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**INFRARED AND ULTRAVIOLET BEHAVIOUR OF  
EFFECTIVE SCALAR FIELD THEORY**

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**Abstract**

We consider the infrared and ultraviolet behaviour of the effective quantum field theory of a single  $Z_2$  symmetric scalar field. In a previous paper we proved to all orders in perturbation theory the renormalizability of massive effective scalar field theory using Wilson's exact renormalization group equation. Here we show that away from exceptional momenta the massless theory is similarly renormalizable, and we prove detailed bounds on Green's functions as arbitrary combinations of exceptional Euclidean momenta are approached. As a corollary we also prove Weinberg's Theorem for the massive effective theory, in the form of bounds on Green's functions at Euclidean momenta much greater than the particle mass but below the naturalness scale of the theory.

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In a previous paper [1] we used the exact renormalization group [2,3] to construct a stable, unitarity and causal effective massive scalar field theory with a  $Z_2$  global symmetry, for which we could then prove perturbative renormalizability — boundedness, convergence and universality — for processes at energy scales similar to the mass of the particle but far below some naturalness scale. These results were trivially extended to theories with arbitrary numbers of particles of spin zero or one half, with linearly realized global symmetries.

However we only really considered theories with two different scales,  $\Lambda_R$  of order the masses of the particles, and  $\Lambda_0$ , the naturalness scale, at which all the interactions could become equally important and the effective theory becomes equivalent to a general S-matrix theory. We might hope that we could be more ambitious than this, since we often in practice wish to consider situations where we have a number of different scales to be dealt with simultaneously. An obvious example would be to have two particles with significantly different masses; for processes at scales of order the light particle mass one could then consider the extent to which the effects of the heavy particle decouple. This is discussed in an accompanying paper [4]. In the present paper we consider instead processes at scales much larger than the mass scale of the theory. We first consider a massless theory, proving bounds on Green's functions corresponding to processes at finite scales  $\Lambda_R \ll \Lambda_0$ , but with arbitrary combinations of (almost) exceptional momenta. It is then relatively straightforward to turn the argument around, and consider high energy processes of a massive theory (the only real difference between the two cases being the renormalization conditions on relevant couplings, as we will see). In this way both infrared and ultraviolet behaviour may be examined within the same framework.

Within the conventional formulation of quantum field theory the behaviour of a regulated Feynman graph when all external Euclidean momenta are large was given by Weinberg's theorem [5]. Historically Weinberg's theorem was the final step in the proof of perturbative renormalizability of local quantum field theory. Similar techniques may be used to deduce heuristically the infrared structure of the unrenormalized theory when there are no exceptional momenta [6]. Much more detailed studies based on similar methods to those used by Landau to find threshold singularities were made by Kinoshita [7], while a rigorous but complicated analysis based on BPHZ subtraction was made by Symanzik [8]. The main reason for this technical complication of what should be, after all, simply a matter of scaling, is that in the conventional formulation of perturbative quantum field theory all the different scales in the theory come into play at the same time, and ultraviolet and infrared problems must be dealt with together. However, in the exact renormalization group approach each scale is treated separately; in perturbation theory the graphs are built up by successively nesting high momentum loops in a strict hierarchy of decreasing scales. In this way ultraviolet and infrared issues can be completely disentangled, and the derivation of bounds on Green's functions in either the massless limit or in the deep Euclidean region becomes relatively straightforward.

We begin with a brief summary of the exact renormalization group, in the formulation used in ref.[1]. In §2 we discuss the definition of a massless scalar theory and its infrared divergences, show how to obtain bounds on the flow equations for arbitrary combinations of exceptional momenta, use these to prove a set of bounds on the renormalized Green's functions, and then use these to discuss the infrared finiteness of physical S-matrix elements. In §3 we derive a similar set of bounds on the Green's functions of a massive effective theory in the deep Euclidean region, which amount to an extension of Weinberg's Theorem, and show further how these bounds may be systematically improved.

## 1. Effective Field Theory and the Exact Renormalization Group.

An effective field theory may be defined through a classical Euclidean action

$$S[\phi; \Lambda] = \frac{1}{2} (\phi, P_{\Lambda}^{-1} \phi) + S_{\text{int}}[\phi; \Lambda]. \quad (1.1)$$

It is assumed that this action is Lorentz invariant, an analytic functional of the fields  $\phi$  and their derivatives, and is natural at some scale  $\Lambda_0$ . The propagator function is taken to be of the form

$$P_{\Lambda}(p) = \frac{K_{\Lambda}(p)}{(p^2 + m^2)}, \quad (1.2)$$

where  $0 < m^2 \ll \Lambda_0^2$ , and the regulating function  $K_{\Lambda}(p)$  is of the form  $K_{\Lambda}(p) \equiv K((p^2 + m^2)/\Lambda^2)$ , with  $K(x)$  real, positive, and monotonically decreasing in  $x$ ,  $K(0) = 1$ , while both  $K(z)$  and  $1/K(z)$  are regular functions, with an essential singularity at the point at infinity. If the order of this singularity is sufficiently large, the S-matrix of the theory will be finite. The interaction may be expanded in the general form

$$S_{\text{int}}[\phi; \Lambda] \equiv \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{g^r}{(2m)!} \int \frac{d^4 p_1 \cdots d^4 p_{2m}}{(2\pi)^{4(2m-1)}} V_{2m}^r(p_1, \dots, p_{2m}; \Lambda) \delta^4(\sum_{i=1}^{2m} p_i) \phi_{p_1} \cdots \phi_{p_{2m}}, \quad (1.3)$$

where the vertex functions  $V_{2m}^r(p_1, \dots, p_{2m}; \Lambda)$  are, for  $\Lambda > 0$ , analytic functions of their arguments, and  $g$  is a coupling constant used to order the perturbation series. We may assume that the theory has a  $Z_2$  global symmetry, so that only even terms occur in the expansion.

It is shown in [1] that it is possible to construct actions of the form (1.1) such that, despite the presence of derivatives of the fields of arbitrarily high order, the classical equations of motion have only the usual vacuum and free-particle solutions, allowing for the construction of in- and out-states, and thus of a perturbative S-matrix. It is shown furthermore that this S-matrix is unitary and causal at all scales.

The quantum theory may be defined by reducing  $\Lambda$  from the naturalness scale  $\Lambda_0$  down to zero; the regulating function in the propagator (1.2) then ensures that all modes are successively

integrated over. The connected amputated Green's functions  $\tilde{G}_{2m}^c$  are invariant under changes of  $\Lambda$  provided the effective interaction satisfies the exact renormalization group equation [2,3]

$$\frac{\partial S_{\text{int}}}{\partial \Lambda} = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{\partial P_\Lambda}{\partial \Lambda} \left[ \frac{\delta S_{\text{int}}}{\delta \phi_p} \frac{\delta S_{\text{int}}}{\delta \phi_{-p}} - \frac{\delta^2 S_{\text{int}}}{\delta \phi_p \delta \phi_{-p}} \right]. \quad (1.4)$$

So starting from a 'bare' interaction  $S_{\text{int}}[\phi; \Lambda_0]$ , solution of (1.4) yields a renormalized interaction  $S_{\text{int}}[\phi; \Lambda]$ ; formally

$$\exp[-S_{\text{int}}[\phi; \Lambda] - \mathcal{E}(\Lambda)] = \exp(\mathcal{P}_{\Lambda_0} - \mathcal{P}_\Lambda) \exp[-S_{\text{int}}[\phi; \Lambda_0]], \quad (1.5)$$

where

$$\mathcal{P}_\Lambda \equiv \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} P_\Lambda(p) \frac{\delta}{\delta \phi_p} \frac{\delta}{\delta \phi_{-p}}, \quad (1.6)$$

and  $\mathcal{E}(\Lambda)$  is a field independent constant. At  $\Lambda = 0$  all modes have been integrated out and the amputated connected Green's functions  $\tilde{G}_{2m}^c$  may be read off from  $S_{\text{int}}[\phi; 0]$ ;

$$\tilde{G}_{2m}^c(p_1, \dots, p_{2m}) \equiv \prod_{i=1}^{2m} \left( -\frac{\delta}{\delta \phi_{p_i}} \right) S_{\text{int}}[\phi; 0] \Big|_{\phi=0}. \quad (1.7)$$

In terms of the vertex functions defined in (1.3) the evolution equation (1.4) becomes

$$\begin{aligned} \Lambda \frac{\partial}{\partial \Lambda} V_{2m}^r(p_1, \dots, p_{2m}; \Lambda) &= -\frac{\Lambda^2}{m(2m-1)} \int \frac{d^4 p}{(2\pi\Lambda)^4} K'_\Lambda(p) V_{2m+2}^r(p, -p, p_1, \dots, p_{2m}; \Lambda) \\ &+ \sum_{l=1}^m \sum_{s=1}^{r-1} \frac{\Lambda^{-2} 2(2m)!}{(2m+1-2l)!(2l-1)!} K'_\Lambda(P) V_{2l}^s(p_1, \dots, p_{2l-1}, P; \Lambda) V_{2m+2-2l}^{r-s}(-P, p_{2l}, \dots, p_{2m+2}; \Lambda), \end{aligned} \quad (1.8)$$

where  $K'_\Lambda(p) \equiv K'((p^2 + m^2)/\Lambda^2)$ ,  $K'(x)$  being the first derivative of the regulating function  $K(x)$ ; for a regulating function with an essential singularity of finite order,  $K'(x) = P(x)K(x)$  for some polynomial  $P(x)$ . The amputated connected Green's functions are then

$$\tilde{G}_{2m}^c = \delta^4 \left( \sum_{i=1}^{2m} p_i \right) \sum_r g^r V_{2m}^r(0), \quad (1.9)$$

and S-matrix elements are given by analytic continuation and addition of external lines. It can be shown that the S-matrix is independent of the choice of regulating function, provided only that  $K(z)$  is chosen such that the order of its essential singularity sufficiently large that the integral in the first term on the right hand side of (1.8) converges.

