of unification, in the same way as did the analysis of gauge couplings. In both cases one uses the renormalisation group to continue the measured values and looks for simplicity appearing at the unification scale; in the case of gauge couplings simplicity is equality between the couplings, in the case of the mass matrices simplicity is simple ratios of quark and lepton masses and the appearance of "texture" zeroes. However the analysis of the mass matrices so far presented falls short of the ideal "bottom-up" approach for it starts by assuming a particular, theoretically motivated, texture for the mass matrices rather than just starting with measured values, continuing them to high energies, and looking for simplicity in the form of a definite texture.

In this paper we will attempt to implement this "bottom-up" approach through a systematic study of possible zeroes in the quark Yukawa couplings. The major difficulty in implementing this program is that laboratory measurements only determine the masses, i.e. the diagonal mass matrix $\mathbf{Y}_u^{Diag}, \mathbf{Y}_d^{Diag}$, and the CKM mass matrix, *not* the full mass matrices, $\mathbf{Y}_u, \mathbf{Y}_d$; i.e. we have

$$\begin{aligned} \mathbf{Y}_u^{Diag} &= \mathbf{R}_u^L.\mathbf{Y}_u.\mathbf{R}_u^{R\dagger} \\ \mathbf{Y}_d^{Diag} &= \mathbf{R}_d^L.\mathbf{Y}_d.\mathbf{R}_d^{R\dagger} \\ \mathbf{V}_{CKM} &= \mathbf{R}_u^L.\mathbf{R}_d^{L\dagger} \end{aligned} \tag{1}$$

If we are to determine $\mathbf{R}_{u,d}^{L,R}$, and hence the mass matrices separately, a simplifying assumption is needed. These mass matrices come from the Yukawa sector of the theory which, in the standard model is given by

$$\mathcal{L}_Y = Q\mathbf{Y}_u\bar{u}H^* + Q\mathbf{Y}_d\bar{d}H + L\mathbf{Y}_e\bar{e}H \tag{2}$$

where Q_i are the three quark doublets, $\bar{u}_i$, $\bar{d}_i$ the three right handed charge $-2/3$, $1/3$ antiquark isodoublets, L_i the three lepton doublets, $\bar{e}_i$ the three right handed antilepton isosinglets, and H is the Higgs doublet normalized to its vacuum value.

Given that the general case is not tractable we will concentrate in this paper on a promising possibility that has been widely studied, namely the case that the matrices in family space $\mathbf{Y}_u$, $\mathbf{Y}_d$, $\mathbf{Y}_e$ are symmetric in family space. At the level of the Standard Model and $SU(5)$, this is not necessarily true, but at the $SO(10)$ level and beyond, where each family appears as a single representation, this assumption is natural (but not inevitable). With this assumption we can diagonalize the Yukawa matrices, or the associated mass matrices, by means of a Schur rotation i.e. $\mathbf{R}^L = \mathbf{R}^{R\star} \equiv \mathbf{R}$.

We can analyze the most general symmetric mass matrix case[1] leading to five or six texture zeroes, because there are just 6 possible forms of symmetric mass matrix with an hierarchy of three non-zero eigenvalues and three texture zeroes (at least one of the up or down quark mass matrices must have three of the texture zeroes). Allowing for the redefinition of the quark fields to absorb phases, these matrices involve just three real parameters. This allows us to determine, up to the six fold discrete ambiguity, the diagonalising matrix $\mathbf{R}_u$ (or $\mathbf{R}_d$) in terms of the masses. Hence, using eq(1), we may compute $\mathbf{R}_d$ (or $\mathbf{R}_u$) in terms of the CKM matrix and hence find $\mathbf{Y}_d$ (or $\mathbf{Y}_u$). Further texture zeroes in $\mathbf{Y}_u$, $\mathbf{Y}_d$ will result in predictions for the mixing angles of the CKM matrix.

The advantage of this technique is that it allows a determination of the down quark mass matrix using experimentally measured quantities without prejudicing the result by the choice of a specific texture. Thus the general problem of searching for structure in mass matrices may be solved with the assumption of symmetric mass matrices for the case that there are 5 or 6 texture zeroes. We will also consider the remaining case of just 4 texture zeroes, although in this instance there are fewer predictions making it somewhat uninteresting given the success of the more predictive forms.

The paper is organised as follows. In Section 2 we introduce the procedure needed for a general analysis including the introduction of a Wolfenstein-like parameterisation for the mass matrices and the inclusion of the radiative corrections needed to continue the mass matrices to high energy. Section 3 gives an explicit example of the analysis and presents the textures consistent with present measurements of masses and mixing angles. Section 4 presents the results of our general analysis in which the radiative corrections are deter-

[1] In fact the analysis applies to hermitian matrices too for a general 3×3 symmetric mass matrix may be transformed to an hermitian matrix through the freedom to redefine the nine phases of the three left-handed doublets and six right-handed singlets of quark fields. These 9 phases may be used to make both the up and down quark mass matrices hermitian since it is always possible to choose a basis in which either the top or the bottom mass matrix is diagonal.

mined numerically by integrating the renormalisation group equations. This allows us to include the effects of thresholds correctly and to perform a complete analysis of the gauge couplings, radiative electroweak breaking, and masses and mixing angles. Section 5 discusses these results in the context of the analytic solutions. Finally Section 6 presents our conclusions.

2 A general analysis for symmetric mass matrix texture.

The procedure we adopt is straightforward:

- We first assume that the up quark mass matrix $\mathbf{Y}_u$ has a certain texture (*i.e.* zeroes in specific places)). After using the freedom to redefine quark phases to make the elements of $\mathbf{Y}_u$ real the diagonalising matrix $\mathbf{R}_u$ is parametrized by three angles[2]. If $\mathbf{Y}_u$ has three zeroes, these angles will be related to quark mass ratios. If $\mathbf{Y}_u$ has just two zeroes there will be one undetermined angle.

- We now form the down quark mixing matrix by computing

$$\mathbf{R}_d = P_d \mathbf{V}_{CKM}^\dagger P_u^\dagger . \mathbf{R}_u \tag{3}$$

  where $P_{u,d}$ are diagonal matrices of phases $e^{i\,\phi^i_{u,d}}$ needed to express the "measured" CKM matrix in a basis in which the quark fields have arbitrary phases. Using $\mathbf{R}_d$ we may form the down quark mass matrix

$$\mathbf{Y}_d = \mathbf{R}_u^\dagger . P_u . \mathbf{V}_{CKM} . P_d^\dagger \mathbf{Y}_d^{Diag} . P_d . \mathbf{V}_{CKM}^\dagger . P_u^\dagger . \mathbf{R}_u \tag{4}$$

  In writing this we have assumed $\mathbf{Y}_d$ is hermitian rather than symmetric and is diagonalised by an hermitian matrix. As discussed above we are free to do this because we may use the freedom to redefine quark phases to change the symmetric matrix to an hermitian matrix.

- We examine the matrix elements of $\mathbf{Y}_d$ and derive relations between quark masses and the CKM mixing angles by requiring that some of these elements be zero (texture zeroes).

[2]With this phase convention it is clear the CP violating phase resides in $\mathbf{R}_d$

We then start the process all over again, *i.e.* input a different texture for $\mathbf{Y}_u$, etc.

- Finally we repeat the whole process starting with the down quark mass matrix $\mathbf{Y}_d$ and computing the up quark mass matrix $\mathbf{Y}_u$. These steps leave us with all possible relations among quark masses and mixing angles, derived from requiring zeroes in the Yukawa matrices.

