RAL-93-033

Science and Engineering Research Council
Rutherford Appleton Laboratory
Chilton DIDCOT Oxon OX11 0QX

RAL-93-033

Classification of (2,2) Compactifications in Fermionic Strings

S A Abel and C M A Scheich

May 1993

Classification of (2,2) Compactifications in Fermionic Strings

S. A. Abel[1] and C. M. A Scheich[2]

[1]Rutherford Appleton Laboratory
Chilton, Didcot
Oxon OX11 0QX
England

[2]Theoretical Physics Department
Keble Road
Oxford OX1 3NP
England

May 21, 1993

Abstract

We present a general scheme for generating $(2,2)$ symmetric fermionic string models and classify the models in $D = 8$ and $D = 6$ space-time dimensions with one twist. It is pointed out that they allow the geometrical interpretation as generalised torus compactifications. Their relation to other compactifications is discussed and the overlap with orbifolds is determined.

Since string theory is an appealing candidate for the unification of gravity with gauge interactions, an extensive knowledge of the possible string theories in four dimensions is of wide interest. The most promising candidate so far is the heterotic string, but its construction can only be considered to be unique in $D = 10$ space-time dimensions (if we choose the more promising version with gauge group $E_8 \otimes E_8'$). Any further reduction of space-time dimensions leads at first glance to an embarrassingly large number of models. This is especially true in the case of $D = 4$. Further studies however, have revealed that there are a suprisingly small number of (potentially phenomenological viable) models. Let us review the present situation for three large classes of models:

Compactifications on Calabi-Yau manifolds have been partially classified in Ref.[2]. Many of the models are related and there are only a few three generation models [3,4]. Of special interest here is the subclass of models that may also be realised by means of minimal $N = 2$ superconformal models [5]. They belong to special Calabi-Yau manifolds with fixed moduli and therefore allow the extraction all information relevant to their phenomenology [6]. In addition, four dimensional string theories may be constructed from superconformal theories directly [7]. However only a little is known about them.

Orbifolds are an especially interesting class of compactified string models. Their phenomenological implications have been studied in great detail [9]. $Z_N \otimes Z_M$ orbifolds seem to correspond to some of the $(2,2)$ compactifications by $N = 2$ minimal superconformal models [10], whilst Z_N orbifolds do not.

The last large class of models uses free fermions to construct models directly in four dimensions, and the task of finding a phenomenologically viable model has received considerable attention [13]. They again belong to compactifications with fixed moduli. This class has been compared to orbifolds in Ref.[17] by using theta-function identities. We will confirm their result from another point of view.

Since the fermionic construction is very well suited for model building, one should try to clarify its relation to the other classes of compactification which to date remains rather obscure. In order to do this we shall attempt to classify fermionic string models, and to compare them to other models systematically. We begin in this letter with the case of $D = 8$ and $D = 6$ and postpone the study of $D = 4$ to Ref.[20]. This is because in these higher dimensions the models are reduced drastically in number, but still show the main features of their respective compactifications.

We furthermore restrict ourselves to symmetric $(2,2)$ models with the maximal gauge group $g \otimes E_{D/2+4} \otimes E_8'$ where g is a model dependent gauge group of rank 2,5,8 for $D = 8,6,4$ respectively (for details see [19]). Further breaking of the gauge group by embeddings of twists should then work in the usual way, and will not spoil the relevance of the classification. Nonabelian embeddings that lower the rank are also not considered. They correspond to using real fermions in the fermionic string formulation. In addition, restricting the discussion to left-right symmetric models also allows a direct interpretation as a compactification.

We begin with a general discussion of the construction of $(2,2)$ fermionic strings that possess left-right symmetry. We shall construct the models using the formulation of Ref.[12] with only complex fermions. The internal fermions then have phases associated with them a_r, b_r, c_r; $r = 1, \cdots 5 - D/2$ which come in triplets for left and right movers and fulfill the constraint

$$a_r + b_r + c_r \in \left\{0, \frac{1}{2}\right\} \qquad \mathrm{mod}(1). \tag{1}$$

This constraint follows from the periodicity or antiperiodicity of the superpartner of the stress-energy tensor. As shown in Ref.[12] one derives the $N = 1$ superconformal algebra on the world-sheet. The extension to $N = 2$ [18] is rather cumbersome, and it is not possible to realise the $N = 2$ algebra for complex fermions in a closed form. In general only matrix elements of the conformal fields are known. The exception to this statement is the case where one complex fermion is only ever periodic or antiperiodic in its boundary condition. It is then possible to immediately form an $N = 2$ superconformal algebra

$$\begin{aligned} T(z) &= \frac{1}{2}\sum_{i=1}^{3} : (:\overline{\psi}_i\partial\psi_i : + : \psi_i\partial\overline{\psi}_i :) \\ G^+(z) &= -i\psi_1(:\overline{\psi}_3\psi_3 : -i : \overline{\psi}_2\psi_2 :) \\ G^-(z) &= -i\overline{\psi}_1(:\overline{\psi}_3\psi_3 : +i : \overline{\psi}_2\psi_2 :) \\ J(z) &= :\overline{\psi}_1\psi_1 : \end{aligned} \tag{2}$$