In [3] and [1] (1.8) was used to obtain bounds on the vertex functions and their derivatives; defining

$$\|V\|_\Lambda \equiv \max_{\{p_1, \dots, p_n\}} \left[ \prod_{i=1}^n [K_\Lambda(p_i)]^{1/4} |V(p_1, \dots, p_n; \Lambda)| \right], \quad (1.10)$$

and using the simple inequalities

$$\begin{aligned} \int \frac{d^4 p}{(2\pi)^4} |[K_\Lambda^{1/2}(p)]| &< C\Lambda^4 |K^{1/2}(m^2/\Lambda^2)|, \\ |K_\Lambda^{-1/2}(p)\partial_p^k K_\Lambda(p)| &< D_k \Lambda^{-k} |K^{1/2}(m^2/\Lambda^2)|, \end{aligned} \quad (1.11)$$

we find for example that, ignoring constants on the right hand side,

$$\begin{aligned} \left\| \frac{\partial}{\partial \Lambda} \left( \partial_p^j V_{2m}^r(\Lambda) \right) \right\|_{\bar{\Lambda}} &\leq \bar{\Lambda} \|\partial_p^j V_{2m+2}^r(\Lambda)\|_{\bar{\Lambda}} \\ &+ \bar{\Lambda}^{-3} \sum_{l=1}^m \sum_{s=1}^{r-1} \sum_{j_i; j_1+j_2+j_3=j} \bar{\Lambda}^{-j_1} \|\partial_p^{j_2} V_{2l}^s(\Lambda)\|_{\bar{\Lambda}} \cdot \|\partial_p^{j_3} V_{2m+2-2l}^{r-s}(\Lambda)\|_{\bar{\Lambda}}, \end{aligned} \quad (1.12)$$

where  $\bar{\Lambda} \equiv \max\{\Lambda, \Lambda_R\}$ ,  $\Lambda_R$  being of order  $m$ . This bound is then used to prove a bound on the amputated connected Green's functions (1.7)(and their derivatives); to all orders in perturbation theory it turns out that

$$\|\partial_p^j \tilde{G}_{2m}^c\|_{\Lambda_R} \leq \Lambda_R^{4-2m-j}. \quad (1.13)$$

## 2. Infrared Behaviour

### 2.1. Defining a Massless Theory.

To define a massless theory it is tempting simply to set  $m = 0$  in the regularized propagator (1.2), and then further insist that at each order in perturbation theory the renormalized two-point vertex function vanishes at  $\Lambda = 0$  and at zero momentum, ie. that  $V_2^r(0, 0; 0) = 0$ . Clearly it would not be sufficient to impose this condition at any finite renormalization scale  $\Lambda_R > 0$ , since then a mass term would be generated from remaining evolution down to  $\Lambda = 0$ , and the particle would not be truly massless. It is thus imperative when dealing with massless particles that we use a formulation of the exact renormalization group (such as that described in [1], and outlined in the previous §) which allows the consideration of arbitrarily small regularization scales  $\Lambda$ .<sup>1</sup>

However we know from experience that when calculating with massless theories we encounter infrared divergences when partial sums of momenta tend to zero. It is therefore necessary to see how these divergences manifest themselves when using the exact renormalization group. From (1.5) we can see that the vertices in the effective action at some scale  $\Lambda < \Lambda_0$  can be constructed from diagrams consisting of the bare vertices and the regular 'propagator'  $P_{\Lambda_0}(p) - P_\Lambda(p)$ . Since there are manifestly no infrared singularities in these diagrams the effective action must be well-defined

<sup>1</sup> In the formulation of the exact renormalization group used in [3] it is not possible to consider vertex functions with external momenta above  $\Lambda$ ; all low energy renormalization scales must then be set at some finite scale  $\Lambda_R$  and it is impossible to consider theories with massless particles.

for all finite  $\Lambda$ . Indeed, this statement is also true nonperturbatively; in the flow equation (1.4) both terms are infrared finite even when  $m = 0$ . Therefore, we could try to regulate the theory in the infrared by defining a minimum value of  $\Lambda$ ,  $\Lambda_{\min}$ , and then attempting to define the massless theory by taking the limit  $\Lambda_{\min} \rightarrow 0$ . We might then obtain a strictly massless theory if we set the renormalization condition  $V_2^r(P, -P; \Lambda_{\min}) \sim \Lambda_{\min}^2$ , where  $|P| \sim \Lambda_{\min}$ .

The problem with such an approach is that in the limit  $\Lambda_{\min} \rightarrow 0$  the vertex functions  $V_{2m}^r(p_i, \Lambda_{\min})$  remain regular functions of momenta (or more specifically of the momentum invariants  $\{z_{ij} \equiv (p_i + p_j)^2; i = 1, \dots, 2m, j = 1, \dots, i\}$ ), while the connected amputated Green's functions must, when analytically continued to the physical region, contain a complicated set of overlapping poles and cuts at  $z_{ij} = 0$ . Indeed the crucial relationship (1.9) between the vertex functions in the limit  $\Lambda \rightarrow 0$  and the amputated connected Green's functions cannot be established when  $m = 0$  because in this limit the regulating function no longer vanishes for all Euclidean momenta  $p$ , but instead has nonvanishing support at  $p^2 = 0$ : if  $\sigma$  is the order of the essential singularity at infinity,

$$\lim_{\Lambda \rightarrow 0} K(p^2/\Lambda^2) = \begin{cases} 0, & \text{if } \Re p^2 > 0; \\ 1, & \text{if } p^2 = 0; \\ 0, & \text{if } \Re p^2 < 0 \text{ and } \sigma \text{ is even;} \\ \infty, & \text{if } \Re p^2 < 0 \text{ and } \sigma \text{ is odd.} \end{cases} \quad (2.1)$$

So the correspondence between S-matrix elements and the Euclidean vertex functions is broken at precisely the points we wish to consider, namely at exceptional momenta where the Green's functions may be infrared divergent.

It is thus necessary to disentangle the infrared divergences at exceptional momenta from the poles and cuts by choosing an infrared regulator which works even as  $\Lambda$  goes to zero. The obvious choice is to keep a small non-zero mass term in the propagator (1.2), but with  $m$  much less than the scale  $\Lambda_R$  at which we want to investigate the physics. The identification (1.9) will then hold for all external momenta, so we can then examine how the amputated connected Green's functions depend on the ratio  $(\Lambda_R/m)$  at exceptional momenta of order  $m$  or less, and thus deduce the infrared behaviour of the theory by considering them in the limit  $m \rightarrow 0$ .

This choice of infrared regulator is by no means unique, but has certain advantages: it makes the extension of the results on the infrared structure of the massless theory to the consideration of Green's functions at large momenta in the massive theory (i.e. to Weinberg's theorem) very straightforward, and furthermore the techniques used here can be applied directly to situations where we have a number of particles with significantly different masses (as is required for the treatment of decoupling in ref.[4]).

To ensure that the theory is massless in the limit  $m \rightarrow 0$  we must simply set the renormalization conditions on the two point vertex at  $\Lambda = 0$  such that it is of order  $m^2$  for momenta of order  $m$ . We thus take the renormalization conditions at  $\Lambda = 0$  to be

$$\begin{aligned} \lim_{\Lambda \rightarrow 0} V_2^r(\tilde{P}, -\tilde{P}; \Lambda) &= \Lambda_m^2 \hat{\lambda}_1^r, \\ \lim_{\Lambda \rightarrow 0} [\partial_{p_\mu} \partial_{p_\nu} V_2^r(p, -p; \Lambda)|_{p=P_0}]_{\delta_{\mu\nu}} &= \hat{\lambda}_2^r, \\ \lim_{\Lambda \rightarrow 0} V_4^r(P_1, P_2, P_3, P_4; \Lambda) &= \hat{\lambda}_3^r. \end{aligned} \tag{2.2}$$

Here  $\Lambda_m$  is some scale of order  $m$ , so  $\Lambda_m \ll \Lambda_R$ , the renormalization scale,  $\tilde{P}$  and  $P_i$  (such that  $\sum_1^4 P_i = 0$ ) are the external momenta at which the renormalization conditions are set, while  $\hat{\lambda}_i^r$  are some renormalization constants chosen independently of  $\Lambda_0$ ,  $\Lambda_R$  and  $\Lambda_m$ . We choose  $P_i$  and all their partial or complete sums with magnitude similar to  $\Lambda_R$ , while  $\tilde{P}$  has magnitude similar to or less than  $\Lambda_m$ . Thus, except for the renormalization condition on the two-point vertex, the renormalization conditions are set for momenta much larger than the mass of the particle, and in particular for non-vanishing momenta as  $m \rightarrow 0$ .<sup>2</sup>

The renormalization conditions on irrelevant vertices at  $\Lambda_0$  are (for convenience) taken to be the same as in §2 of [1], namely

$$\partial_p^j V_{2m}^r(\Lambda_0) = 0 \quad 2m + j > 4. \tag{2.3}$$

They will later be relaxed in order to prove the universality of the massless theory.

## 2.2. Exceptional Momenta.

Now that the theory is well defined, we can begin to investigate its infrared structure, by considering the behaviour of the vertices when various combinations of their external momenta become small. However, before we do this it will be convenient to introduce some new terminology, in order to specify precisely what we mean by this; it is not only the magnitudes of single momenta that are important in general but that of the sum of a set of momenta. Consider a particular set  $\{p_{\sigma(1)}, \dots, p_{\sigma(n)}\}$  of  $n$  momenta (where  $\sigma(i)$  is some permutation of  $i = 1, \dots, 2m$ ). We define this set<sup>3</sup> to be ‘exceptional’ if the magnitude of the sum of its momenta is less than a certain

<sup>2</sup> It would be possible to set the renormalization condition on the second momentum derivative of the two point vertex at  $P$  also. However, this would lead to the necessity of a wave-function renormalization which behaves like  $P \log(\Lambda_R/\Lambda_m)$ . As long as we were to define our Green’s functions via this wavefunction renormalization before taking the limit  $m \rightarrow 0$ , we would still only obtain infrared divergences at exceptional momenta, but we believe that the bounding argument is far clearer if we avoid this complication.

<sup>3</sup> In order to prove perturbative renormalizability and infrared finiteness of a massless scalar field theory using renormalization group flow, it will be necessary to consider all possible combinations of exceptional momenta. The proof of infrared finiteness in ref.[9], in which momenta are only allowed to become exceptional in pairs, so that each exceptional set contains only two elements, is thus incomplete.

value, which in practice we choose to be the scale at which the renormalization conditions are set,  $\Lambda_R$ . In fact it will also prove useful to impose a lower bound  $E$  on the momenta, so that in fact  $E^2 < (\sum_{i=1}^n p_{\sigma(i)})^2 < \Lambda_R^2$ .

For a vertex with  $2m$  legs we always have at least one exceptional set  $\{p_1, \dots, p_{2m}\}$  simply because the sum of the momenta entering the vertex is in practice equal to zero because of the delta function in the definition (1.3) which enforces momentum conservation. We can also see that because of momentum conservation as soon as we have one exceptional set we automatically have two; if there is an exceptional set  $\{p_{\sigma(1)}, \dots, p_{\sigma(n)}\}$  containing  $n$  momenta there is also another one  $\{p_{\sigma(2m-n+1)}, \dots, p_{\sigma(2m)}\}$  containing  $2m - n$  momenta.