- Before comparing with experiment, we run each of these relations through the renormalization group machine to include the radiative corrections.

This outlines the scheme of analysis we adopt. Its implementation requires the determination of the possible texture structures, the parameterisation of the mass matrices in a manner that allows for a systematic analysis of the predictions, and a determination of the radiative corrections. We turn now to a discussion of each of these points.

2.1 Possible texture structures

As we mentioned above the analysis of the general case is possible because there are just 6 possible forms of symmetric mass matrices with just three non-zero eigenvalues and the maximum number (three) of texture zeroes capable of describing the hierarchy of up or down quark mass matrices. These are

1.

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix} \tag{5}$$

2. [3]

$$\begin{pmatrix} 0 & a_2 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix} \tag{6}$$

[3]This is the form used in refs [3, 4, 5, 7, 11]

3.

$$\begin{pmatrix} a_3 & 0 & 0 \\ 0 & 0 & b_3 \\ 0 & b_3 & c_3 \end{pmatrix} \tag{7}$$

4. [4]

$$\begin{pmatrix} 0 & 0 & a_4 \\ 0 & b_4 & 0 \\ a_4 & 0 & c_4 \end{pmatrix} \tag{8}$$

5. [5].

$$\begin{pmatrix} 0 & a_5 & 0 \\ a_5 & 0 & b_5 \\ 0 & b_5 & c_5 \end{pmatrix} \tag{9}$$

6.

$$\begin{pmatrix} 0 & a_6 & b_6 \\ a_6 & 0 & 0 \\ b_6 & 0 & c_6 \end{pmatrix} \tag{10}$$

where $a_i \lesssim b_i \lesssim c_i$ are constants determined by the quark masses. Note that we have chosen the axes so that the largest entry (approximately equal to the heaviest quark mass) is in the (3,3) position. It may readily be verified that these six forms are the complete set of possibilities up to relabeling of the axes.

We will analyse all of these possibilities in turn for the up (or the down) quark mass matrices. Then using eq(4) we may study the implications of three, two or one further zeroes in the down (or the up) quark mass matrices, corresponding to a total of 6, 5 or 4 texture zeroes.

The most predictive Ansatz has a total of 6 zeroes (3 in the up and 3 in the down quark matrices) reducing the number of parameters needed to specify the mass matrix. In terms of these the 6 up and down quark masses

[4]This is the form used for up quarks in ref. [8].

[5]This is the Fritzsch [10] matrix for both up and down quarks. This form is used for the up quarks only in refs. [3, 4, 7, 11]

and the CKM matrix elements must be determined. In principle there are 6 measurable quantities in the unitary CKM matrix but for small mixing angles this is reduced to 4 leaving a total of 10 experimentally measurable quantities. Mass matrices with texture are overconstrained leading to the prediction of relations between these quantities and it is our task to find which, if any, of these textures is viable.

We are also able completely to analyse the case of five texture zeroes, although one of the up or down quark rotations is not completely determined. The analysis in this case proceeds exactly as before because in this case too one of the up or down mass matrices has three zeroes and is given by one of the set above.

The analysis of the case of four texture zeroes using the set of matrices above is incomplete because four zeroes may also occur when both the up and down matrices have just two zeroes. Below we will discuss this case too, but we have not analysed its implications fully because of the residual uncertainty in determining both the up and down current quark basis.

2.2 Parameterization of the mass matrices

As we will see, in the analytic analysis of possible textures it is useful to parameterize the quark mass matrices in a way that keeps track of the order of magnitude of the various components of the mass matrices. This was done for the CKM matrix by Wolfenstein [12]

$$\mathbf{V}_{CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho+i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho+i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{11}$$

where the small expansion parameter is $\lambda \approx .2$ the (1,2) matrix element of the CKM matrix (approximately the Cabibbo angle) and $A \approx .9 \pm .1$. The assignment of the CP phase in eqn(11) is arbitrary, the only constraint is the invariance of J^{CP}. We now go one step further and parametrize the down quark masses *à la* Wolfenstein. We choose[6]

[6]The parameterisation of quark masses *à la* Wolfenstein is not new, although here we take it to apply at the Unification scale rather than at low energies. The first references are given in [13].

$$\mathbf{Y}_d^{Diag} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & \hat{m}_s/\lambda^2 & 0 \\ 0 & 0 & \hat{m}_b/\lambda^4 \end{pmatrix} \tag{12}$$

where

$$\begin{aligned} \hat{m}_s &\approx m_s\, \lambda^2 \\ \hat{m}_b &\approx m_b\, \lambda^4 \end{aligned} \tag{13}$$

are both of order m_d. This is a general parameterization of the down quark mass matrix which is useful because it exhibits the order of magnitude of the various elements. The analogous form for the up quarks is

$$\mathbf{Y}_u^{Diag} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & \hat{m}_c/\lambda^4 & 0 \\ 0 & 0 & \hat{m}_t/\lambda^8 \end{pmatrix} \tag{14}$$

where

$$\begin{aligned} \hat{m}_c &\approx m_c \lambda^4 \\ \hat{m}_t &\approx m_t \lambda^8 \end{aligned} \tag{15}$$

2.3 Evolution of the CKM matrix and the mass eigenvalues.

We have stressed that the experimentally observed quantities used in our analysis are the CKM matrix and the eigenvalues of the mass matrix and not the full mass matrices. For the analysis presented above we would like to know these quantities at the unification scale and thus we need to consider their radiative corrections. In an elegant paper, Olechowski and Pokorski [15] studied the renormalisation group evolution of the CKM matrix and the masses and we can use their results to get a rough idea of the size of the radiative corrections.

Keeping the top and bottom Yukawa couplings only and neglecting thresholds, the CKM matrix elements evolve as

$$\begin{aligned} 16\pi^2 \frac{d \mid V_{ij} \mid}{dt} &= -\frac{3c}{2}(h_t^2 + h_b^2) \mid V_{ij} \mid, \; ij = 13, 31, 23, 32 \\ \frac{d \mid V_{12} \mid}{dt} &\sim \mathcal{O}(\lambda^4) \end{aligned} \tag{16}$$

Here $t = ln(Q/Q_0)$ where the elements are evaluated at the scale Q, h_t and h_b are the Yukawa couplings and c is a constant determined by the couplings of the theory and is 2/3 in the MSSM. The irreducible phase of the CKM matrix also evolves via the equation

$$16\pi^2 \frac{d \mid J_{CP} \mid}{dt} = -3c(h_t^2 + h_b^2) \mid J_{CP} \mid \tag{17}$$

where we can choose $\mid J_{CP} \mid= \mathrm{Im}(V_{23}V_{12}V_{13}^\star V_{22}^\star)$. If we substitute the Wolfenstein parameterisation, eq(11) in eqs(16) and (17) we find $\mid J_{CP} \mid= A^2\lambda^6(1 - \lambda^2/2)(\rho + i\eta)$ and

$$\begin{aligned} 16\pi^2 \frac{dA}{dt} &= -\frac{3c}{2}(h_t^2 + h_b^2)A \\ \frac{d\lambda}{dt} &\approx 0 \\ \frac{d\rho}{dt} &= 0 \\ \frac{d\eta}{dt} &= 0 \end{aligned} \tag{18}$$

The beauty of this result is that only the A parameter in $\mathbf{V}_{CKM}$ evolves on going from low to high scales making the analysis including radiative corrections quite straightforward. The remaining radiative corrections are to the diagonal Yukawa couplings and have the form

$$16\pi^2 \frac{d(h_u/h_t)}{dt} = -\frac{3}{2}(bh_t^2 + ch_b^2)(h_u/h_t) \tag{19}$$

and similarly for (h_c/h_t).

$$16\pi^2 \frac{d(h_d/h_b)}{dt} = -\frac{3}{2}(ch_t^2 + bh_b^2)(h_d/h_b) \tag{20}$$

and similarly for (h_s/h_b). Here b=2 for the MSSM.