in a complex notation related to the real one by $\psi = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2)$, $X = \frac{1}{\sqrt{2}}(X^1 + iX^2)$. Furthermore the space-time supersymmetry generator may only be realised in a closed in the simplest of cases. In order to compare models with only free fermions to any of the models with internal manifold coordinates we can simply bosonise two of the complex fermions into one complex boson for each triplet (here we also require left-right symmetry);

$$\psi_i =: e^{iH} : \tag{3}$$

With this bosonisation we get immediately a torus compactification. But this torus with its coordinates H_i does not always have the usual compactified former ten dimensional coordinates as such. This is already obvious from the fact that supersymmetry requires that a compactified coordinate X_i and its supersymmetric partner ψ_i fulfill the following boundary conditions:

$$\begin{aligned} gX_i(\sigma_1,\sigma_2) &= X_i(\sigma_1+2\pi,\sigma_2) &&= e^{-2\pi i\theta_i}X_i(\sigma_1,\sigma_2) + \pi v_i \\ hX_i(\sigma_1,\sigma_2) &= X_i(\sigma_1,\sigma_2+2\pi) &&= e^{-2\pi i\phi_i}X_i(\sigma_1,\sigma_2) + \pi u_i \\ g^{-1}S_i(\sigma_1,\sigma_2) &= S_i(\sigma_1+2\pi,\sigma_2) &&= e^{+2\pi i\theta_i}S_i(\sigma_1,\sigma_2) \\ h^{-1}S_i(\sigma_1,\sigma_2) &= S_i(\sigma_1,\sigma_2+2\pi) &&= e^{+2\pi i\phi_i}S_i(\sigma_1,\sigma_2), \end{aligned} \tag{4}$$

where for an abelian orbifold g, h are commuting elements of an abelian discrete group and u_i, v_i are bosonic shifts that define a lattice. Here only fermions with diagonal boundary conditions are used, and so a model without twists on the H_i is created. The above relations including twists *may* be fulfilled using fermions with periodic and anti-periodic boundary conditions however. In the case of several compactified dimensions, it has been conjectured Z_2, Z_4 and Z_8 orbifolds are realised in this way [17,16]. This fact also gives a simple interpretation for the connections between the theta-functions found in Ref.[17].

Only for the special choice of boundary conditions above (i.e. $\theta_1,\ \phi_1\ \in\ \{0, \frac{1}{2}\}$) do we have the usual bosonic interpretation, by making the identification

$$\sqrt{2}\partial X \equiv i(:\overline{\psi}_3\psi_3 : -i : \overline{\psi}_2\psi_2 :). \tag{5}$$

The algebra is then

$$\begin{aligned} T(z) &= \frac{1}{2} :\overline{\psi}_1\partial\psi_1 : + : \psi_1\partial\overline{\psi}_1 : - : \partial\overline{X}\partial X : \\ G^+(z) &= -\sqrt{2}\psi_1\partial X \\ G^-(z) &= -\sqrt{2}\overline{\psi}_1\partial\overline{X} \\ J(z) &= :\overline{\psi}_1\psi_1 : \end{aligned} \tag{6}$$

as required. Nevertheless our approach is in the above sense more general than that of Ref.[8] in that we do not require the relations in Eq.(4) to hold for fermionic and bosonic coordinates simultaneously. We simply include all cases where, in the algebra in Eq.(2), twist in the fermionic ψ_i's are compensated by phases in the other fermions or after bosonisation by lattice shifts rather than twists in the bosonic coordinates. This corresponds to an embedding of the shifts u_i, v_i into the gauge group for the left movers. We are effectively therefore considering tori with more complicated spin-structures[1].

It is these spin structures which can lead to more complicated models possibly related to orbifolds. To be more explicit, what we require for an orbifold interpretation are identifications along the lines of Eq.(5), but such identifications are of course only possible if the partition function remains the same. This has been shown for a Z_2 orbifold in for example Ref.[14]. In fact searching directly for this equivalence of partition functions was exactly the approach adopted in Ref.[17].

At this point we should also state the relation to the orbifold construction of Ref. [11]. Here one uses the heterotic string in a fully bosonised form for the left movers and also requires (4). Our interpretation belongs to using mixed $R-V$ vectors of the type $R-V=(0,\Theta_k,(0,v_k)\mid(\Theta_I,v_I))$, $k=1,2,3$, $I=1,\cdots,11$ in the authors notation.

In addition to the Z_2, Z_4, Z_8 orbifolds there is another class of models that may overlap with fermionic strings. As already pointed out in Ref.[16], there is the possibility of fermionic strings with nondiagonal boundary conditions. In the case that the boundary conditions of the fermions transforming into each other are the same, we find permutational orbits of fermions. Now it is possible that such a permutational modding may be absorbed into different diagonal boundary conditions in the fermionic string. On the other hand it could belong to a permutational modding of the compactified bosonic coordinates via Eq.(3), or to a phase modding in a diagonalised bosonic basis or all of the above. This open question will be discussed further in Ref.[20].