We also wish to define a quantity  $e$  which we may think of (loosely) as the number of exceptional momenta for a given vertex; if we do this appropriately then  $e$  will actually turn out to give the degree of infrared divergence at the vertex. To do this we first define an irreducible exceptional set to be an exceptional set with no exceptional subsets. We then may define the number of exceptional momenta  $e$  for a given vertex with  $2m$  legs to be the total number of distinct momenta contained in the irreducible exceptional sets excluding those in the largest irreducible exceptional set.<sup>4</sup> Note the the number of exceptional momenta  $e$  is not necessarily the same as the number of irreducible exceptional sets; indeed it is not difficult to see that if the largest irreducible exceptional set contains  $n$  momenta, then  $e = 2m - n$  irrespective of the details of the other exceptional sets. Thus, for the case where only the total sum of momenta is exceptional we would say that the vertex has  $e = 0$ ; no exceptional momenta. If there are only two exceptional sets, we must have the number of exceptional momenta  $e \leq m$ , since if one set contains  $n$  momenta, the other must contain  $2m - n$ . At the other end of the scale, the maximum of value of  $e$  is  $2m - 1$ ; this can only occur if all the momenta are individually exceptional, falling into  $2m$  exceptional sets with one element each.

In order to illustrate this definition further we consider the simplest cases. For the two point vertex the two external momenta sum to zero. Thus for large values of  $p^2$ , we have a single irreducible exceptional set of two momenta,  $\{p, -p\}$ , and we take  $e = 0$ ; we have no exceptional momenta. If on the other hand  $p^2 < \Lambda_R^2$ , then we have two irreducible sets of exceptional momenta,  $\{p\}$  and  $\{-p\}$ , and we take  $e = 1$ . The possible combinations of irreducible sets (up to permutations) and the corresponding number of exceptional momenta for the four point vertex are:

$\{p_1, p_2, p_3, p_4\}$	$e = 0,$
$\{p_1, p_2, p_3\}\{p_4\}$	$e = 1,$
$\{p_1, p_2\}\{p_3, p_4\}$	$e = 2,$
$\{p_1, p_2\}\{p_3\}\{p_4\}$	$e = 2,$
$\{p_1, p_2\}\{p_3, p_4\}\{p_2, p_3\}\{p_1, p_4\}$	$e = 2,$
$\{p_1\}\{p_2\}\{p_3\}\{p_4\}$	$e = 3.$

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<sup>4</sup> This definition means that  $e$  is actually equal to the minimum value of the total number of distinct momenta within exceptional sets after the imposition of overall momentum conservation (which renders the vertex a function of only  $2m - 1$  independent momenta).

### 2.3. Bounding the Flow Equations.

We now consider the bounding of the vertices. As in §2.2 of [1] we construct a norm  $\|V\|_\Lambda$  of a vertex function  $V(p_1, \dots, p_n)$  by multiplying each leg of the vertex by  $[K_\Lambda(p_i)]^{1/4}$  and finding the maximum with respect to all the momenta, as in (1.10). However, in order to investigate the dependence of the vertices on momenta with magnitude less than  $\Lambda_R$  we now also introduce the idea of a restricted norm in which we find the maximum over a restricted range of momenta. In practice we consider taking the maximum over sets of momenta  $\Pi(n, e; \Lambda_R, E)$  defined as the set  $\{p_1, \dots, p_n\}$  such that there are  $e$  exceptional momenta, the total momentum in each exceptional set (including the largest) being constrained to lie in the range  $(E, \Lambda_R)$ . The restricted norm is thus

$$\|V_{2m}(\Lambda)\|_{\Lambda_R}^{E,e} \equiv \max_{\Pi(n,e;\Lambda_R,E)} \left[ \prod_{i=1}^n [K_{\Lambda_R}(p_i)]^{1/4} |V_{2m}(p_1, \dots, p_n; \Lambda)| \right]. \quad (2.4)$$

It is not difficult to see that this is indeed a norm. If  $e = 0$  we will omit the superscripts  $E$  and  $e$ .

We can use this definition of a restricted norm to bound both sides of the flow equations (1.8). However, in order to do this effectively we must also suitably extend the inequalities (1.11). The first of these is only used for bounding integrals over internal loop momenta, independent of all external momenta, and thus does not change if we put restrictions on the external momenta. The argument  $p$  in the second inequality (1.11) is however to be identified with  $P$  in (1.8), which is a sum of external momenta. If we restrict the form of the external momenta, we therefore restrict the possible values of  $P$ , and thus require an improved form of this inequality for such restricted  $p$ . In fact we want to consider the left-hand side of this inequality as a maximum over a range of  $p$  greater than a minimum value,  $E \in [0, \Lambda_R)$ . Remembering the form of  $K_\Lambda(p)$ , as described following (1.2), and in particular its dependence on the mass  $m$ , it is not difficult to show that

$$\max_{p>E} \left| \Lambda^{-n} K_\Lambda^{-1/2}(p) \partial_p^j K_\Lambda(p) \right| \leq \bar{\Lambda}^{-n-j}, \quad (2.5)$$

where  $\bar{\Lambda} = \max(\Lambda_m, \Lambda, E)$ , and  $n$  is a non-negative integer.

We are now nearly ready to bound the left-hand side of (1.8), or more precisely, the left-hand side of (1.8) after it has been differentiated  $j$  times with respect to external momenta. For  $\Lambda \in [\Lambda_R, \Lambda_0]$  we proceed in exactly the same way as in §2 of [1]; we take the norm with respect to  $\Lambda$  and do not concern ourselves with the values of the external momenta, to give the bounded flow equation (1.12) with  $\bar{\Lambda} = \Lambda$ .

In the range  $\Lambda \in [0, \Lambda_R]$  we must be rather more careful in our bounding. In this case we consider the number of sets of momenta with magnitudes as low as  $E \in [0, \Lambda_R)$ . (In fact, as in (2.5),  $E$  has an effective lower cut-off of  $\Lambda$  if  $\Lambda > \Lambda_m$  and  $\Lambda_m$  if  $\Lambda \leq \Lambda_m$ , so it is only really necessary to consider  $E \in [\max(\Lambda, \Lambda_m), \Lambda_R)$  as we will see shortly.) To this end we need to consider what type of norms we have on the right-hand side of the  $j$ th momentum derivative of (1.8) once we

have specified the type of norm on the left-hand side (i.e. the number of exceptional momenta in contains).

We first consider the first term on the right-hand side of (1.8). If  $p_1, \dots, p_{2m}$  contain  $e$  exceptional momenta then  $\partial_p^j V_{2m+2}(p, -p, p_1 \dots p_{2m}; \Lambda)$  contains exactly  $e+2$  exceptional momenta. The extra two exceptional momenta are only due to the fact that  $p$  and  $-p$  sum to zero, and therefore obviously form a set of two which is exceptional. ( $p$  and  $-p$  can take all values and can therefore also form two exceptional sets of one. We will see that this creates no new effects.) We might think that since  $p$  can take any value, then whenever  $p$  is equal to the sum of some set of momenta, and thus  $p_{i_1} + \dots + p_{i_k} - p = 0$ , we would have another exceptional configuration. This is strictly true, but rather misleading: since  $V_{2m+2}(p, -p, p_1 \dots p_{2m}; \Lambda)$  is invariant not only under permutations of momenta, but also under  $p \rightarrow -p$ , the vertex can only depend on  $p$  through the invariants  $p^2$  or  $((p_{i_1} + \dots + p_{i_k}) \cdot p)^2$ , and not on  $(p_{i_1} + \dots + p_{i_k} \pm p)^2$ . Thus, since the vertex does not depend on  $p_{i_1} + \dots + p_{i_k} - p$ , its value has no significance and we do not class it as an exceptional set.

The second term on the right-hand side of the  $j_{th}$  momentum derivative of (1.8) is a little more difficult to deal with. We first consider the  $\Lambda$ -derivative of the propagator which links the two vertices in this term. The argument of this propagator is  $P = p_1 + \dots + p_{2l-1}$ , which is the sum of all the external momenta on either of the two vertices. In order to obtain the terms with the largest value on the right-hand side of the  $j_{th}$  momentum derivative once we take the restricted norms, we must consider an exceptional value of  $P$ . This is because when taking norms of both sides of (1.8) we will obtain a factor of  $\bar{\Lambda}^{-3}$  from  $\max_{p \leq E} |\Lambda^{-3} K_{\Lambda}^{-\frac{1}{2}}(p) \partial^j K_{\Lambda}(p)|$  if we have exceptional  $P$ , rather than the smaller value of  $\Lambda_R^{-3}$  we would obtain for non-exceptional  $P$  (as we see from (2.5)). This exceptional value of  $P$  is obtained if all the external momenta on each vertex comprise an exceptional set, since all the external momenta for the vertices will then sum to give  $P$  with magnitude less than or equal to  $\Lambda_R$ . This requirement is equivalent to demanding that none of the irreducible sets of exceptional momenta for  $V_{2m}^*$ , including the largest, are split between the two vertices.

We now consider the number of exceptional momenta we may obtain for each of the vertices in the second term on the right of (1.8) for a given number on the left-hand side. Since the bounds on the restricted norms become larger for larger numbers of exceptional momenta (as we will soon discover), in order to find the dominant terms on the right-hand side we need to have the largest value of the sum of the number of exceptional momenta on each of the vertices that we can. We first consider the case of exceptional  $P$ . In this case no sets of exceptional momenta are split between the two vertices. Therefore, the largest set of exceptional momenta for the vertex on the left-hand side resides on one of the vertices in the term on the right-hand side. This largest set contains  $2m - e$  momenta, and must necessarily be the largest set on the vertex on which it now resides. The largest set on the other vertex on the right-hand side must contain at least one momentum. Therefore, the largest sets on each vertex contain at least  $2m - e + 1$  momenta between them. The

two vertices have in all  $2m + 2$  legs, so the maximum value of  $e_1 + e_2$ , the sum of the number of exceptional momenta on each vertex, is  $e + 1$ . If  $P$  is non-exceptional then some of the exceptional sets on the left hand side must be split between the two vertices on the right-hand side. If the largest set is not split, then the largest set on one vertex must contain  $2m - e$  momenta. The largest set on the other vertex must contain at least 2 momenta (else none of the original sets could have been split), and using the same argument as above, the maximum value of  $e_1 + e_2$  is  $e$ . If the largest set is split so that  $n \leq 2m - e$  momenta go on one vertex and the remainder on another vertex, then the largest set on one vertex must contain at least  $n + 1$  momenta, and the largest set on the other vertex must contain at least  $2m - e - n + 1$  momenta. The maximum value of  $e_1 + e_2$  is then  $e$  again.