2.3.1 Quantitative estimates.

As we discuss in the next section, if we impose the GUT relation $m_b = m_\tau$ at the GUT scale together with reasonable boundary conditions on the SUSY breaking parameters, we need a relatively large value of h_t to get the required value of the running mass $m_b = 4.25 \pm 0.1$ GeV evaluated at m_b. We can take h_t to be approximately constant over the range M_X down to M_Z. It is then convenient to introduce the parameter χ defined by $\chi = (M_X/M_Z)^{-h_t^2/(16\pi^2)}$. We have

$$\begin{aligned}
\frac{A(M_X)}{A(M_z)} &= \chi \\
\frac{(h_d/h_b)(M_X)}{(h_d/h_b)(M_Z)} &= \chi \\
\frac{(h_u/h_t)(M_X)}{(h_u/h_t)(M_Z)} &= \chi^3
\end{aligned} \tag{21}$$

With the value of $h_t \approx 1.25$ used in our favourite analysis of electroweak breaking (cf. Section 4.2) we have $\chi \approx 0.7$. With this value the evolution of $\mathbf{V}_{CKM}$ up to M_X requires A should be reduced by $\approx$30%. Relative to h_d and h_s, h_b should be increased by $\approx$30% at M_X and relative to h_u and h_c, h_t is increased by a factor 2.5−3 at M_X.

The beauty of this form is that the radiative corrections are entirely specified in terms of the single parameter χ. Simply by varying χ it is easy to adjust for other possible values of the top quark coupling.

3 Texture analysis

We turn now to the results of applying the analytic analysis of texture zeroes to the symmetric mass matrices. To illustrate the method of Section 2 we consider the case where we start with a specific texture structure with just *two* zeroes for the down quark mass matrix. This will illustrate the general method capable of dealing with the 4 zero case and also, by setting one of the elements zero, the 5 and 6 zero cases too.

$$\mathbf{Y}_d = \begin{pmatrix} 0 & a & 0 \\ a & b & c \\ 0 & c & d \end{pmatrix} \tag{22}$$

Note that choosing c=0 gives the structure 2 (eq(6)) and b=0 gives the structure 5 (eq(9)). In this equation we have used five of the eight relative quark phases to make the elements of $\mathbf{Y}_d$ all real. For $a \lesssim b, c \lesssim d$ the rotation $\mathbf{R}_d$ is approximately given by

$$\mathbf{R}_d \approx \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & -s_3 \\ 0 & s_3 & c_3 \end{pmatrix} \tag{23}$$

where $c_{1,3} = \cos(\theta_{1,3})$, $s_{1,3} = \sin(\theta_{1,3})$ and

$$\begin{aligned} s_1 &\approx \frac{ad}{c^2 - bd} \\ s_3 &\approx \frac{c}{d} \end{aligned} \tag{24}$$

and

$$\begin{aligned} m_b &\approx d \\ m_s &\approx -b + \frac{c^2}{d} \\ m_d &\approx a^2/m_s \end{aligned} \tag{25}$$

Then applying the analogue of eq(25) for the up quark mass matrix we have

$$\begin{aligned} \mathbf{Y}_u &= \mathbf{R}_d^\dagger . P_d \,.\, \mathbf{V}_{CKM}^\dagger \,.\, P_u^\dagger . \mathbf{Y}_u^{Diag} . P_u \,.\, \mathbf{V}_{CKM} \,.\, P_d^\dagger . \mathbf{R}_d \\ &= P_d . P_d^\dagger . \mathbf{R}_d^\dagger . P_d \,.\, \mathbf{V}_{CKM}^\dagger \,.\, P_u^\dagger . \mathbf{Y}_u^{Diag} . P_u \,.\, \mathbf{V}_{CKM} \,.\, P_d^\dagger . \mathbf{R}_d . P_d . P_d^\dagger \end{aligned} \tag{26}$$

It is convenient to compute $\mathbf{Y}_u$ in a different basis $\mathbf{Y}'_u = P_d^\dagger . \mathbf{Y}_u . P_d$, and to absorb the effect of $P_d^\dagger . \mathbf{R}_d . P_d$ by allowing the off diagonal elements to be complex with phases ϕ and θ. It is now straightforward to determine the implications of zeroes in the $\mathbf{Y}'_u$ matrix. To illustrate the method we quote the result for $\mathbf{Y}'_u$ setting $s_3 = 0$ to keep the algebra manageable

$$\begin{aligned} \mathbf{Y}'_u(1,1) &= \frac{\hat{m}_c(1-\epsilon)^2 + A^2 \hat{m}_t(\epsilon - (1 - \rho + i\eta))^2}{\lambda^2} + \hat{m}_c \epsilon (1-\epsilon) + m_u + \mathrm{O}(\lambda) \\ \mathbf{Y}'_u(1,2) &= \frac{-\hat{m}_c(1-\epsilon) + A^2 \hat{m}_t(\epsilon - (1 - \rho + i\eta))}{\lambda^3} + \mathrm{O}(1) \end{aligned}$$

$$
\begin{aligned}
\mathbf{Y}'_u(1,3) &= \frac{-A\,\hat{m}_t(\epsilon-(1-\rho+i\,\eta))}{\lambda^5}+O(\lambda)^{-1}\\
\mathbf{Y}'_u(2,2) &= \frac{\hat{m}_c+A^2\hat{m}_t}{\lambda^4}+O(\lambda)^{-2}\\
\mathbf{Y}'_u(2,3) &= -\frac{A\,\hat{m}_t}{\lambda^6}+O(\lambda)^{-4}\\
\mathbf{Y}'_u(3,3) &= \frac{\hat{m}_t}{\lambda^8}+O(1) \qquad (27)
\end{aligned}
$$

where, following the discussion of Section 2.2, we have parameterised the up quark mass matrix in terms of $\hat{m}_c$, $\hat{m}_t$ and ϵ is defined to be of order one, $\epsilon \equiv s_1 e^{i\phi}/\lambda$, so that the expansion in λ is well ordered.

It is now straightforward to use the form of eq(27) to find the implications of one or more texture zeroes in the up quark mass matrix. From eq(27) we see that it is not possible for $\mathbf{Y}'_u(3,3)$ or $\mathbf{Y}'_u(2,3)$ to be zero. The other matrix elements may vanish for special values of the parameters. For example if the (1,3) matrix element vanishes then, to leading order in the Maclaurin series in λ

$$1-\rho+i\,\eta=\epsilon \qquad (28)$$

It is possible for the (1,1) matrix element to vanish simultaneously with the (1,3) element. Inserting eq(28) in $\mathbf{Y}'_u(1,1)$ and working to the next non-vanishing order in the expansion in powers of λ gives

$$\frac{\hat{m}_c(1-\epsilon)^2}{\lambda^2}+m_u=0 \qquad (29)$$

At leading order this gives

$$\hat{m}_c(1-\epsilon)^2=0 \qquad (30)$$

which, from eqs(24) and (25) gives the Gatto- Sartori-Tonin-Oakes (GSTO) [16] relation

$$\lambda=V_{us}=\sqrt{m_d/m_s} \qquad (31)$$

Solving eq(29) to the next order gives

$$\mid V_{us}\mid=\lambda=\left(\frac{m_d}{m_s}+\frac{m_u}{m_c}+2\sqrt{\frac{m_d\,m_u}{m_s\,m_c}}\cos\phi\right)^{\frac{1}{2}} \qquad (32)$$

where the phase ϕ comes from the phase of ϵ.