Having concluded our general discussion we now set up the framework and the conventions for the classification. We shall use the notation of Ref.[12] in which the light cone gauge is chosen to render the string action down to that of a free field theory on the world sheet, and all the internal degrees of freedom are expressed as fermions. In general one expects the fermions to transform into each other after being parallel transported around the world sheet. However we will restrict ourselves here to a basis in which these boundary conditions become simple phase shifts on the fermions. In the case of complex boundary conditions we are restricted to gauge groups of a fixed rank of 18,20,22 in $D=8,6,4$ respectively. A set of linearly independent 'basis vectors' $\mathbf{W}_i$ is chosen to generate all the possible boundary conditions for the fermionic degrees of freedom in a particular model. Sectors are represented by different points on the lattice 'spanned' by the basis vectors, $\overline{\alpha\mathbf{W}}=\sum_i\alpha_i\mathbf{W}_i$ mod(1), where α_i may take only integer values. For the heterotic string in D dimensions, the vectors $\mathbf{W}_i$ may take the form

$$\mathbf{W}_i=\left[\ (s_i)^{D/2-1},(a_i^r,b_i^r,c_i^r)\ \Big|\ w_i^l\ \right], \tag{7}$$

where $r=(1,\cdots,(10-D)/2)$ and $l=(1,\cdots,26-D)$. The explicit form of the world sheet supercurrent also implies the triplet constraint,

$$a_i^r+b_i^r+c_i^r=s_i \qquad \text{mod}(1). \tag{8}$$

[1]Such an embedding implies a nontrivial cohomology for the exterior derivative, dH, of the field strength associated with the antisymmetric tensor B. Furthermore this requires additional terms in the 10 dimensional supergravity action used in the σ model approach, reflecting the non-perturbative character of the string solution.

Imposing modular invariance yields a set of vector constraints,

$$\begin{aligned} k_{ij} + k_{ji} &= \mathbf{W}_i \cdot \mathbf{W}_j \\ m_j k_{ij} &= 0 \\ k_{ii} + k_{i0} + s_i - \frac{1}{2}\mathbf{W}_i \cdot \mathbf{W}_i &= 0 \qquad \mathrm{mod}(1), \end{aligned} \tag{9}$$

where the k_{ij} are the structure constants, and m_j is the least common denominator of the boundary conditions appearing in $\mathbf{W}_j$. The dot product is defined by

$$\mathbf{W}_i \cdot \mathbf{W}_j = \sum_{\text{left } k} \mathbf{W}_i^k \mathbf{W}_j^k - \sum_{\text{right } k} \mathbf{W}_i^k \mathbf{W}_j^k. \tag{10}$$

In addition to the above, modular invariance gives us a set of GSO projections, which require that physical states must satisfy

$$\mathbf{W}_i \cdot N_{\alpha\mathbf{W}} = k_{ij}\alpha_j + s_i + k_{0i} - \mathbf{W}_i \cdot \overline{\alpha\mathbf{W}} \qquad \mathrm{mod}(1), \tag{11}$$

where $N_{\alpha\mathbf{W}}$ are the number operators, and a summation over repeated indices is implied. In terms of the charge operator

$$Q_{\alpha\mathbf{W}} = N_{\alpha\mathbf{W}} + \overline{\alpha\mathbf{W}} - \mathbf{W}_0, \tag{12}$$

we have

$$\mathbf{W}_i \cdot Q_{\alpha\mathbf{W}} = s_i + k_{ij}\alpha_j - k_{i0} \qquad \mathrm{mod}(1). \tag{13}$$

Our purpose here is to generate $(2,2)$ models. Without loss of generality (as we shall show presently) we choose the first four vectors to be of the form,

$$\begin{aligned} \mathbf{W}_0 &= \left[\ (\tfrac{1}{2})^{D/2-1}, (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^{(10-D)/2}\ \middle|\ (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^{(10-D)/2}, (\tfrac{1}{2})^{(6+D)/2}, (\tfrac{1}{2})^{8}\ \right] \\ \mathbf{W}_1 &= \left[\ (\tfrac{1}{2})^{D/2-1}, (a_1^r, b_1^r, c_1^r)\ \middle|\ (0,0,0)^{(10-D)/2}, (0)^{(6+D)/2}, (0)^{8}\ \right] \\ \mathbf{W}_2 &= \left[\ (0)^{D/2-1}, (0,0,0)^{(10-D)/2}\ \middle|\ (a_1^r, b_1^r, c_1^r), (\tfrac{1}{2})^{(6+D)/2}, (0)^{8}\ \right] \\ \mathbf{W}_3 &= \left[\ (0)^{D/2-1}, (0,0,0)^{(10-D)/2}\ \middle|\ (0,0,0)^{(10-D)/2}, (0)^{(6+D)/2}, (\tfrac{1}{2})^{8}\ \right]. \end{aligned} \tag{14}$$

The $\mathbf{W}_0$ vector is needed to have a modular invariant theory, and to give the gravity multiplet. The $\mathbf{W}_1$ and $\mathbf{W}_2$ vectors implement supersymmetry on the right and left side (which for the heterotic string implies an $E_{D/2+4}$ gauge group). Finally, in order to give a second E_8' factor in the gauge group we have the $\mathbf{W}_3$ vector. Thus we get copies of $N = 2$ algebras on each side, establishing a $(2,2)$ model. In this sense the models can be viewed as a Gepner-like construction with internal copies of the $N = 2$ minimal superconformal models from free fields.