Finally, we must consider how momentum derivatives acting on the left-hand side affects the right-hand side of the bounded flow equation. The effect can occur in two ways: the momentum derivative may act on one of the two vertices in the second term on the right of (1.8), or may act on the propagator linking these two vertices. As in the case of the  $\Lambda$ -derivative of the propagator, we clearly obtain the largest factors when bounding the momentum derivatives of the  $\Lambda$ -derivative of the propagator when  $P$  is exceptional, in which case each momentum derivative gives a factor of  $\bar{\Lambda}^{-1}$ , rather than  $\Lambda_R^{-1}$  for unexceptional  $P$ . Thus, the criterion of not splitting the sets of exceptional momenta between the two vertices on the right-hand side leads to the largest factors for the  $\Lambda$ -derivative of the propagator linking the two vertices, whether there are momentum derivatives or not, and also to the largest value of  $e_1 + e_2$ .

These results now allow us to obtain bounded flow equations that are useful for bounding the vertices in ranges  $\Lambda \in [0, \Lambda_R]$ . Considering all  $\Lambda \in [0, \Lambda_R]$ , we may act with  $j$  momentum derivatives on the flow equation (1.8), then take the norms, and using (1.11) and (2.5) to bound the left-hand side, we obtain

$$\begin{aligned}
& \left\| \frac{\partial}{\partial \Lambda} \left( \partial_P^j V_{2m}^r(\Lambda) \right) \right\|_{\Lambda_R}^{E,e} \leq \left( \Lambda \|\partial_P^j V_{2m+2}^r(\Lambda)\|_{\Lambda_R}^{E,e;0,2} \right. \\
& + \sum_l \sum_{s=1}^{r-1} \sum_{j_i; j_1+j_2+j_3=j} \left[ \bar{\Lambda}^{-3-j_1} (1 - \delta_{l1}(\delta_{j_21} + \delta_{j_20}) - \delta_{(m+1-l)1}(\delta_{j_31} + \delta_{j_30})) \|\partial_P^{j_2} V_{2l}^s(\Lambda)\|_{\Lambda_R}^{E,e_1} \right. \\
& \quad \times \|\partial_P^{j_3} V_{2m+2-2l}^{r-s}(\Lambda)\|_{\Lambda_R}^{E,e_2} \\
& \left. + (\delta_{l1}(\delta_{j_21} + \delta_{j_20}) + \delta_{(m+1-l)1}(\delta_{j_31} + \delta_{j_30})) \sum_i \max_{p_i \geq E} \left( \Lambda^{-3} |\partial^{j_1} K_\Lambda(p_i)| \cdot |\partial_P^{j_2} V_2^s(p_i; \Lambda)| \right) \right. \\
& \quad \left. \times \|\partial_P^{j_3} V_{2m}^{r-s}(\Lambda)\|_{\Lambda_R}^{E,e} \right] + \text{other terms},
\end{aligned} \tag{2.6}$$

for  $e \geq 1$ , and  $m > 1$  or  $m = 1, j > 1$ . The terms on the right-hand side with two vertices which are written explicitly are those where the exceptional sets of momenta are not split between the two vertices, and where at the maximum  $e_1 + e_2 = e + 1$  (this maximum being reached automatically

if we have a two point vertex). The sum over  $l$  does not run from 1 to  $m$  as it does in (1.12) because for a given configuration of sets of exceptional momenta not all these values of  $l$  may be consistent with the requirement that no sets of exceptional momenta are split. The special terms for  $m = 1, j_2 < 2$  occur because these vertices have rather special bounds. We already have some hint of this in the fact that the renormalization condition for  $V_2^r$  is rather special. These special terms will be simplified later. The ‘other terms’ correspond to ways of putting together two vertices such that at least one set of exceptional momenta is split; such terms, as we will prove, make smaller contributions to the bound than the terms written explicitly. The form of the superscript on the norm for the first term on the right of this equation is a little different to the other superscripts. It signifies that since, as we have already mentioned, this term has two momenta which may each become as low as zero we denote it as above. We will discuss this term in more detail later.

For the special case  $m = 1, j = 0, p_1 \leq \Lambda_R$ , we have a slightly different type of inequality. (It is not necessary to derive an inequality for  $\partial_{p_\mu} V_2^r(\Lambda)$  since it can be entirely constructed from  $\partial_p^2 V_2^r(\Lambda)$  using the Taylor formula about  $p = 0$ .) This time we simply take the maximum of the modulus of both sides of (1.8) obtaining

$$\begin{aligned} \max_{p \leq E} \left| \frac{\partial}{\partial \Lambda} \left( V_2^r(p; \Lambda) \right) \right| &\leq \max_{p \leq E} \left| \int d^4 p' \Lambda^{-3} K'_\Lambda(p) V_4^r(p', -p', p, -p; \Lambda) \right| \\ &+ \sum_{s=1}^{r-1} \max_{p \leq E} \left| \Lambda^{-3} K'_\Lambda(p) V_2^s(p; \Lambda) V_2^{r-s}(p; \Lambda) \right|. \end{aligned} \quad (2.7)$$

Since for  $p \leq E$  factors of  $K_{\Lambda_R}^{-1/4}(p) \sim 1$ , and thus the first term on the right-hand side of (2.7) is  $\leq \Lambda \|V_4^r(\Lambda)\|_{\Lambda_R}^{0,3}$  and we may write (2.7) as

$$\max_{p \leq E} \left| \frac{\partial}{\partial \Lambda} \left( V_2^r(p; \Lambda) \right) \right| \leq \Lambda \|V_4^r(\Lambda)\|_{\Lambda_R}^{0,3} + \sum_{s=1}^{r-1} \max_{p \leq E} \left| \Lambda^{-3} K'_\Lambda(p) V_2^s(p; \Lambda) V_2^{r-s}(p; \Lambda) \right|. \quad (2.8)$$

If  $e = 0$  then bounding the left hand side of (1.8) is of course much easier, since there are then no exceptional momenta on either side of the flow equation, except for the pair  $\{p\}\{-p\}$  in the term  $\partial_p^j V_{2m+2}^r(p, -p, p_1 \dots p_{2m}; \Lambda)$ . Taking norms we therefore obtain

$$\begin{aligned} \left\| \frac{\partial}{\partial \Lambda} \left( \partial_p^j V_{2m}^r(\Lambda) \right) \right\|_{\Lambda_R} &\leq \left( \Lambda \| \partial_p^j V_{2m+2}^r(\Lambda) \|_{\Lambda_R}^{0,2} \right. \\ &\left. + \sum_{l=1}^m \sum_{s=1}^{r-1} \sum_{j_i; j_1+j_2+j_3=j} \Lambda_R^{-3-j_1} \| \partial_p^{j_2} V_{2l}^s(\Lambda) \|_{\Lambda_R} \cdot \| \partial_p^{j_3} V_{2m+2-2l}^{r-s}(\Lambda) \|_{\Lambda_R} \right). \end{aligned} \quad (2.9)$$

#### 2.4. Boundedness

These equations together with the boundary conditions on the relevant couplings, (2.5), and the trivial boundary conditions (2.3) on the irrelevant couplings, are all we need to prove<sup>5</sup>

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<sup>5</sup> To avoid confusion the lemmas in this paper are numbered sequentially with those of our previous paper [1].

*Lemma 5:*

i) For all  $\Lambda \in [\Lambda_R, \Lambda_0]$ ,

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda} \leq \Lambda^{4-2m-j} \left( P \log\left(\frac{\Lambda}{\Lambda_R}\right) + \frac{\Lambda}{\Lambda_0} P \log\left(\frac{\Lambda_0}{\Lambda_R}\right) \right). \quad (2.10)$$

ii) For all  $\Lambda \in [0, \Lambda_R]$ , and for  $e = 0$ , we have

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_R} \leq \Lambda_R^{4-2m-j}. \quad (2.11)$$

iii) For all  $\Lambda \in [0, \Lambda_R]$ , and  $1 \leq e \leq 2m - 3$ ,  $m > 1$  or  $m = 1$ ,  $j > 1$

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_R}^{E,e} \leq \begin{cases} \Lambda_R^{4-2m-j} \left(\frac{\Lambda_R}{\Lambda}\right)^{e+j-1} P \log\left(\frac{\Lambda_R}{\Lambda}\right) & e \text{ odd,} \\ \Lambda_R^{4-2m-j} \left(\frac{\Lambda_R}{\Lambda}\right)^{e+j-2} P \log\left(\frac{\Lambda_R}{\Lambda}\right) & e \text{ even.} \end{cases} \quad (2.12)$$

For  $e > 2m - 3$  the bound is simply  $\bar{\Lambda}^{4-2m-j} P \log\left(\frac{\Lambda_R}{\Lambda}\right)$ .

iv) The case  $\Lambda \in [0, \Lambda_R]$ ,  $e = 1$ ,  $m = 1$  and  $j < 2$ , is a little different, since we do not have a restricted norm, but an inequality reflecting the inequality (2.7):

$$\max_{p \leq E} |\partial_p^j V_2^r(\Lambda)| \leq \bar{\Lambda}^{2-j} P \log\left(\frac{\Lambda_R}{\Lambda}\right) \quad (2.13)$$

As for lemmas 1-4 of [1], the method of proof is induction, and the induction scheme is exactly the same as for the proof of lemma 1. Again we assume that the lemma are true up to order  $r - 1$  in the expansion coefficient  $g$ , and that at order  $r$  in  $g$  they true down to  $m + 1$ .

a) We first consider the irrelevant vertices for  $\Lambda \in [\Lambda_R, \Lambda_0]$ . This step is identical to the first step in the proof of lemma 1 in §2.2 of [1], since the flow equation, the lemma and the boundary conditions on the irrelevant vertices are exactly the same as they were there. Thus, we verify i) simply by using the flow equation and integrating from  $\Lambda$  up to  $\Lambda_0$ .

b) When considering the irrelevant vertices for  $\Lambda \in [0, \Lambda_R]$  we have to be more careful about the number of exceptional momenta in external legs. We first consider iii) for  $\Lambda > \Lambda_m$ , with  $e$  odd and  $\leq 2m - 3$ . Integrating from  $\Lambda$  up to  $\Lambda_R$ , taking norms, and using the derived boundary condition obtained by evaluating (2.10) at  $\Lambda_R$  we obtain

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_R}^{E,e} \leq \Lambda_R^{4-2m-j} + \int_{\Lambda}^{\Lambda_R} d\Lambda' \left\| \frac{\partial}{\partial \Lambda'} \left( \partial_p^j V_{2m}^r(\Lambda') \right) \right\|_{\Lambda_R}^{E,e}. \quad (2.14)$$

We are able to bound the integrand in the second term on the right using the bounds we already have on vertices at orders  $r - 1$  and less in  $g$  and vertices of  $m + 1$  or more legs, i.e. those in ii),

iii) and iv). However, before we do this it is necessary to use the lemmas in order to simplify (2.6) a little.