Eq(28) with eq(29) gives

$$\rho^2 + \eta^2 = \lambda \mid \frac{m_u}{\hat{m}_c} \mid = \lambda^{-3} \mid \frac{m_u}{m_c} \mid \tag{33}$$

Eqs(32) and (33) are the consequences of two texture zeroes in the (1,1) and (1,3) positions. From eq(32) we see that the expansion parameter, λ, is of the correct order given by the GSTO relation but with a small correction which determines the phase, ϕ, of ϵ. From eq(33) we see that $\mid \rho + i\eta \mid$ is predicted in terms of this expansion parameter to be small, of order $\sqrt{\lambda}$. From eq(28) and the determination of ϕ, ρ and η may separately be determined. This gives an example of a texture with 5 zeroes and it may readily be verified that no further zeroes can be obtained. We will return to a discussion of these results in Section 5.

3.1 Results of the analytic analysis

The discussion presented in the last section illustrates the method for studying 5 and 6 texture zeroes. Including $r \neq 0$ in eq(26) the most general case of 4 texture zeroes can be also be analysed. Using this method we have surveyed all structures with 6 or 5 texture zeroes. A complete discussion of these results will appear elsewhere but here we list just those solutions that are consistent with present measurements of the CKM matrix elements. Encouragingly there are solutions, consistent with the hoped for simplicity in the mass matrices, but the number of possible solutions is limited; no solutions with 6 texture zeroes were found[7] and only five 5 texture zero solutions are consistent with the measured masses and mixings. These are given in Table 1.

Note that the example presented above corresponds to Solution 4 with 5 texture zeroes (pairs of off-diagonal zeroes are counted as one zero due to

[7]In reference [5] a solution with 6 texture zeroes was presented corresponding to Solution 2 with $E' = 0$. We do not include this in our acceptable set of solutions as it leads to the prediction $\mathbf{V}_{cb} = \sqrt{m_c/m_t}$ which is larger than present indications that $m_t \leq 180$GeV from LEP data [14]. The non-zero entry in the (2,3) position of the $\mathbf{Y}_d$ matrix of Solution 2 reduces the value of $\mathbf{V}_{cb}$. However this is the nearest we get to a 6 zero solution, setting one of elements to zero in any of the other solutions leads to conflict with data. Note also that our solution no.3 corresponds to the Ansatz of ref([8]).

Solution	$\mathbf{Y}_u$	$\mathbf{Y}_d$
1	$\begin{pmatrix} 0 & C & 0 \\ C & B & 0 \\ 0 & 0 & A \end{pmatrix}$	$\begin{pmatrix} 0 & F & 0 \\ F^\star & E & E' \\ 0 & E' & D \end{pmatrix}$
2	$\begin{pmatrix} 0 & C & 0 \\ C & 0 & B \\ 0 & B & A \end{pmatrix}$	$\begin{pmatrix} 0 & F & 0 \\ F^\star & E & E' \\ 0 & E'^\star & D \end{pmatrix}$
3	$\begin{pmatrix} 0 & 0 & C \\ 0 & B & 0 \\ C & 0 & A \end{pmatrix}$	$\begin{pmatrix} 0 & F & 0 \\ F^\star & E & E' \\ 0 & E' & D \end{pmatrix}$
4	$\begin{pmatrix} 0 & C & 0 \\ C & B & B' \\ 0 & B' & A \end{pmatrix}$	$\begin{pmatrix} 0 & F & 0 \\ F^\star & E & 0 \\ 0 & 0 & D \end{pmatrix}$
5	$\begin{pmatrix} 0 & 0 & C \\ 0 & B & B' \\ C & B' & A \end{pmatrix}$	$\begin{pmatrix} 0 & F & 0 \\ F^\star & E & 0 \\ 0 & 0 & D \end{pmatrix}$

Table 1: Symmetric textures. Approximate forms for the parameters extracted from the various fits are given in Table 2

the symmetric structure assumed). After using the freedom to choose the 9 independent quark phases there is one phase left which we take to be the phase of the complex parameter F. Thus we see this solution needs 7 real parameters and one phase to describe the mass matrices and hence the 6 quark masses and 4 CKM parameters. This implies two relations consistent with our analysis showing ρ and η were determined. As a bonus we found that the magnitude of the Cabibbo angle was determined by the approximate GSTO relation.

The other solutions may be analysed in an equivalent way. Solutions 1,3,4 and 5 have 8 parameters and hence yield two relations. Solution 2 has an additional phase and hence only gives one relation. The structure of these relations may be algebraically complex (for example try solving the conditions that the (1,1) and (1,2) matrix elements of eq(27) simultaneously vanish - Solution 5). In order to check the predictions it is convenient to make a best fit to the masses and CKM matrix elements in terms of the non-zero elements. We will do this in the next section in a more complete analysis including the threshold effects in the renormalisation group analysis.

4 Numerical Analysis of Yukawa Matrices

In the last section we discussed the analytic determination of the implications of texture zeroes using the analytic form of the radiative corrections to CKM matrix and the quark masses. In this section we present the results of a more complete and exact analysis where we study all possible 5 or 6 texture zero choices including radiative corrections using a numerical solution to the renormalisation group equations which correctly includes the effects of thresholds in the analysis, omitted in our approximate analytic treatment. In this we begin at the scale M_X with a given pair of Yukawa matrices $\mathbf{Y}_u$ and $\mathbf{Y}_d$ and evolve down to low energies to examine the resulting fermion masses and CKM matrix elements. Again we present the five solutions which exhibit different textures but are all consistent with the data.

4.1 Details of the Procedure

The evolution of the $\mathbf{Y}_u$ mass matrix, for example, is

$$16\pi^2\frac{d\mathbf{Y}_u}{dt} = \frac{3}{2}(b\mathbf{Y}_u\mathbf{Y}_u^\dagger + c\mathbf{Y}_d\mathbf{Y}_d^\dagger)\mathbf{Y}_u + (\Sigma_u - A_u)\mathbf{Y}_u \tag{34}$$

where $t = \ln Q$, $\Sigma_u = Tr[3\mathbf{Y}_u\mathbf{Y}_u^\dagger + 3a\mathbf{Y}_d\mathbf{Y}_d^\dagger + a\mathbf{Y}_e\mathbf{Y}_e^\dagger]$, $A_u = A_{u3}g_3^2 + A_{u2}g_2^2 + A_{u1}g_1^2$. Here $\mathbf{Y}_e$ is the lepton mass matrix and the values coefficients a, b, c, A_{ui} depend on whether MSSM or SM is relevant. Analogous to eq(34) there are equations for evolving the $\mathbf{Y}_{d,l}$ matrices. Details may be found in ref [17] for example. In ref [18] we derived solutions for the SUSY spectrum by combining MSSM and unification of the gauge couplings and demanding that the electroweak symmetry is spontaneously broken by radiative corrections. We continue with this description here but ensuring that across each individual threshold of SUSY particles, Higgs and quarks etc., not only do the gauge couplings alter but also the values of the coefficients in eq(34) and indeed individual elements in $\mathbf{Y}_{u,d}$ and $\mathbf{Y}_e$.