The numerical survey of the spectra generated by the above vectors reveals the remarkable fact that, for any choice of (a_1^r, b_1^r, c_1^r), the theory generated has the maximal supersymmetry (and so $E_8 \otimes E_8'$ gauge groups), and therefore corresponds to a torus compactification in the usual sense. Specifically this means that in $D = 8, 6, 4$ we find $N = 1, 2, 4$ space-time supersymmetry respectively, and an $E_8 \otimes E_8'$ gauge group.

Before continuing, we need to show that with such a choice of vectors one may obtain all possible left-right symmetric models. To do this we prove a useful general result, which is that a set of $\{\mathbf{W}_i\}$ is equivalent to any other $\{\mathbf{W}'_i\}$, (i.e. gives the same set of models) provided that the boundary conditions generated are identical. This will then ensure that the choice of vectors in Eq.(14) is sufficient to cover all possible models without twists if one includes all values of k_{ij}'s.

The proof is as follows. Since we wish to generate the same boundary conditions we must be able to express each $\mathbf{W}'_i$ as a linear combination of $\{\mathbf{W}_i\}$. So consider the case where we have

$$\begin{aligned} \mathbf{W}'_i &= \mathbf{W}_i \quad \text{for} \quad i \neq j, \\ \mathbf{W}'_j &= \overline{\mathbf{W}_l + \mathbf{W}_j}, \; l \in i. \end{aligned} \tag{15}$$

Clearly the $\mathbf{W}'_i$, $i \neq j$ projections in Eq.(11) for any sector $\overline{\alpha \mathbf{W}}$ are unchanged if $k'_{ij} = k_{ij} + k_{il}$. The k'_{ji} are then determined by Eq.(9). The remaining projection is

$$\begin{aligned} \mathbf{W}'_j \cdot N_{\alpha\mathbf{W}} &= k'_{ji}\alpha_i + s'_j + k'_{0j} - \mathbf{W}'_j \cdot \overline{\alpha\mathbf{W}} \\ &= -(k_{ij} + k_{il})\alpha_i + s_j + s_l + k_{0j} + k_{0l} - \mathbf{W}'_j \cdot (\overline{\alpha\mathbf{W}} - \alpha\mathbf{W}) \\ &= -(k_{ij} + k_{il})\alpha_i + s_j + s_l + k_{0j} + k_{0l} - (\overline{\mathbf{W}_j + \mathbf{W}_l}) \cdot (\overline{\alpha\mathbf{W}} - \alpha\mathbf{W}) \\ &= \mathbf{W}_j \cdot N_{\alpha\mathbf{W}} + \mathbf{W}_l \cdot N_{\alpha\mathbf{W}} \qquad \text{mod}(1). \end{aligned} \tag{16}$$

Thus the modular invariance conditions are also satisfied in the new basis. Since $k'_{ij} = k_{ij} + k_{il}$ is satisfied for one choice of structure constant, any model generated in the old basis is also generated in the new basis. Extrapolation to general linear combinations then follows trivially.

To go beyond torus compactification, we will need to add more vectors to break down supersymmetry. Such additional vectors, which we shall refer to as compactification vectors, may be either left–right symmetric,

$$\mathbf{W}_4 = \left[\; (0)^{D/2-1}, (a^r_4, b^r_4, c^r_4) \;\middle|\; (a^r_4, b^r_4, c^r_4), (0)^{(6+D)/2}, (0)^8 \;\right], \tag{17}$$

or may occur in left–right symmetric pairs,

$$\begin{aligned} \mathbf{W}_4 &= \left[\; (0)^{D/2-1}, (a^r_4, b^r_4, c^r_4) \;\middle|\; (a^r_5, b^r_5, c^r_5), (0)^{(6+D)/2}, (0)^8 \;\right] \\ \mathbf{W}_5 &= \left[\; (0)^{D/2-1}, (a^r_5, b^r_5, c^r_5) \;\middle|\; (a^r_4, b^r_4, c^r_4), (0)^{(6+D)/2}, (0)^8 \;\right], \end{aligned} \tag{18}$$

and so on. Such a model including $\mathbf{W}_4$ or $\mathbf{W}_4$, $\mathbf{W}_5$ will be called "twisted", since it has the properties we already associate with a twist in other compactifications.For example we have an additional projection on the already existing untwisted sectors and the appearance of additional twisted sectors. For $N = 1$ the theories generated have the gauge group