To begin we may use iv) to simplify the terms containing  $V_2^s(\Lambda)$  in (2.6). We first consider  $E \geq \Lambda$ . From ii) we know that for  $p \geq \Lambda_R$ ,  $\partial_p^{j_2} V_2^s(\Lambda) \leq K_{\Lambda_R}^{-1/2}(p) \Lambda_R^{-j_2}$ , which grows less quickly than  $\partial_p^{j_1} K_\Lambda(p)$  falls. Hence, up to the usual multiplicative constant, the maximum value of  $(\Lambda^{-3} |\partial_p^{j_1} K_\Lambda(p_i)| \cdot |\partial_p^{j_2} V_2^s(p_i; \Lambda)|)$  in the range  $p \geq \Lambda_R$  is the value at  $\Lambda_R$ , i.e.  $\Lambda_R^{-1-j_1-j_2}$ . For  $p = E'$  where  $E \leq E' \leq \Lambda_R$ , from iv) we know that  $|\partial_p^{j_2} V_2^s(\Lambda)| \leq (E')^{2-j_2} P \log(\Lambda_R/E')$ , while  $\Lambda^{-3} |\partial_p^{j_1} K_\Lambda(p)| \leq (E')^{-3-j_1}$  as we see from (2.5). So the combined effect is that  $(\Lambda^{-3} |\partial_p^{j_1} K_\Lambda(p_i)| \cdot |\partial_p^{j_2} V_2^s(p_i; \Lambda)|) \leq (E')^{-1-j_1-j_2} P \log(\Lambda_R/E')$  in this range. Therefore, whether  $j_2$  is zero or one, and whatever the value of  $j_1$ , the maximum value of  $(\Lambda^{-3} |\partial_p^{j_1} K_\Lambda(p_i)| \cdot |\partial_p^{j_2} V_2^s(p_i; \Lambda)|)$  for  $p_1 \geq E$  occurs when  $p_1 = E$  and is equal to  $E^{-1-j_1-j_2} P \log(\Lambda_R/E)$ . In this case we have the normal factor of  $E^{-3-j_1}$ , as we do for the other explicit terms with two vertices in (2.6), and the two-point vertex and its first derivative behaves as though it has a bound  $E^{2-j_2} P \log(\Lambda_R/E)$ , i.e. the same type of bound as its higher derivatives. The terms on the right-hand side of (2.6) with the two-point function and its first derivative having exceptional momenta thus behave like the other explicitly written terms with two vertices. For  $E < \Lambda$  the bound on  $(\Lambda^{-3} |\partial_p^{j_1} K_\Lambda(p_i)| \cdot |\partial_p^{j_2} V_2^s(p_i; \Lambda)|)$  is unchanged below  $\Lambda$ , being equal to  $\Lambda^{-1-j_1-j_2} P \log(\Lambda_R/\Lambda)$ , which is larger than that for all  $E' > \Lambda$ . This therefore gives us our maximum, and when  $E \leq \Lambda$  we obtain the normal factor of  $\Lambda^{-3-j_1}$  as the other explicit terms with two vertices in (2.6), i.e. the result for the last term in (2.6) generalizes in the same way as for  $E \geq \Lambda$ . This is clearly also true when both  $E$  and  $\Lambda$  are  $\leq \Lambda_m$ .

We also consider the first term on the right of (2.6), i.e. the bound where the two internal momenta are equal and opposite. Since the growth of the vertices for small momenta depends only on the number of exceptional momenta, it is only the value of the arguments  $\sum_i p_i$  of the vertices that determine this infrared behaviour. In general, a given vertex  $V_{2m+2}^r(p_a, p_b, p_1, \dots, p_{2m}; \Lambda)$  will depend on all possible partial sums of momenta. However, as we have already stated, if two of the momenta sum to zero, i.e.  $p_a = -p_b = p$ , then invariance under permutations of the momenta guarantees there is no dependence on  $(\pm p + p_{i_1} + \dots + p_{i_k})$ . Imposing  $p_a + p_b = 0$  (and thus  $\sum_{i=1}^{2m} p_i = 0$ ), while keeping all subsets non-exceptional, we induce an infrared factor of  $P \log(\frac{\Lambda_R}{\Lambda})$  because we have a sum of two momenta which is exceptional with a value lower than  $\Lambda$ , in fact a value of zero. (Of course, if  $\Lambda \leq \Lambda_m$  we replace  $\Lambda$  by  $\Lambda_m$  in the above.) From iii) this behaviour will not change if we allow  $p$  to become small on its own; we still have only two exceptional momenta. The vertex is now a function of  $\Lambda$ ,  $\Lambda_R$  and  $\{p_1, \dots, p_{2m}\}$  (and  $\Lambda_0$ , to inverse powers, of course), and we can only obtain the further infrared behaviour by letting partial sums of  $\{p_1, \dots, p_{2m}\}$  become small, and this further infrared behaviour must therefore depend only on the value of these partial sums. Thus, if we now let subsets of  $\{p_1, \dots, p_{2m}\}$  become exceptional as low as  $E > \Lambda$ , any new infrared behaviour can only depend on the value of these subsets, i.e. on  $E$ . Similarly, any

derivatives with respect to  $p_i \in \{p_1, \dots, p_{2m}\}$  can only bring about infrared behaviour depending on  $E$ .

We must verify the form of this  $E$ -dependent behaviour. We could consider increasing  $p_a$ ,  $p_b$  and  $p_a + p_b$  to a value  $\sim E$ , clearly having to alter other momenta do so; but only by order  $E$ . It is obviously possible to do this in such a way that the number and type of subsets of  $\{p_1, \dots, p_{2m}\}$  exceptional with respect to  $E$  is unchanged, and such that no partial sum of  $\{p_a, p_b, p_1, \dots, p_{2m}\}$  sums to less than  $E$ . When we set  $p_a = -p_b$ , the terms in the expression  $V_{2m+2}^r(p_a, p_b, p_1, \dots, p_{2m}; \Lambda)$  involving partial sums including some of both  $\{p_a, p_b\}$  and  $\{p_1, \dots, p_{2m}\}$  all reduced to partial sums involving just the  $\{p_1, \dots, p_{2m}\}$ . Removing this restriction on  $p_a$  and  $p_b$ , as described above, we do not qualitatively alter the value of any of these partial sums; if they were exceptional as low as  $E$  then they remain so, if not, then changing their value by  $\sim E$  does not make them so. Similarly, all partial sums involving the  $\{p_1, \dots, p_{2m}\}$  in both cases remain of the same order of magnitude. Thus, all the infrared behaviour which was exhibited by  $V_{2m+2}^r(p, -p, p_1, \dots, p_{2m}; \Lambda)$  must remain qualitatively the same when we make the type of change of momenta outlined above, except that the logarithmic term in  $(\Lambda_R/\Lambda)$  will necessarily become a logarithmic term in  $(\Lambda_R/E)$ . The whole must now satisfy  $\|V_{2m+2}^r(\Lambda')\|_{\Lambda_R}^{E, e+2}$ , and in this way we see that the powerlike  $E$ -dependence of  $V_{2m+2}^r(p, -p, p_1, \dots, p_{2m}; \Lambda)$  is the same as that for  $V_{2m+2}^r(p_a, p_b, p_1, \dots, p_{2m}; \Lambda')$  in the infrared region, and thus, from iii) we must have the bound  $\|V_{2m+2}^r(\Lambda)\|_{\Lambda_R}^{E, e; 0, 2} \leq \Lambda_R^{3+e-2m} E^{-1-e-j} P \log(\frac{\Lambda_R}{\Lambda})$  for odd  $e$  (with obvious alteration for even  $e$ ). Hence, the bound on the first term on the right-hand side of (2.6) will be  $\Lambda \Lambda_R^{3+e-2m} E^{-1-e-j} P \log(\frac{\Lambda_R}{\Lambda})$ .<sup>6</sup> But finally, we see that  $\Lambda P \log(\frac{\Lambda_R}{\Lambda}) \leq E P \log(\frac{\Lambda_R}{E})$  for  $E \geq \Lambda$  and this term may be written as  $\Lambda_R^{3-2m+e} E^{-e-j} P \log(\frac{\Lambda_R}{E})$ .