In this procedure the parameters at the unification scale determining the supersymmetric mass spectrum are the soft SUSY breaking parameters $m_{1/2}, m_0, \mu_0, A, B$. The allowed ranges of these parameters may be constrained by the need to obtain the correct scale of electroweak breaking without fine tuning, consistency with dark matter abundance and unification of gauge couplings. In our estimates of threshold effects we will use values consistent with these constraints.

We choose forms for $\mathbf{Y}_{u,d}$ at M_X, either one of the six three-zero types given by eqns (5)-(10) or one of the 12 possible two-zero types derived from them. Provided $\mathbf{Y}_e$ is small it plays a negligible role in the evolution equation 34 and can be ignored in an analysis of the quark structure. However, as has been stressed in [5], the simple Georgi-Jarlskog Ansatz is very successful in describing the lepton mass matrix and so, for completeness, we analyse this structure too. Motivated by this Ansatz the form for $\mathbf{Y}_e$ we take is governed by the choice for $\mathbf{Y}_d$, taking the same zero texture but multiplying the (2,2) element by the factor 3 in order to give the relations for the eigenvalues at the scale $Q = M_X$: $h_\tau = h_b, h_\mu = 3h_s, h_e = h_d/3$[8]

[8]We have chosen to include a discussion of the lepton masses to illustrate that all the quark mass structures of interest here may be combined with reasonable lepton masses,

Given the initial conditions we now evolve down to low energies via the renormalisation group equations all quantities including the matrix elements of $\mathbf{Y}_{u,d}$ and $\mathbf{Y}_e$ and attempt to fit the values of $m_e, m_\mu, m_\tau, m_d, m_s, m_b, m_u, m_c$ and the CKM matrix elements. We have included the analysis of lepton masses for completeness but we stress once more that our assumption of a Georgi-Jarlskog-like Ansatz in the lepton sector plays a negligible role in determining the quark structure.

4.2 Results of analysis

As discussed above, we actually include the lepton masses in generating our solutions. In particular we assume $h_b/h_\tau = 1$ at $Q = M_X$ and to achieve a value for this ratio at low energy around 2.4 we find we need h_t to be relatively large. Depending on our particular choice for the parameters of the MSSM at $Q = M_X$ ($m_{1/2}$, m_0, etc.) the values of h_t are in the range 1$-$1.5 and this value enters into all the relevant evolution equations of the masses. Predictions for $\tan\beta$ depend in particular on the assumed values for A, B at $Q = M_X$ and this uncertainty (together with the uncertainty on m_b/m_τ) translates into a range of values for m_t of 145$-$185 GeV. The masses of the other fermions are independent of the precise value of m_t however.

Altogether we find five solutions for the Yukawa matrices at $Q = M_X$ which are consistent with the low energy fermion masses and CKM matrix elements. The structure of the five forms are listed in table 1. Rather than list the numeric values we present, in table 2, approximations to the matrix elements in powers of $\lambda(\simeq 0.22)$, the small parameter in eqn (11). In table 3 we list the results of the five solutions for the fermion masses and CKM matrix elements.

Note that in our five solutions there is a single candidate structure, for $\mathbf{Y}_d$, namely the form of eq(22), which for certain cases reduces to the form of eq(6). For $\mathbf{Y}_u$ the five solutions correspond to the forms eqs(6, 8, 9, 10, (6), (8), (9), (19) and one related to eq(22) by an interchange of axes.

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given the simple G-J Ansatz. However, the analysis of of quark mass matrices is essentially independent of this Ansatz and does not change if we choose not to include lepton masses in the analysis.

Solution	$\mathbf{Y}_u$	$\mathbf{Y}_d$
1	$\begin{pmatrix} 0 & \sqrt{2}\lambda^6 & 0 \\ \sqrt{2}\lambda^6 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 4\lambda^3 \\ 0 & 4\lambda^3 & 1 \end{pmatrix}$
2	$\begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & 0 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 2\lambda^3 \\ 0 & 2\lambda^3 & 1 \end{pmatrix}$
3	$\begin{pmatrix} 0 & 0 & \sqrt{2}\lambda^4 \\ 0 & \lambda^4 & 0 \\ \sqrt{2}\lambda^4 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 4\lambda^3 \\ 0 & 4\lambda^3 & 1 \end{pmatrix}$
4	$\begin{pmatrix} 0 & \sqrt{2}\lambda^6 & 0 \\ \sqrt{2}\lambda^6 & \sqrt{3}\lambda^4 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
5	$\begin{pmatrix} 0 & 0 & \lambda^4 \\ 0 & \sqrt{2}\lambda^4 & \frac{\lambda^2}{\sqrt{2}} \\ \lambda^4 & \frac{\lambda^2}{\sqrt{2}} & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Table 2: Approximate forms for the symmetric textures using the parameterisation of Section 2.

Solution	1	2	3	4	5	Experiment [19, 20]
m_d	.0075	.0072	.0082	.0071	.0071	.0055 – .0115
m_s	0.159	0.170	0.164	0.169	0.169	0.105 – 0.230
m_b	4.23	4.23	4.23	4.19	4.20	4.15 – 4.35
m_u	.0042	.0040	.0045	.0044	.0040	.0031 – .0064
m_c	1.27	1.28	1.27	1.30	1.20	1.22 – 1.32
$\|V_{ud}\|$	.9756	.9754	.9759	.9756	.9758	.9747 – .9759
$\|V_{us}\|$	.2197	.2203	.2180	.2197	.2185	.218 – .224
$\|V_{ub}\|$	.0029	.0029	.0034	.0029	.0040	.003 – .008
$\|V_{cd}\|$	.2195	.2201	.2178	.2195	.2184	.218 – .224
$\|V_{cs}\|$	.9744	.9743	.9747	.9744	.9748	.9734 – .9752
$\|V_{cb}\|$	.0483	.0471	.0500	.0490	.0448	.035 – .047
$\|V_{td}\|$	.0100	.0096	.0110	.0098	.0078	.006 – .018
$\|V_{ts}\|$	.0474	.0462	.0488	.0481	.0443	.035 – .047
$\|V_{tb}\|$	.9988	.9989	.9987	.9988	.9990	.9987 – .9994
ϕ_{CP}	111^0	65^0	96^0	117^0	49^0	$25^0 - 160^0$

Table 3: Resulting values for masses and CKM matrix elements for the five solutions of table 1. Masses in GeV. For the solutions considered in this analysis, $m_t \sim 180$ GeV but this value could easily be decreased to around 150 GeV as a result of the uncertainty in the value of $\tan\beta$ and m_b.