$$G = g \otimes E_{(8+D)/2} \otimes E'_8, \tag{19}$$

where the first group, g, is some product of low rank subgroups coming from the compactified degrees of freedom. The selection of vectors above is not sufficient to guarantee a (2,2) compactification since we still have to choose the structure constants. A poor choice of k_{ij} can spoil the ($N = 2$) algebra by projecting out some of the supersymmetry generators via the modular invariance conditions in Eq.(11). This can lead to (0,2) or (2,0) models. Thus even at this stage we can have the equivalent of Wilson line breaking. Other compactifications like Calabi-Yau and Gepner models allow analogous breakings. In order to guarantee a (2,2) model we need to impose a condition on the structure constants. We do this by insisting that, given a gauge group G, there are the requisite number of gravitino degrees of freedom.

Let us now show that this is always possible by a suitable choice of k_{ij}'s. The adjoints needed for building up G are always found in the $\mathbf{W}_0$ sector. Thus we need only consider the sectors of the form

$$\overline{\alpha\mathbf{W}} = \left[\ (\frac{1}{2})^{D/2-1}, (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^{(10-D)/2}\ \middle|\ (a^r, b^r, c^r), (0)^{(6+D)/2}, (\frac{1}{2})^8\ \right]. \tag{20}$$

Since the vacuum energy is $[E_R, E_L] = [-1/2, e_L]$, they give the fermionic representation vectors for building up the exceptional group. These are of the form

$$A^\mu = b^\mu_{-1/2}|0\rangle \otimes \hat{O}|a\rangle, \tag{21}$$

where $\hat{O}$ is some combination of excitations and $|a\rangle$ is a spinorial ground state. There always exist the sectors in which the boundary conditions of the left and right movers are swopped (except for the fermions building up E_8'), which we refer to as reflected and denote with a tilde,

$$\overline{\alpha\tilde{\mathbf{W}}} = \left[\ (0)^{D/2-1}, (a^r, b^r, c^r)\ \middle|\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^{(10-D)/2}, (\frac{1}{2})^{(6+D)/2}, (\frac{1}{2})^8\ \right]. \tag{22}$$

As the vacuum energy of these sectors is $[E_R, E_L] = [e_L, -1]$, we potentially have the gravitinos as the reflected states of Eq.(21);

$$\psi^\mu = \tilde{\hat{O}}|a\rangle \otimes a^\mu_{-1}|0\rangle. \tag{23}$$

All that we need to do is to ensure modular invariance. We know that the physical states in Eq.(21) satisfy

$$\mathbf{W}_i \cdot N_{\alpha\mathbf{W}} = k_{ij}\alpha_j + s_i + k_{0i} - \mathbf{W}_i \cdot \overline{\alpha\mathbf{W}} \qquad \mathrm{mod}(1) \tag{24}$$

for all i, so that it is sufficient to show that

$$\tilde{\mathbf{W}}_i \cdot N_{\tilde{\alpha}\mathbf{W}} = \tilde{k}_{ij}\tilde{\alpha}_j + \tilde{s}_i + \tilde{k}_{0i} - \tilde{\mathbf{W}}_i \cdot \overline{\alpha\tilde{\mathbf{W}}} \qquad \mathrm{mod}(1). \tag{25}$$

for states of the type shown in Eq.(23). Since $\mathbf{W}_i \cdot \overline{\alpha\mathbf{W}} + \tilde{\mathbf{W}}_i \cdot \overline{\alpha\tilde{\mathbf{W}}} = 0$ and $\mathbf{W}_i \cdot N_{\alpha\mathbf{W}} + \tilde{\mathbf{W}}_i \cdot N_{\alpha\tilde{\mathbf{W}}} = 0$ this translates into the condition

$$k_{ij}\alpha_j + \tilde{k}_{ij}\tilde{\alpha}_j + k_{0i} + \tilde{k}_{0i} + s_i + \tilde{s}_i = 0 \qquad \mathrm{mod}(1). \tag{26}$$

In addition $\mathbf{W}_i \cdot \mathbf{W}_j + \tilde{\mathbf{W}}_i \cdot \tilde{\mathbf{W}}_j = 0$ implies that $k_{ij} + k_{ji} + \tilde{k}_{ij} + \tilde{k}_{ji} = 0$ by Eq.(9). Thus a sufficient condition for a (2,2) compactification is

$$k_{ij} + \tilde{k}_{ij} = \delta_{j0}(s_i + \tilde{s}_i) \qquad \mathrm{mod}(1). \tag{27}$$

So in general, for any gauge group G, the appropriate gravitino states exist by a suitable choice of k_{ij}.

Let us now discuss the results of the systematic numerical study of models with one twist, (i.e. boundary vectors $\mathbf{W}_0$ to $\mathbf{W}_4$) in $D = 8$ and $D = 6$ space-time dimensions along the lines just given.