Substituting the bounds in iii) and ii) and the results derived above into (2.6) and looking at the case where  $E \geq \Lambda'$ , we obtain

$$\begin{aligned} \left\| \frac{\partial}{\partial \Lambda'} \left( \partial_p^j V_{2m}^r(\Lambda') \right) \right\|_{\Lambda_R}^{E, e} &\leq \Lambda_R^{3-2m+e} E^{-e-j} P \log\left(\frac{\Lambda_R}{E}\right) \\ &+ \sum \Lambda_R^{4-2m+e_A+e_B} E^{-1-e_A-e_B-j} P \log\left(\frac{\Lambda_R}{E}\right). \end{aligned} \quad (2.15)$$

The first term comes from the first term on the right-hand side of (2.6), as described above. The second term on the right-hand side of (2.15) comes from those terms on the right of (2.6) explicitly involving two vertices, where the vertex on which resides the original largest exceptional

<sup>6</sup> It would be possible to verify this result in a rather direct manner by deriving bounds for vertices where we let some sets of momenta become as small as one value  $E$  and other sets as small as another value  $\bar{E}$ . The methods used to prove such bounds would be very similar to those used in this paper, but the argument would clearly be rather more complicated. We therefore believe the above explanation is more suitable for this paper.

set we call vertex  $A$ , letting the other be vertex  $B$ , and where the sum is over the possible values of  $e_A + e_B$ , and the maximum value of  $e_A + e_B$  is  $e + 1$ . Since  $e$  is odd, we see that so too is  $e_A$ . However, we are not able to simply substitute  $e_A + e_B = e + 1$  into (2.15) and take this to be the dominant term, because we derived the result  $e_A + e_B = e + 1$  by assuming that the largest exceptional set on vertex  $B$  contained only one momentum. If this is the case then  $e_B = 2m_B - 1$ , where  $2m_B$  is the number of legs on vertex  $B$ . But from iii) the greatest value of  $e_B$  as far as the bounding is concerned is  $2m_B - 3$  (even including the special case  $m_B = 1$  when we consider the result above); all terms including two vertices on the right-hand side of (2.6) contribute the same effective value of  $e_A + e_B = e - 1$  so long as  $e_A + e_B \geq e - 1$ , so the dominant term gives  $e_A + e_B = e - 1$ . Thus both terms on the right-hand side of (2.15) give an equally large contribution and we have

$$\left\| \frac{\partial}{\partial \Lambda'} \left( \partial_p^j V_{2m}^r(\Lambda') \right) \right\|_{\Lambda_R}^{E,e} \leq \Lambda_R^{3-2m+e} E^{-e-j} P \log \left( \frac{\Lambda_R}{E} \right). \quad (2.16)$$

It is relatively easy to convince oneself that the 'other terms' in (2.6), which have a factor of  $\Lambda_R^{-3}$  rather than  $E^{-3}$ , contribute terms the same as those explicitly displayed, but with factors of  $(E/\Lambda_R)$  to a positive power, and may therefore be absorbed into the leading term. If we did not split the exceptional sets we would have a maximum effective value of  $e_1 + e_2 = e - 1$ , as compared to a maximum of  $e_1 + e_2 = e$  if we had split them (actually, only if we split the largest set), and thus have lost a factor of  $(\Lambda_R/E)$ . However, we gain a factor of  $(\Lambda_R/E)^3$ , or more if we have derivatives, from the propagator linking the two vertices.

Considering  $\Lambda' > E$ , as it will be for part of the range of integration, then, as we see from the lemma, the value of  $E$  becomes unimportant; the bound on the vertex is the same for all  $E$  and is as though  $E$  were equal to  $\Lambda'$ . Thus, for  $\Lambda' > E$ , we replace (2.16) by

$$\left\| \frac{\partial}{\partial \Lambda'} \left( \partial_p^j V_{2m}^r(\Lambda') \right) \right\|_{\Lambda_R}^{E,e} \leq \Lambda_R^{3-2m+e} (\Lambda')^{-e-j} P \log \left( \frac{\Lambda_R}{\Lambda'} \right), \quad (2.17)$$

We can therefore evaluate (2.14) by splitting the integral to find

$$\| \partial_p^j V_{2m}^r(\Lambda) \|_{\Lambda_R}^{E,e} \leq \Lambda_R^{4-2m-j} + \Lambda_R^{3-2m+e} E^{1-e-j} P \log \left( \frac{\Lambda_R}{E} \right), \quad (2.18)$$

which verifies iii) for  $\Lambda \in [\Lambda_m, \Lambda_R]$ ,  $e \geq \Lambda$ , and  $e$  odd and  $\leq 2m - 3$ . If  $E < \Lambda$  then we still obtain (2.14), but in this case  $E < \Lambda'$  for the whole range of integration. Thus, we can use the same arguments as those above in order to verify iii) for  $\Lambda \in [\Lambda_m, \Lambda_R]$ ,  $E < \Lambda$ , and  $e$  odd and  $\leq 2m - 3$ .

The extensions to even  $e$  and  $e > 2m - 3$  are now very easy. If  $e$  is even then so is  $e + 2$  and  $e_A$ . Thus, both of the explicitly written contributions on the right-hand side of (2.15), or its equivalent for  $E < \Lambda'$ , will increase their power of  $E$ , or  $\Lambda'$ , by one and, correspondingly, decrease their power of  $\Lambda_R$  by one. It is clear from the rest of the above argument that this increase in the

power of  $E$  and decrease in the power of  $\Lambda_R$  will carry through to the bound on  $\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_R}^{E,e}$ , and thus (2.12) is verified for even  $e$  for  $e \leq 2m - 3$ . If  $e > 2m - 3$  then since  $2m_A = 2m$  at the most, then the effective value of  $e_A$  must decrease by  $e - 2m + 3$ . Again, it is clear that this will carry through to the bound on  $\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_R}^{E,e}$  and consequently the bound on the  $2m$ -point vertex remains the same for  $e > 2m - 3$  as it was for  $e = 2m - 3$ ; exactly the result we want. (If we were to split the largest exceptional set between the vertices, then it is easy to see that the maximum effective value of  $e_1 + e_2$  is unchanged in this case, and such splitting still leads to sub-dominant contributions.) We have therefore now verified iii) for all  $\Lambda \in [\Lambda_m, \Lambda_R]$  and for all  $e$ .

The case  $\Lambda \in [0, \Lambda_m]$  is treated in a similar manner. Considering  $E \geq \Lambda_m$  and odd  $e$  which is  $\leq 2m - 3$ , and integrating from  $\Lambda$  up to  $\Lambda_m$ , taking norms, and using the derived boundary condition obtained by evaluating (2.12) at  $\Lambda_m$  we obtain

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_R}^{E,e} \leq \Lambda_R^{3-2m+e} E^{1-e-j} P \log\left(\frac{\Lambda_R}{E}\right) + \int_{\Lambda}^{\Lambda_m} d\Lambda' \left\| \frac{\partial}{\partial \Lambda'} \left( \partial_p^j V_{2m,e}^r(\Lambda') \right) \right\|_{\Lambda_R, E}. \quad (2.19)$$

This time, since  $\Lambda'$  is always less than  $E$ , we bound the integrand in the second term on the right obtaining (2.15) for all  $\Lambda'$ . We can then easily evaluate the integral in (2.19) to find that the second term on the right is  $(\Lambda_m - \Lambda) \Lambda_R^{2+2e} E^{1-2e-j} P \log\left(\frac{\Lambda_R}{E}\right)$ . This is clearly less than or equal to the bound in iii) for all  $\Lambda$  and  $E$ , and substituting it into (2.19) we immediately verify iii) for  $\Lambda \in [0, \Lambda_m]$ ,  $E \geq \Lambda_m$ ,  $e$  odd and  $\leq 2m - 3$ . If  $E < \Lambda_m$  then the argument is exactly the same once we replace  $E$  by  $\Lambda_m$ . Thus, we easily verify iii) for  $\Lambda \in [0, \Lambda_m]$ ,  $E < \Lambda_m$ ,  $e$  odd and  $\leq 2m - 3$ . For both  $E \geq \Lambda_m$  and  $E < \Lambda_m$  the extension to even  $e$ , or  $e > 2m - 3$  is obvious.

It is now comparatively simple to verify ii) for the irrelevant vertices. Considering any  $\Lambda \in [0, \Lambda_R]$ , integrating from  $\Lambda$  up to  $\Lambda_R$ , taking norms, and using the derived boundary condition obtained by evaluating i) at  $\Lambda_R$  we obtain

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_R} \leq \Lambda_R^{4-2m-j} + \int_{\Lambda}^{\Lambda_R} d\Lambda' \left\| \frac{\partial}{\partial \Lambda'} \left( \partial_p^j V_{2m}^r(\Lambda') \right) \right\|_{\Lambda_R}. \quad (2.20)$$

Again, we are able to bound the integrand in the second term on the right using the bounds we already have on vertices at orders  $r - 1$  and less in  $g$  and vertices of  $m + 1$  or more legs, i.e. those in ii) and iii) (the bound iv) not being needed this time). So substituting the bounds in ii) and iii) into (2.9) and using our result for the first term on the right-hand side we see that the integrand in (2.20) is  $\leq \Lambda_R^{3-2m-j}$  for all  $\Lambda' \in [0, \Lambda_R]$ . Substituting this result into (2.20) and performing the integral it is very easy to see that (2.11) is verified. Thus, we have verified lemma 5 for all irrelevant vertices at order  $r$  in  $g$ .

c) The proof of the lemma for the relevant vertices proceeds in a similar way to the proof of lemma 1 for the relevant vertices in §2.2 in [1]. We first consider the 4-point vertex. Since the momenta at which the renormalization condition for this vertex was set are non-exceptional, we

can use (2.9) and the bounds already obtained for the vertices at lower order in  $g$  or equal order in  $g$ , but with greater  $m$ , to write

$$\left| \frac{\partial}{\partial \Lambda} V_4^r(P_i; \Lambda) \right| \leq \Lambda_R^{-1}, \quad (2.21)$$

and subsequently,

$$|V_4^r(P_i; \Lambda_R) - V_4^r(P_i; 0)| \leq \int_0^{\Lambda_R} d\Lambda' \left| \frac{\partial}{\partial \Lambda'} V_4^r(P_i; \Lambda') \right| \leq c. \quad (2.22)$$

So, using the renormalization condition on  $V_4^r(P_i; 0)$ , we can say that  $|V_4^r(P_i; \Lambda_R)| \leq c$  and we have a bound on the vertex defined at  $\Lambda_R$  for the particular momenta at which the renormalization condition is set. Using Taylor's formula, as in [1], for  $\Lambda = \Lambda_R$  we can verify ii) and iii) (which are identical for  $\Lambda = \Lambda_R$ ) and thus obtain a boundary condition on the vertex at  $\Lambda_R$ ;  $\|V_4^r(\Lambda_R)\| \leq c$ . It is now straightforward to verify ii) and iii) for  $V_4^r(\Lambda)$  for  $\Lambda \in [0, \Lambda_R]$  in exactly the same way as these bounds were verified for the irrelevant vertices. Thus, ii) and iii) are verified for the four-point vertex at order  $r$  in  $g$ .

The verification of ii) and iii) for the second momentum derivative of the two-point vertex is performed in exactly the same manner as for the four-point vertex. The cases of the two-point vertex and its first momentum derivative are not quite so simple due to the fact that they obey a different lemma to all the other vertices if  $\Lambda \leq \Lambda_R$  and  $p \leq \Lambda_R$ . However, iv) may be verified using Taylor's formula in a rather simple manner.