5 Discussion of the supersymmetric textures.

5.1 Structure of the $\mathbf{Y}_d$ matrix at M_X.

All of the textures found in the last section have the form

$$\mathbf{Y}_d = m_b \begin{pmatrix} 0 & \alpha\,\lambda^4 & 0 \\ \alpha\,\lambda^4 & \beta\,\lambda^3 & \gamma\,\lambda^3 \\ 0 & \gamma\,\lambda^3 & 1 \end{pmatrix} \tag{35}$$

where the parameters α, β, and γ are of $O(1)$

$$\begin{aligned} \alpha &= \frac{\sqrt{\hat{m}_s m_d}}{\hat{m}_b} \\ \beta &= \frac{\hat{m}_s}{\hat{m}_b} \end{aligned} \tag{36}$$

The results of the last section determine the constants α, β and γ : $\alpha = \beta = 2$, $\gamma = 2N$ for $N = 0, 1, 2$. To make a connection with the analysis of Section 3 we note that

$$\begin{aligned} \frac{\hat{m}_s}{\hat{m}_b} &= 2 \\ \frac{\hat{m}_s}{m_d} &= 1 \\ s_3 &= \frac{1}{2}N\lambda^2 \\ s_1 &= \lambda \end{aligned} \tag{37}$$

where s_1 and s_3 are the mixing angles of eq(23), the general structure of $\mathbf{R}_d$ given in this equation being relevant because the $\mathbf{Y}_d$ matrix of Table 1 is always of the form of eq(22).

5.2 Structure of the $\mathbf{Y}_u$ matrix at M_X.

Similarly the solutions 1,2,and 4 for the $\mathbf{Y}_u$ matrix have the form

$$\mathbf{Y}_u = m_t \begin{pmatrix} 0 & \alpha'\,\lambda^6 & 0 \\ \alpha'\,\lambda^6 & \beta'\,\lambda^4 & \gamma'\,\lambda^2 \\ 0 & \gamma'\,\lambda^2 & 1 \end{pmatrix} \tag{38}$$

where

$$\begin{aligned} \alpha' &= \frac{\sqrt{m_c' m_u}}{m_t'} \\ \beta' &= \frac{m_c'}{m_t'} \end{aligned} \tag{39}$$

and we have chosen the (2,3) matrix element in eq(39) as is found in our fits. The results of the last section determine the constants α', β' and γ'

With the phases chosen so that the $\mathbf{Y}_u$ matrix is real the matrix needed to diagonalise it has the form

$$\mathbf{R}_u = \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix} \tag{40}$$

Using $\mathbf{R}_d$ given in eq(23) we may determine the CKM matrix in the basis in which $\mathbf{Y}_d$ is diagonal, by

$$\mathbf{V}_{CKM} = \mathbf{R}_u\, P_u\, \mathbf{R}_d^\dagger \tag{41}$$

where P^u is the diagonal matrix of phases which relates the basis in which $\mathbf{Y}_u$ is diagonal to the basis in which $\mathbf{Y}_d$ is real. Just how many phases are required to determine P^u depends on the number of texture zeroes. An overall phase is irrelevant so we may write

$$P_u = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \tag{42}$$

Thus there are at most two phases (ϕ and θ) giving for $\mathbf{V}_{CKM}$

$$\mathbf{V}_{CKM} \approx \begin{pmatrix} c_1c_2 - s_1s_2e^{-i\phi} & s_1e^{i\phi} + c_1s_2 & s_2(s_3 - s_4e^{i\theta}) \\ -c_1s_2 - s_1e^{-i\phi} & -s_1s_2e^{i\phi} + c_1c_2c_3c_4 + s_3s_4e^{-i\theta} & s_3 - s_4e^{i\theta} \\ s_1e^{-i\phi}(s_3 - s_4e^{-i\theta}) & -c_1(s_3 - s_4e^{i\theta}) & c_3c_4 + s_3s_4e^{i\theta} \end{pmatrix} \tag{43}$$

giving

$$\begin{aligned} \mid V_{us} \mid &= (s_1^2 + s_2^2 + 2s_1s_2\cos\phi)^{\frac{1}{2}} \\ \mid V_{cb} \mid &= (s_3^2 + s_4^2 + 2s_3s_4\cos\theta)^{\frac{1}{2}} \\ \frac{|V_{ub}|}{|V_{cb}|} &= s_2 \end{aligned} \tag{44}$$

Solution	m_c/m_t	m_u/m_c	V_{cb}	V_{ub}/V_{cb}
1	$\frac{\lambda^4}{\chi^3}$	$2\,\lambda^4$	$\frac{4\,\lambda^3}{\chi}$	$\sqrt{2}\,\lambda^2$
2	$\frac{\lambda^4}{\chi^3}$	λ^4	$\frac{\lambda^2(1-2\,\lambda)}{\chi}$	λ^2
3	$\frac{\lambda^4}{\chi^3}$	$2\,\lambda^4$	$\frac{4\,\lambda^3}{\chi}$	$\frac{1}{2\sqrt{2}}\,\lambda$
4	$\frac{(\sqrt{3}-1)\lambda^4}{\chi}$	$(\sqrt{3}+2)\lambda^4$	$\frac{\lambda^2}{\chi^3}$	$\frac{\sqrt{2}}{\sqrt{3}-1}\,\lambda^2$
5	$\frac{(\sqrt{2}-\frac{1}{2})\,\lambda^4}{\chi^3}$	$\frac{\sqrt{2}}{(\sqrt{2}-\frac{1}{2})^2}\lambda^4$	$\frac{\frac{1}{\sqrt{2}}\lambda^2}{\chi}$	$\sqrt{2}\,\lambda^2$

Table 4: Results following from the five symmetric texture solutions.

To illustrate the structure we derive for the $\mathbf{V}_{CKM}$ matrix that results from one of the texture structures of Section 3 we consider solution 4. In this case there are two phases left after field redefinition. The up mixing angles are $s_3 = 0$ and $s_2 = \sqrt{\frac{m_u}{m_c}}$ and (cf. eq(43)) the vanishing of s_3 means that one of the phases is irrelevant. From eqs(24) and (44) we may then derive eq(32) as must be the case since $\mathbf{Y}_u$ has a zero in the (1,1) position. We also have from eq(44) the result $\frac{\mathbf{V}_{ub}}{\mathbf{V}_{cb}} = \sqrt{\frac{m_u}{m_c}}$. This result also follows from eq(33) again as is expected since $\mathbf{Y}_u$ has a zero in the (1,3) position.

5.2.1 Comparison with analytic analysis.

Here we compare the full results of Section 4 with the expectation following from the analytic solution of the renormalisation group equations given in Section 2.3.

For all five solutions we have, for the parameters evaluated at a scale M_Z

$$\begin{aligned} V_{us} \approx V_{cd} &\approx \lambda \\ \frac{m_s}{m_b} &= \frac{2\,\lambda^3}{\chi} \end{aligned}$$

Solution	m_c/m_t	m_u/m_c	V_{cb}	V_{ub}/V_{cb}
1	0.0067	0.0046	0.060	0.068
2	0.0067	0.0023	0.038	0.048
3	0.0067	0.0046	0.060	0.078
4	0.0049	0.0087	0.068	0.040
5	0.0061	0.0030	0.048	0.068

Table 5: Results following from the five symmetric texture solutions using λ=0.22.

$$\frac{m_d}{m_s} = \lambda^2 \tag{45}$$

where $\chi = (M_X/M_Z)^{-h_t^2/(16\pi^2)}$ and we have taken h_t approximately constant as in eq(21). The results for the remaining elements of $\mathbf{V}_{CKM}$ are given in Table 4.

These are the values evaluated at the scale M_Z. For comparison we quote the experimental results [20]

$$\begin{aligned}
\frac{m_s}{m_b} &= 0.03 - 0.07 \\
\frac{m_d}{m_s} &= 0.04 - 0.067 \\
V_{cb} &= 0.025 - 0.050 \\
\frac{V_{ub}}{V_{cb}} &= 0.05 - 0.13 \\
\frac{m_c}{m_t} &\approx 0.0072 \text{ (for } m_t = 180 \text{ GeV)} \\
\frac{m_u}{m_c} &= 0.003 - 0.005
\end{aligned} \tag{46}$$

This is to be compared with the texture results obtained using the best value for $\lambda = 0.22$. All solutions give V_{us}=0.22 and $\frac{m_d}{m_s}$=0.05 and $\frac{m_s}{m_b}$=0.03, in agreement with the experimental results.