$D = 8$ Dimensions

The highest number of supersymmetries in $D < 10$ dimensions is $N = 2^{(8-D)/2}$ and may be achieved using a torus compactification. The models with the highest N in any dimension have no matter fields, only gauge and gravity multiplets. Thus in $D = 8$, since there is only $N = 1$ supersymmetry, singlets together with gauge bosons and their superpartners form full gauge supermultiplets, and the compactification is essentially trivial.

Nevertheless it will prove useful to examine this case, since it will allow us to observe some general aspects of our scheme which we can apply to the more complicated lower dimensional models. The most obvious feature of (2,2) compactifications is the dramatic reduction in the number of models. In fact searching over the $\sim 10^5$ models up to order 20 (which very probably contain all possible distinct models), we find only two supersymmetric ones. These models, which have $E_8 \otimes E_8'$ symmetry are shown in table (1).

In addition we find that the different models are generated by the compactification vectors, regardless of the supersymmetry vectors. Thus given the order of the supersymmetry, a model (i.e. the gauge group, number of generations and singlets) is defined mainly by the compactification vectors.

Let us now compare the results of table (1) with other compactifications. Because of the requirement of absence of tachyons, there exist no orbifolds. From the table it is clear that there is no equivalent to the maximal Gepner model with $g = SU(3)$ [5], but an additional model with $g = SU(2) \otimes U(1)$. In fact the first configuration for an $SU(3)$would require the boundary conditions of the internal degrees of freedom to be degenerate. This is disallowed by the modular invariance conditions. Despite this it does seem that fermionic strings and Gepner models in $D = 8$ correspond to the torus at different points in moduli space. However a direct bosonisation as discussed in the introduction is only possible with periodic and anti-periodic boundary conditions resulting in model 1. Here one also realises that no orbifolds are possible due to the modular invariance conditions.

Alternatively the fermionic strings could also be interpreted as generalised tori as outlined above. Furthermore, we find that the supersymmetry may be broken entirely by adding just one $\mathbf{W}_4$, if the phases chosen are complicated enough. This contrasts with the techniques used by most model builders who frequently introduce many vectors with the simplest twistings.

$D = 6$ Dimensions

In this case we may have $N = 2$ with $G = g \otimes E_8 \otimes E_8'$, or $N = 1$ with $G = g \otimes E_7 \otimes E_8'$. We searched through $\sim 10^6$ models and find only 37 distinct cases. These are displayed in table (2).

As in the case of $D = 8$, we find that imposing $(2,2)$ symmetry drastically reduces the number of available models. The models are repeated many times with vectors of arbitrarily high order. The minimal allowed gauge group is $U(1)^5$, and the number of generations is nearly always less than ten which corresponds to the K3 manifold and the Z_N orbifolds. We observe that, with the exception of three models, the number of singlets is a multiple of the number of generations.

Clearly for the first six models the spectrum of the the T_2^2 torus with enlarged gauge group emerges. Bosonisation is restricted as above and gives models 1 and 3. In the fully bosonised formalism simple torus compactification corresponds to having only one vector V_0 (in which all the entries are $\frac{1}{2}$ in the notation of Ref.[11]) and gives $N = 1, 2, 4$ and $E_8 \otimes E_8'$ in $D = 8, 6, 4$ dimensions respectively). In the case of the Gepner models one obtains $g = SU(3) \otimes SU(2)^2$.

However, at this point we are also able to identify four spectra in models 9–12 in which the number of generations, and of untwisted generations match those of the four Z_N orbifolds in $D = 6$ [15]. These orbifolds have already been studied in Ref.[15] and it was found that all of them may be blown up into the $K3$ manifold. For example, the choice of vectors which have the non-zero entries

$$c_1^r = \frac{1}{2},\ a_4^r = \frac{1}{6},\ b_4^r = c_4^r = \frac{11}{12} \qquad (r = 1, 2) \tag{28}$$

generates model 10. In addition to the single untwisted generation, we find single generations coming from the sectors $\overline{\mathbf{W}_0 \pm 2\mathbf{W}_4}$, $\overline{\mathbf{W}_0 \pm 4\mathbf{W}_4}$ and $\overline{\mathbf{W}_1 + \mathbf{W}_2 \pm 2\mathbf{W}_4}$, and 3 generations from each of the sectors $\overline{\mathbf{W}_0 + 6\mathbf{W}_4}$ and $\overline{\mathbf{W}_1 + \mathbf{W}_2 + 6\mathbf{W}_4}$. (These sectors give the $\underline{12}$ representation and acting with the analog of the supersymmetry on the left side ($\mathbf{W}_2$) gives the $\underline{32}$ to build up $\underline{56}$ of E_7.) The generations in model 10 are always distributed in this way. In fact there is a particular characteristic distribution of generations for each of the models 9–12.