Using the result  $\|\partial_p^2 V_2^r(\Lambda)\|_{\Lambda_R}^{E,1} \leq P \log(\frac{\Lambda_R}{\Lambda})$ , i.e. iii) for the second derivative of the two point function, we immediately see that for all  $p \leq \Lambda_R$  that  $\partial_p^2 V_2^r(p, -p; \Lambda) \leq P \log(\frac{\Lambda_R}{\Lambda})$  for  $p = E \geq \Lambda$  and that  $\partial_p^2 V_2^r(p, -p; \Lambda) \leq P \log(\frac{\Lambda_R}{\Lambda})$  for  $p \leq \Lambda$ . We may construct  $\partial_{p_\mu} V_2^r(p, -p; \Lambda)$  using Taylor's formula;

$$\partial_{p_\mu} V_2^r(p, -p; \Lambda) = p_\nu \int_0^1 d\rho \partial_{k_\nu} \partial_{k_\mu} V_2^r(k, -k; \Lambda), \quad (2.23)$$

where  $k = \rho p$ . Taking the modulus of both sides and adopting the same notation as in Appendix B of [1], i.e. letting  $\bar{p}$  denote any particular component of momenta and noting that the sum over components does not change our qualitative result, we obtain

$$|\partial_{p_\mu} V_2^r(p, -p; \Lambda)| \leq |\bar{p}| \int_0^1 d\rho |\partial_{k_\nu} \partial_{k_\mu} V_2^r(k, -k; \Lambda)|. \quad (2.24)$$

If  $E \leq \Lambda$ , the right-hand side of this inequality is clearly  $\Lambda P \log(\frac{\Lambda_R}{\Lambda})$ . If  $\Lambda \leq E \leq \Lambda_R$  then the right-hand side is  $\leq E \int_0^1 d\rho P \log(\frac{\Lambda_R}{\rho E})$ . The integral over  $\rho$  gives  $P \log(\frac{\Lambda_R}{E})$  and the right-hand side of the inequality is  $E P \log(\frac{\Lambda_R}{E})$ . So, for all  $0 \leq E \leq \Lambda_R$  we have verified iv) for the first momentum derivative of the two-point function at order  $r$  in  $g$  simply by using Taylor's formula (Appendix B of [1] not being required). This is clearly true for  $\Lambda \in [0, \Lambda_m]$  as well as for  $\Lambda \in [\Lambda_m, \Lambda_R]$ .

In order to verify ii) for  $\partial_p V_2^r(\Lambda)$  we must again use Taylor's formula in conjunction with the techniques of Appendix B. In fact, since  $\partial_p V_2^r(\Lambda)$  vanishes at zero momentum simply due to

Lorentz invariance, we can construct it entirely from  $\partial_p^2 V_2^r(\Lambda)$  using Taylor's formula about zero momentum. Doing this for  $\Lambda = \Lambda_R$  we verify ii) for  $\partial_p V_2^r(\Lambda_R)$ . It is then easy to verify ii) for all  $\Lambda \in [0, \Lambda_R]$  in the same way as for  $V_4^r(\Lambda)$  and the irrelevant vertices.

Finally we consider the two-point vertex with no momentum derivatives. We look at  $\Lambda \in [0, \Lambda_R]$  for the vertex at the momentum where the renormalization condition is set. Integrating from  $\Lambda$  down to 0, we obtain

$$|V_2^r(\tilde{P}, -\tilde{P}; \Lambda)| \leq |V_2^r(\tilde{P}, -\tilde{P}; 0)| \leq \int_0^\Lambda d\Lambda' \left| \frac{\partial}{\partial \Lambda'} V_2^r(\tilde{P}, -\tilde{P}; \Lambda') \right|. \quad (2.25)$$

Since this momentum is exceptional, we must use (2.8), where in this case  $E \sim \Lambda_m$ , and the bounds already obtained for the two-point vertex at lower order in  $g$  and the four-point vertex at equal order in  $g$ . The first term on the right of (2.8) is then clearly  $\leq \Lambda P \log(\frac{\Lambda_R}{\Lambda})$ . The maximum value of  $\Lambda^{-3} K'_\Lambda(p_1)$  is  $\leq \Lambda^{-3}$ , and from iv) the maximum values of  $V_2^s(\Lambda)$  and  $V_2^{r-s}(\Lambda)$  are both  $\leq \Lambda^2 P \log(\frac{\Lambda_R}{\Lambda})$ . The second term on the right-hand side of (2.8) is therefore also  $\leq \Lambda P \log(\Lambda_R/\Lambda)$  (or  $\Lambda_m P \log(\Lambda_R/\Lambda_m)$  when  $\Lambda \leq \Lambda_m$ ). Substituting this into (2.25), performing the integral and using the renormalization condition (2.5) on  $V_2^r(\tilde{P}, -\tilde{P}; 0)$ , we find that  $|V_2^r(\tilde{P}, -\tilde{P}; \Lambda)| \leq \Lambda^2 P \log(\frac{\Lambda_R}{\Lambda})$ , and we have a bound on the vertex defined at  $\Lambda$  for the particular momenta at which the renormalization condition is set. We can now verify iv) using the Taylor formula, as for the first derivative of the two-point function, i.e. using the equation

$$|V_2^r(p, -p; \Lambda)| \leq \Lambda^2 P \log\left(\frac{\Lambda_R}{\Lambda}\right) + |\bar{p} + \bar{q}| \int_0^1 d\rho |\partial_k V_2^r(k, -k; \Lambda)|, \quad (2.26)$$

with  $k = q + \rho(p - q)$ . The argument is clearly the same as that used for the first momentum derivative if  $q = \tilde{P} = 0$ , with the additional term from the value of the vertex at  $p = \tilde{P}$  obviously satisfying (2.13). We can see that the argument is essentially unchanged if  $\bar{p} \neq 0$ , since it is only of the order of  $\Lambda_m$ . Thus,  $\Lambda$  or  $E$  in (2.26) will only be replaced with  $\Lambda \pm \Lambda_m$  or  $E \pm \Lambda_m$  respectively, in comparison with the  $\tilde{P} = 0$  case, and these are of order  $\Lambda$  and  $E$ , and act the same way as far as bounds are concerned. iv) is therefore verified for the two-point vertex at order  $r$  in  $g$  for  $\Lambda \in [0, \Lambda_R]$ .

We may verify ii) for the two-point vertex in a straightforward manner. Since  $E P \log(\frac{\Lambda_R}{E}) \leq \Lambda_R$  when  $E \leq \Lambda_R$  we know that  $\|\partial_p V_2^r(p, -p; \Lambda)\|_{\Lambda_R}^{E,1} \leq \Lambda_R$  for all  $p_1$ . We can therefore use Taylor's formula and the result in appendix B of [1] to prove that  $\|V_2^r(\Lambda)\|_{\Lambda_R}^{E,1} \leq \Lambda_R^2$ . This verifies (2.11) for the two-point vertex, and is consistent with, but weaker than, (2.13) for this vertex. Thus, we have now verified ii), iii) and iv) for all the vertices at order  $r$  in  $g$ .

d) Using the derived boundary condition for the relevant couplings at  $\Lambda_R$  for some momenta with magnitude  $\leq \Lambda_R$ , e.g. all momenta zero or momenta equal to those at which the renormalization conditions at  $\Lambda = 0$  were set, we can verify i) for the relevant vertices by integrating the

coupling constants from  $\Lambda \in [\Lambda_R, \Lambda_0]$  down to  $\Lambda_R$  and using Taylor's formula, as in §3.1; working downwards in  $m$  and, for given  $m$ , downwards in number of derivatives. Thus, lemma 5 is verified for the relevant vertices at order  $r$  in  $g$ , and therefore for all vertices at this order. By induction, since it is trivially satisfied at zeroth order in  $g$ , the lemma is therefore true at all orders in  $g$ . ■

In particular, we consider ii), iii) and iv) at  $\Lambda = 0$ . Remembering the direct relationship (1.9) between the vertices defined at  $\Lambda = 0$  and the Green's functions we find that for  $e = 0$

$$\|\partial_p^j \tilde{G}_{2m}^c(p_1, \dots, p_{2m}; \Lambda_0, \lambda_i)\|_{\Lambda_R} \leq \Lambda_R^{4-2m-j}. \quad (2.27)$$

For  $e \geq 1$ , and  $m > 1$  or  $m = 1, j > 1$ ,<sup>7</sup>

$$\|\partial_p^j \tilde{G}_{2m}^c(p_1, \dots, p_{2m}; \Lambda_0, \lambda_i)\|_{\Lambda_R}^{E,e} \leq \begin{cases} \Lambda_R^{4-2m-j} \left(\frac{\Lambda_R}{\Lambda}\right)^{e+j-1} P \log\left(\frac{\Lambda_R}{\Lambda}\right) & e \text{ odd,} \\ \Lambda_R^{4-2m-j} \left(\frac{\Lambda_R}{\Lambda}\right)^{e+j-2} P \log\left(\frac{\Lambda_R}{\Lambda}\right) & e \text{ even.} \end{cases} \quad (2.28)$$

up to a maximum of  $e = 2m - 3$ , where for  $e$  greater than this value (even for  $e = 1, m = 1$ ) the bound is  $\bar{\Lambda}^{4-2m-j} P \log(\frac{\Lambda_R}{\Lambda})$ . For  $e = 1, m = 1$  and  $j < 2$ , we have

$$\max_{p \leq E} |\partial_p^j \tilde{G}_2^c(p, -p; \Lambda_0, \lambda_i)| \leq \bar{\Lambda}^{2-j} P \log\left(\frac{\Lambda_R}{\Lambda}\right). \quad (2.29)$$

In fact we can actually improve the right-hand side of these bounds if we change the way in which we take the norms slightly, and consider the momentum derivatives more carefully. All the norms considered above were entirely general; we never considered which of the momenta became exceptional, only that certain types of sets did. For the vertices themselves this is all it makes sense to do anyway since the vertices are invariant under permutations of momenta due to the Bose symmetry. However, when we differentiate with respect to a particular momentum we break this Bose symmetry, and should thus use a norm which distinguishes between different momenta, taking account of whether the derivatives are with respect to those momenta that are within exceptional sets, or not. It is then possible to show that the bounds (2.28) may be improved by a factor of  $\bar{\Lambda}/\Lambda_R$  for each momentum derivative which acts on a momentum which is not in some irreducible exceptional set.