For the other quantities we have the results given in Table 5.

It may be seen all quantities are reasonably close to the experimental values, given the simple analytic form, which ignores threshold effects, taken for the evolution of the CKM matrix elements and Yukawa couplings. Thus we see the analytic approximation gives quite a reliable guide to the results of the complete numerical analysis for all quantities.

6 Discussion and Conclusions

We have determined all possible forms of symmetric quark mass matrices having a total of five or six texture zeroes which are consistent with the measured values of the quark masses and mixing angles. It is encouraging for the idea of unification that there are such solutions corresponding to simplicity at the unification scale. This simplicity extends to the lepton quark matrices for the lepton masses are consistent with the Georgi Jarlskog relations at the unification scale.

With the present measurements of quark masses and mixing angles, we find several candidate solutions, corresponding to the discrete ambiguity in determining the current quark basis. In detail these solutions give different predictions relating the masses and mixing angles and so may be distinguished by improved measurements of the CKM matrix elements. Perhaps the most useful outcome of this analysis is the determination, cf. Table 3, of the level of accuracy needed for the discrimination between solutions.

The obvious question raised by these textures is what underlying theory can lead to such structure? Although a detailed answer to this question lies beyond the scope of this paper we cannot resist drawing some conclusions from the general structure found in the various solutions. The parameterisation of the quark (and lepton) mass matrices that was suggested by the hierarchy of masses and mixing angles is (cf eqs(11),(12), and (14)) a perturbative expansion in λ. This structure strongly suggests to us an underlying symmetry broken by terms of $O(\lambda)$. In the limit the symmetry is exact only the third generation is massive and all mixing angles are zero. Symmetry breaking terms gradually fill in the mass matrices generating an hierarchy of mass scales and mixing angles. Of course the idea of such a symmetry structure is not new [22] and essentially all attempts to provide an explanation of the quark and lepton masses rely on broken symmetry, although they may not emphasise its role. However the realisation of the importance of an underlying symmetry leads to a discussion of the general properties such a solution must have and indeed these properties seem to be quite promising in explaining the form of the structures found.

To illustrate this let us first discuss the two generation case. We consider the simplest possible symmetry based on an $U(1)$ symmetry or on a Z_N discrete subgroup. (Such symmetries are common in Grand Unification or compactified string theories. Of course the structure that emerges may also be derived from larger symmetries.) With our assumption that the mass matrices are symmetric we must take the left- and the right- handed components to transform in the same way under this symmetry. Moreover the $SU(2)$ symmetry requires that the up and down quarks must have the same transformation properties. Thus, without any loss of generality we may take the transformation properties of the quarks to be

$$c_{L,R},\ s_{L,R} \rightarrow c_{L,R},\ s_{L,R}$$
$$u_{L,R},\ d_{L,R} \rightarrow \alpha\ u_{L,R},\ \alpha\ d_{L,R}$$

where $\alpha = exp(i2\pi/N)$.

The transformation properties of the elements of the quark mass matrix then have the form

$$\begin{pmatrix} u_L & c_L \end{pmatrix} \cdot \begin{pmatrix} \alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix} \cdot \begin{pmatrix} u_R \\ c_R \end{pmatrix} \quad \begin{pmatrix} d_L & s_L \end{pmatrix} \cdot \begin{pmatrix} \alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix} \cdot \begin{pmatrix} d_R \\ s_R \end{pmatrix} \qquad (47)$$

The form of the mass matrices depends on the transformation properties of the Higgs, $H_{1,2}$, which couple to the up and down quarks respectively and generate their masses once the electroweak symmetry is broken. If they are singlets under the Z_N group then, from eq(47), we see that only the c and s quarks acquire mass. If, however, the Z_N symmetry is broken by the vacuum expectation value, x, of a field Θ transforming as $\Theta \rightarrow \bar{\alpha}\Theta$ then we may expect corrections to the mass matrix to occur due to higher dimension terms coupling the quarks to the combination of fields $H_{1,2}\Theta^n$ for some integer n, giving a mass matrix of the form

$$\mathbf{Y} = \begin{pmatrix} \frac{x^2}{M_1^2} & \frac{x}{M_2} \\ \frac{x}{M_2} & 1 \end{pmatrix} \qquad (48)$$

where $M_{1,2}$ are the masses associated with the scale of new physics generating the higher dimensional terms.

Another way that such an hierarchy may arise is through the mixing of $H_{1,2}$ with other Higgs states $H_{1,2}^{a,b}$ transforming as $\bar{\alpha}$ and $\bar{\alpha}^2$. Then the light

Higgs state will be a mixture $H_{1,2}^{Light} \approx H_{1,2} + \frac{x}{M_2} H_{1,2}^a + \frac{x^2}{M_1^2} H_{1,2}^b$, where x is now the vev of a field $\bar{\Theta}$ transforming as $\bar{\Theta} \rightarrow \alpha\bar{\Theta}$ and $M_{1,2}$ are the masses of intermediate states mixing $H_{1,2}$ with the other Higgs states. This again gives the structure of eq(48).

It is now easy to see how all of the structures of Table 2 may arise. Assuming, for simplicity, a single scale of new physics, $M_{1,2} = M$, then

$$\mathbf{Y} = \begin{pmatrix} \lambda^2 & \lambda \\ \lambda & 1 \end{pmatrix} \quad \lambda = \frac{x}{M} \tag{49}$$

If there should be no additional field $H_{1,2}^b$ transforming as $\bar{\alpha}^2$, then

$$\mathbf{Y} = \begin{pmatrix} 0 & \lambda \\ \lambda & 1 \end{pmatrix} \tag{50}$$

If instead there is no additional field $H_{1,2}^a$ transforming as $\bar{\alpha}$ then

$$\mathbf{Y} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{51}$$

The remaining structure encountered in Table 2 has the form

$$\mathbf{Y} = \begin{pmatrix} \lambda & \lambda \\ \lambda & 1 \end{pmatrix} \tag{52}$$

and it too may also be generated. For example if the discrete symmetry is Z_3 then $\alpha^2 = \bar{\alpha}$ in eq(47). Then the structure of eq(52) results if the vev of Θ develops along a "D-flat" direction $< \Theta > = < \bar{\Theta} > = x$, where $\bar{\Theta}$ is a field in the conjugate representation to Θ[9]. Another possibility discussed below is that the form of eq(52) comes from eq(48) because the masses M_i are not degenerate.

Thus we may easily generate any of the substructures found in Table 2. Indeed such structures are the natural expectations in any underlying theory, such as a Grand Unified theory or a compactified string theory, which possesses additional symmetries of the type discussed.

[9]The spontaneous breaking of symmetries at high scales in supersymmetric theories must proceed along such flat directions.