This is an intriguing connection, but direct reformulation as an orbifold along the lines given in the introduction is only possible for the case of the Z_2 orbifold. This is acheived using vectors which have only periodic or anti-periodic boundary conditions. There are only two possible non-trivial cases. The first has a single $\mathbf{W}_4$ vector with the entries

$$c_1^r = \frac{1}{2},\ b_4^r = c_4^r = \frac{1}{2} \qquad (r = 1, 2). \tag{29}$$

The second can be made from the first by adding an additional symmetric vector, $\mathbf{W}_5$, to the above, with entries

$$a_5^1 = c_5^1 = b_5^2 = c_5^2 = \frac{1}{2} \tag{30}$$

and all others zero. Both models have exactly the same spectrum as the Z_2 orbifold [15] accompanied by additional, matching pairs of gauge bosons and singlets. Thus we see that the relation of these fermionic string models to the Z_2 orbifold is identical to that between the superconformal models and the K3 manifold. We therefore deduce that the fermionic string belongs to a Z_2 orbifold on a point in moduli space with enlarged symmetries. Using real fermions one may then break the rank of the gauge group. We should stress that in $D = 6$ the modular invariance conditions prohibit the construction of the Z_4 orbifold in a similar manner even with real fermions.

In addition to the above, we have two rather peculiar models with 13 and 17 generations. These have particularly symmetric configurations. For example the series of models with the non-zero boundary conditions

$$c_1^r = \frac{1}{2},\ b_4^1 = \frac{1}{m},\ c_4^1 = c_4^2 = \frac{m-1}{m},\ a_4^2 = \frac{m+1}{3m},\ b_4^2 = \frac{2(m+1)}{3m} \qquad m = 2^{(2n+1)} \tag{31}$$

generates only these spectra.

Having made a systematic study of models with one compactification vector of the $\mathbf{W}_4$ type, let us make some remarks about the case of symmetric pairs (18). If we choose a $\mathbf{W}_4$, $\mathbf{W}_5$ with $(a_5^r, b_5^r, c_5^r) = (0, 0, 0)$, then it is possible to generate more vectors of the $\mathbf{W}_1$ variety. In this case the $\mathbf{W}_4$ and $\mathbf{W}_5$ vectors usually project out as many gravitino degrees of freedom as new ones are generated, and the order of the supersymmetry is unchanged. The gravitinos may then appear in more complicated sectors of the form $\overline{\alpha\mathbf{W}} = \overline{\mathbf{W}_0 + \alpha_1\mathbf{W}_1 + \alpha_4\mathbf{W}_4}$, which depend on the choice of the structure constants k_{ij}. We have found this to be true for various cases and we believe this to be a general mechanism.

For the general case of several compactification vectors no simple pattern is obvious, but the classification of models so far suggests that there should be only models with a smaller number of generations than the those already found.

The results of our classification of fermionic strings in $D = 8$ and $D = 6$ dimensions are the following. In $D = 8$ we find only the torus with different gauge groups g. In $D = 6$ we find nearly all the possible generation numbers below 10, which is the result for the $K3$ manifold and Z_N orbifolds, and in addition models with 13 and 17 generations. The overlap with orbifolds consists of the Z_2 compactification, for which a direct bosonisation procedure exists even in this case of purely complex fermions. The spectrum of the Z_2 orbifold is generated in many more cases where direct bosonisation is not possible, and this leads us to conjecture that the three additional 10 generation models may be linked to the Z_3, Z_4 and Z_6 orbifolds. This will possibly be explained by a study of fermionic strings with permutational moddings. Either way it seems to be always possible to interpret fermionic strings as tori with generalised spin structures.

A similar study for $D = 4$ is under way and should shed more light on these questions. There a systematic study is much more complicated because of the huge number of possible models. Nevertheless our observations in higher dimensions have given some idea as to what we can expect to find.

Acknowledgement We thank D. C. Dunbar and H-P. Nilles for useful discussions. We would also like to thank the RAL computer division, especially Dick Roberts.

Table Captions

Table 1 Supersymmetric (2,2) models in $D = 8$. U is the number of untwisted generations, and n_s is the number of singlets. The gauge group is $g \otimes E_8 \otimes E_8'$. The singlets together with the gauge bosons and their superpartners form full gauge supermultiplets. Where possible we express g as a product of special unitary groups.

Table 2 Supersymmetric (2,2) models in $D = 6$. The gauge group is $g \otimes E_{6+N} \otimes E_8'$. For $N = 2$ the singlets are incorporated in gauge multiplets.

References

[1] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B256 (1985) 46

[2] P. Candelas, A. M. Dale, C. A. Lütken, R. Schimmrigk, Nucl. Phys. B298 (1988) 499

[3] S. T. Yau, in Proc. Argonne Symp. on Anomalies, Geometry and Topology (World Scientific 1985)

[4] P. Candelas, M. Lynker, R. Schimmrigk, Nucl. Phys. B341 (1990) 383

[5] D. Gepner Nucl. Phys. B296 (1988) 757; Phys. Lett. 199B (1987) 380; J. Fuchs, A. Klemm, C. M. A. Scheich and M. G. Schmidt, Ann. Phys. 204 (1990) 1

[6] C. M. A. Scheich, M. G. Schmidt, Int. J. of Mod. Phys. A7 (1992) 8021

[7] Y. Kazama, H. Suzuki, Nucl. Phys. B321 (1989) 232

[8] L. Dixon, J. A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678; Nucl. Phys. B274 (1986) 285

[9] L. E. Ibanez, J. E. Kim, H.-P. Nilles, F. Quevedo, Phys. Lett. B191 (1987) 282; A. Font, L. E. Ibanez,F. Quevedo, A. Sierra, Nucl. Phys. B (1990) 421

[10] A. Font, L. E. Ibañez and F. Quevedo, Phys. Lett. 217B (1989) 272

[11] A. H. Chamseddine and J-P. Derendinger, Nucl. Phys. B301 (1988) 381

[12] H. Kawai, D. C. Lewellyn and S-H. H. Tye, Nucl. Phys. B288 (1987) 1; I. Antoniades, C. P. Bachas and C. Kounnas, Nucl. Phys. B289 (1987) 87

[13] I. Antoniadis, J. Ellis, J. S. Hagelin, D. V. Nanoupolos, Phys. Lett.231 (1989) 65

[14] P. Ginsparg, in Lectures given at Les Houches Summer Session 1988 (World Scientific 1989)

[15] J. A. Harvey and J. Minahan, Phys. Lett. 188B (1987) 44; M. A. Walton, Phys. Rev. D37 (1988) 377

[16] D. Bailin, D. C. Dunbar and A. Love, Nucl. Phys. B330 (1990) 124; Int. J. Mod. Phys. A5 (1990) 939; D. C. Dunbar, Mod. Phys. Lett. A4 (1989) 2339

[17] R. Peschanski and C. A. Savoy, Phys. Lett. 221B (1989) 276

[18] T. Banks, L. Dixon, D. Friedan, E. Martinec, Nucl. Phys. B299 (1988) 613

[19] D. Lüst and S. Theisen, Int. J. Mod. Phys. A4 (1989) 4513

[20] S. A. Abel, C. M. A. Scheich, work in progress

Table 1:

number	N	group g	gen	$\overline{\text{gen}}$	U	n_s
1	1	$SU(2)^2$	2	2	4	6
1	1	$SU(2) \otimes U(1)$	2	2	4	4

Table 2:

number	N	group g	gen	$\overline{\text{gen}}$	U	n_s
1	2	$SO(8)$	2	2	4	34
2	2	$SU(4) \otimes U(1)$	2	2	4	22
3	2	$SU(2)^4$	2	2	4	18
4	2	$SU(2)^3 \otimes U(1)$	2	2	4	16
5	2	$SU(2)^3 \otimes U(1)$	2	2	4	16
6	2	$SU(2)^2 \otimes U(1)^2$	2	2	4	14
7	1	$U(1)^5$	17	0	1	51
8	1	$U(1)^5$	13	0	1	52
9	1	$SU(2)^5$	10	0	2	80
10	1	$SU(2)^2 \otimes U(1)^3$	10	0	1	32
11	1	$SU(2) \otimes U(1)^4$	10	0	1	54
12	1	$U(1)^5$	10	0	1	30
13	1	$U(1)^5$	9	0	1	36
14	1	$U(1)^5$	9	0	1	27
15	1	$SU(2)^3 \otimes U(1)^2$	8	0	2	40
16	1	$SU(2)^2 \otimes U(1)^3$	7	0	1	20
17	1	$SU(2) \otimes U(1)^4$	6	0	2	30
18	1	$U(1)^5$	5	0	1	25
19	1	$U(1)^5$	5	0	1	20
20	1	$U(1)^5$	5	0	1	15
21	1	$SU(4) \otimes U(1)^2$	2	0	1	14
22	1	$SU(4) \otimes U(1)^2$	2	0	1	12
23	1	$SU(2)^2 \otimes U(1)^3$	2	0	1	10
24	1	$SU(2)^2 \otimes U(1)^3$	2	0	1	8
25	1	$SU(2)^2 \otimes U(1)^3$	2	0	2	6
26	1	$SU(2) \otimes U(1)^4$	2	0	1	6
27	1	$SU(4) \otimes U(1)^2$	1	0	1	6
28	1	$SU(3) \otimes U(1)^3$	1	0	1	10
29	1	$SU(2)^3 \otimes U(1)^2$	1	0	1	10
30	1	$SU(2)^3 \otimes U(1)^2$	1	0	1	8
31	1	$SU(2)^2 \otimes U(1)^3$	1	0	1	5
32	1	$SU(2)^2 \otimes U(1)^3$	1	0	1	4
33	1	$SU(2) \otimes U(1)^4$	1	0	1	6
34	1	$SU(2) \otimes U(1)^4$	1	0	1	5
35	1	$SU(2) \otimes U(1)^4$	1	0	1	3
36	1	$U(1)^5$	1	0	1	4
37	1	$U(1)^5$	1	0	1	3