It is straightforward to extend this discussion to the three generation case. Rather than present a general analysis let us just illustrate the possibilities by presenting a symmetry structure which leads to one of the solutions found in Table 2. We assume that the underlying theory yields a three family spectrum of quarks with transformation properties

$$t_{L,R},\ b_{L,R} \rightarrow t_{L,R},\ b_{L,R}$$
$$c_{L,R},\ s_{L,R} \rightarrow \alpha c_{L,R},\ \alpha s_{L,R}$$
$$u_{L,R},\ d_{L,R} \rightarrow \bar{\alpha}^4\ u_{L,R},\ \bar{\alpha}^4\ d_{L,R}$$

We further assume that, in addition to the light Z_N singlet Higgs field, H_1, needed for electroweak symmetry breaking, there are massive Higgs fields $H_1^{a,b}$ transforming under Z_N as $\bar{\alpha}$ and α^3 with which the singlet field may mix. Assuming only a single scale of new physics the resulting light Higgs has the form $H_1^{Light} \approx H_1 + \bar{\theta} H_1^a/M + \theta^3 H_1^b/M^3$ giving for the up quark mass matrix the form (with $\lambda^2 = x/M$)

$$\mathbf{Y}_u \approx \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & 0 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix} \tag{53}$$

which is of the form of the up quark mass matrix of case 2 in Table 2. (In deriving this result we have assumed that the vev of Θ develops along a "D-flat" direction $< \Theta > = < \bar{\Theta} > = x$.)

What this example shows is how the pattern of fermion masses and mixing angles, as encoded in the mass matrices, may simply be related to the multiplet structure of the underlying theory. As we have seen the apparently complicated pattern may result from a relatively simple multiplet structure. To complete this example we need to explain why the down quarks have a different mass matrix. Again this proves to be relatively simple to explain on the basis of the multiplet structure. Any difference between the up and down quark mass matrices comes from different mixings of the Higgs, $H_{1,2}$, giving masses to the up and down quarks respectively. For example we suppose that in the H_2 sector there is an additional massive Higgs field H_2^c transforming as α^2. We further assume that the massive Higgs fields transforming as $\bar{\alpha}^3$ and α^2 receive *their* mass only at order x via the Θ vev[10], so that the mixing

[10]This will happen naturally if the fields with which they couple to obtain their mass have the appropriate transformation properties under the Z_N group.

of these fields to the light singlet H_2 will be enhanced by a factor M/x. The resulting light Higgs field is $H_2^{Light} \approx H_2 + \bar{\theta} H_2^a/M + \theta^3 H_2^b/xM + \theta^2 H_2^c/xM^2$ giving a down quark mass matrix of the form

$$\mathbf{Y}_d \approx \begin{pmatrix} 0 & \lambda^4 & 0 \\ \lambda^4 & \lambda^2 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix} \tag{54}$$

Up to corrections of $O(1)$, which may be expected in generating the higher dimension terms, this is of the form of the down quark mass matrix in Table 2. Thus we have constructed an example which, by assuming a definite multiplet and symmetry structure, generates the full texture structure of case 2 of Table 2 in terms of a single expansion parameter, x.

Of course the next step is to identify the underlying theory which leads to the appropriate multiplet and symmetry structure. It is known in compactified string theories that definite family structures for quarks and leptons may emerge and that they possess definite transformation properties under discrete symmetries of the type discussed above. Indeed specific models have been analyzed which do lead to structures in the fermion mass matrices of the type just discussed [21]. Similarly additional broken gauge symmetries may lead to the type of mass structure discussed above [23].

In conclusion, the fact that simple textures for the quark and lepton masses can describe in detail the quark and lepton masses and mixings lends further circumstantial evidence in favour of an underlying unified theory. It is to be hoped that improvements in experimental measurements of the CKM matrix elements will further refine this evidence. To us the resultant structure strongly suggests a (broken) symmetry explanation of the structure of the type which naturally arises in GUTs or compactified string unification and encourages us in the search for a definite theory.

References

[1] M.S. Chanowitz, J. Ellis and M.K. Gaillard, Nucl. Phys. B **128** (1977) 506.
A. Buras, J. Ellis, M.K. Gaillard and D.V. Nanopoulos, Nucl. Phys. B **135** (1978) 66.

[2] H. Arason, D.J. Castaño, B. Keszthelyi, S. Mikaelian, E.J. Piard, P. Ramond and B.D. Wright, Phys. Rev. Lett. **67** (1991) 2933.
B. Ananthan, G. Lazarides and Q. Shafi, Phys. Rev. D **44** (1991) 1613.

[3] H. Georgi and C. Jarlskog, Phys. Lett. B **86** (1979) 297.

[4] J. Harvey, P. Ramond and D. Reiss, Phys. Lett. B **92** (1980) 309.

[5] S. Dimopoulos, L.J. Hall and S. Raby, Phys. Rev. Lett. **68** (1992) 1984; Phys. Rev. D **45** (1992) 4195.

[6] P. Ramond, in "On Klauder's Path : a Field Trip", eds. G. Emch, G. Hegerfeldt, L. Streit (World Scientific, Singapore) 1992.

[7] H. Arason, D.J. Castaño, P. Ramond and E.J. Piard, Phys. Rev. D **47** (1993) 232.

[8] G.F. Giudice, Mod. Phys. Lett. **A7** (1992) 2429.

[9] S. Weinberg, in " A Festschrift for I.I. Rabi" [Trans. N.Y. Acad. Sci., Ser. II (1977), v. 38], p. 185;
F. Wilczek and A. Zee, Phys. Lett. B **70** (1977) 418.

[10] H. Fritsch,Phys. Lett. B **70** (1977) 436; Phys. Lett. B **73** (1978) 317;
P. Kaus and S. Meshkov, Mod. Phys. Lett. **A3**(1988) 1251.
F.J. Gilman and Y. Nir, Ann. Rev. Nucl. Part. Sci. **40** (1990) 213;

[11] X.G. He and W.S. Hou, Phys. Rev. D **41** (1990) 1517.

[12] L. Wolfenstein, Phys. Rev. Lett. **51** (1983) 1945.

[13] P. Fishbane and P.Q. Hung, Phys. Rev. D **45** (1992) 293.
Y. Koide, H. Fusaoka and C. Habe, Phys. Rev. D **46** (1992) R4813.

[14] L. Rolandi, Talk given at 26th Int. Conf. on HEP (ICHEP) Dallas, CERN-PPE-92-175, 1992.

[15] M. Olechowski and S. Pokorski, Phys. Lett. B **257** (1991) 388.

[16] R. Gatto, G. Sartori and M. Tonin, Phys. Lett. B **28** (1968) 128; R.J. Oakes, Phys. Lett. B **29** (1969) 683; Phys. Lett. B **31** (1970) 620(E); Phys. Lett. B **30** (1970) 262.

[17] B. Grzadkowski, M. Lindner and S. Theisen, Phys. Lett. B **198** (1987) 64.

[18] G.G. Ross and R.G. Roberts, Nucl. Phys. B **377** (1992) 571.

[19] J. Gasser and H. Leutwyler, Phys. Rep. **87** C (1982) 77. H. Leutwyler, Nucl. Phys. B **337** (1990) 108. S. Narison, Phys. Lett. B **216** (1989) 191.

[20] A. Buras and M.K. Harlander, Munich preprint MPI- PAE/PTh 1/92; J.L. Rosner, Jour. Phys. **G18**(1992) 1575. K. Peach, Jour. Phys. **G16**(1990) 131.

[21] B.R. Greene, K.H. Kirklin, P.J. Miron, and G.G. Ross, Nucl. Phys. B **278** (1986) 667.

[22] H. Fritzsch and J. Plankl,Phys. Lett. B **237** (1990) 451 and references therein.

[23] S. Dimopoulos, L.J. Hall and S. Raby, talk presented by L.J.Hall, Erice 1992.