We could also derive very similar bounds on a theory with no  $Z_2$  symmetry. The bounds for all vertices with no exceptional momenta, or with numbers of legs plus derivatives,  $n + j$ , greater than 3, would be as in equations (2.27) and (2.28), with  $2m + j$  obviously replaced by  $n + j$ . We would again have (2.29) for the two-point function, but would also need the bound

$$\max_{p_1, p_2 \leq E} |\tilde{G}_3^c(p_1, p_2, -p_1 - p_2; \Lambda_0, \lambda_i)| \leq \bar{\Lambda} P \log\left(\frac{\Lambda_R}{\Lambda}\right), \quad (2.30)$$

<sup>7</sup>  $\bar{\Lambda}$  is now effectively  $\max(E, \Lambda_m)$  since  $\Lambda = 0$ .

which can only be obtained by setting the renormalization condition that the three point vertex has magnitude  $\sim \Lambda_m$  for external momenta with magnitude  $\sim \Lambda_m$ . In particular, as  $m \rightarrow 0$  the three point function must be set to be zero for zero external momenta.<sup>8</sup> This is a reflection of the well-known result that infrared finite massless theories are not allowed to have super-renormalizable couplings. However, we see this more clearly as a restriction on the three-point Green's function than one on the bare three-point vertex.

### 2.5. Infrared Finiteness.

In §2.4 we have provided a rather detailed description of the infrared behaviour in the Euclidean region of the theory defined in §2.1, where 'infrared' is taken to mean momenta with magnitudes below those at which the renormalization conditions (other than that on the mass) were set, i.e. below  $\Lambda_R$ . We can now consider the limit  $m \rightarrow 0$  in order to investigate the behaviour of a strictly massless theory. We see immediately that all amputated connected Green's functions at non-exceptional momenta remain finite in this limit to all orders in perturbation theory, and we can also see how quickly the Green's functions may diverge for different combinations of momenta tending to zero.

The proof of convergence in the low mass case is a straightforward combination of the boundedness argument in the previous section and that in §2.3 of [1]. The  $\Lambda_0$ -derivative of the Green's functions away from exceptional momenta is infrared finite at low energies, and is weighted by an extra factor of  $(\frac{\Lambda_R}{\Lambda_0})^P \log(\Lambda_R/\Lambda_0)$ ; as exceptional momenta are approached, the  $\Lambda_0$ -derivative of the Green's functions displays the same degree of infrared divergence as the Green's function itself. Similarly if we were to introduce natural bare irrelevant couplings (as in §3.1 of [1]), the nature of the infrared divergences is unchanged; the massless theory is convergent and universal in just the same way as the massive theory. It can also be systematically improved (see §3.2 of [1]) by setting more low energy renormalization conditions (obviously in a manner consistent with the bounds, and for momenta with magnitudes  $\sim \Lambda_R$ ); the low energy Green's functions will then only change by amounts of order the Green's functions themselves, weighted by further powers of  $(\Lambda_R/\Lambda_0)$ . Thus, we can draw exactly the same conclusions for a massless theory as we did for the massive theory; if we set renormalization conditions on the low energy relevant couplings which are independent of the naturalness scale, then the low energy physics is extremely insensitive to both the value of this naturalness scale and to the irrelevant bare couplings, provided only that the naturalness scale is much larger than the scale of the physics.

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<sup>8</sup> For a conventional quantum field theory this would mean that the massless theory must be  $Z_2$ -symmetric, since all the low energy renormalization conditions maintain this symmetry. In the effective theory we may still have  $Z_2$  breaking terms in the form of irrelevant bare vertices. Hence, due to the insensitivity of the low energy theory to these irrelevant vertices the  $Z_2$  symmetry is only very weakly broken at low energies.

Since the analytic structure of our effective theory allows a well-defined analytic continuation (as discussed in §2.4 of [1]) we may also consider the significance of our results for Minkowski space Green's functions. Using the Landau rules one may readily demonstrate that singularities due to the vanishing of masses only occur due to the momenta associated with internal lines becoming either soft or collinear [7]. (For an accessible discussion of this result see [10].) However, this simple result does not distinguish between internal loop momenta and external momenta, and thus does not tell us whether or not diagrams may in fact be divergent due to internal momenta only becoming soft and/or collinear (indeed in two dimensions we know by experience that they do). In this context, our result proves that in four dimensions Green's functions may not become divergent in the massless limit simply due to internal lines becoming soft and/or collinear, and therefore that any singularities must result from the external momenta becoming soft and/or collinear. The inequality (2.28) then shows us the type of divergences we may (and do) obtain when partial sums of external momenta become soft.

From the Landau rules we also expect (and indeed obtain) singularities on the boundary of the physical region when external momenta become collinear. However, by avoiding these singularities it is possible to perform the same type of analytic continuation as described in §2.4 of [1]. We may thus see that all Green's functions away from the physical singularities are strictly finite, and therefore so are their discontinuities across the cuts (which are traditionally taken to lie along the timelike axis).<sup>9</sup> Also this analytic continuation shows that the Green's functions on the boundary of the physical region are only very weakly dependent on  $\Lambda_0$  and the renormalization conditions on irrelevant operators, for energies much less than  $\Lambda_0$ . For example, if we were to perform an analytic continuation from a Euclidean region where all partial sums of momenta have magnitudes  $\sim \Lambda_R$  and analytically continue to the boundary of the physical region along a path which avoids the singularities given by the Landau rules by a distance  $\sim \Lambda_R$ , we can also see that  $\partial_p^j \tilde{G}_{2m}^{c,r}(p_1, \dots, p_{2m}; \Lambda_0, \lambda_i) \leq \Lambda_R^{4-2m-j}$  for partial sums of momenta with magnitudes of order  $\Lambda_R$  on the boundary of the physical region.

Using the analytic continuation we can also consider the S-matrix. At first sight it appears that this is not strictly well-defined, since there are singularities for processes involving either very soft particles or energetic but collinear particles. Nevertheless, the rather formal argument of [11] proves that for any theory which is unitary for finite mass (as our effective theory is, as was shown in §5.3 of [1]) the transition probabilities, and hence cross-sections, for particle scattering are always finite in the massless limit provided that one sums over degenerate initial and final states containing

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<sup>9</sup> This result for the photon two point function, which is true for all timelike momenta, may be used, for example, to show that the total decay rate for a virtual photon (obtained for example from  $e^+e^-$ -annihilation) into quarks plus arbitrary numbers of soft gluons is finite.

arbitrary numbers of particles.<sup>10</sup> Our bounds on the Green's functions may thus also be applied to S-matrix elements.

### 3. Ultraviolet Behaviour.

#### 3.1. Weinberg's Theorem.

Within the conventional formulation of quantum field theory Weinberg's theorem[5] is a result telling us about the behaviour of Euclidean Green's functions when all the external momenta have magnitudes much greater than the mass of the particles (the 'deep Euclidean' region). It states that if the magnitude of all partial sums of the momenta of a Green's function are of the order  $E$ , then a given amputated Green's function is bounded by a constant times  $E^d$  up to logarithmic corrections, where  $d$  is the power given by naive dimensional analysis ( $d = 4 - 2m$  for our example of the scalar field theory with  $Z_2$  symmetry.) In our case we have a naturalness scale  $\Lambda_0$ , which we take to be finite, and thus we can only hope to show that Weinberg's theorem is true for energy scales up to  $\Lambda_0$ .

In a certain sense, we have already proved this result in the previous section. Setting  $\Lambda_R = E$  in (2.28) we find that the value of the Green's functions (and their momentum derivatives) are bounded by  $E$  to the appropriate power, without even the logarithmic corrections of  $P \log(E/\Lambda_m)$  we might naively expect. We made no restrictions on the value of the scale  $\Lambda_R$ , other than it be greater than  $m$ , so it can span the full range of values for which we would expect Weinberg's theorem to hold. However, since we were interested in the behaviour of the theory in the massless limit we chose to specify that the renormalization conditions on the second momentum derivative of the two-point function and on the four-point function were set for momenta (and partial sums of momenta in the latter case) with magnitudes of order  $\Lambda_R \gg m$ . Here, on the other hand, we are interested not in the massless limit, but in the behaviour of Green's functions for a massive theory in the deep Euclidean region. We thus want to set renormalization conditions for momenta not of order  $E$ , but rather of order  $m$ . In particular, we want to be able to set on-shell renormalization conditions, as described in §2.4 of [1]. The arguments of §2 must then be altered in order to prove Weinberg's theorem.

We define our theory in the same way as in §2.1 of [1], i.e. in exactly the same way as in §2.1 above except for the renormalization conditions (2.5) on  $[\partial_{p_\mu} \partial_{p_\nu} V_2^r(p, -p; 0)|_{p=P_0}]_{\delta\mu\nu}$  and

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<sup>10</sup> A rather less formal proof of this theorem, using cut diagrams, is presented in [10]. Since this proof proceeds from the cutting relations, we may adapt it to our effective theory by employing the 'unitary representation' discussed in §5.3 of [1], in which the propagator assumes its usual unregulated form. Then when the cut lines go on-shell, the regulating functions at the ends of these lines are set to unity, and thus their essential singularity presents no obstruction to the proof.

$V_4^r(P_1, P_2, P_3, P_4; 0)$ , which are now set with the  $P_i$  having all having magnitudes of order  $\sim m$ . We also define a scale  $\Lambda_H$ , which is the scale at which we wish to investigate the physics ( $\Lambda_R$  is the scale at which the renormalization conditions are set, so in this section  $\Lambda_R \equiv \Lambda_m$ ). Exceptional momenta are then defined in exactly the same way as in §2.2 (once we replace  $\Lambda_R$  by  $\Lambda_H$ ), as are the restricted norms. This means that the bounded flow equation for  $\Lambda \in [\Lambda_H, \Lambda_0]$  is the same as (1.12) in §1, while the bounded flow equations for  $\Lambda \in [\Lambda_m, \Lambda_H]$  and  $\Lambda \in [0, \Lambda_m]$  are the same as (2.6), (2.8) and (2.9) in §2.3. If we were to suggest that our lemma for the bounds on the vertices was identical to lemma 5 however, we would find that the proof would break down as soon as we tried to prove the lemma for the relevant vertices (the first obstruction would be that (2.21) would be no longer true if the  $P_i$  had magnitudes of order  $\Lambda_m$ ). We must thus set up a new lemma, fully expecting there to be logarithmic corrections to the bounds, even when  $E = \Lambda_H$ : we thus propose

*Lemma 6:*

i) For all  $\Lambda \in [\Lambda_H, \Lambda_0]$ ,

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_\Lambda \leq \Lambda^{4-2m-j} \left( P \log\left(\frac{\Lambda}{\Lambda_m}\right) + \frac{\Lambda}{\Lambda_0} P \log\left(\frac{\Lambda_0}{\Lambda_m}\right) \right). \quad (3.1)$$

ii) For all  $\Lambda \in [0, \Lambda_H]$ , and for  $e = 0$ , we have

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_H} \leq \Lambda_H^{4-2m-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right). \quad (3.2)$$

iii) For all  $\Lambda \in [0, \Lambda_H]$  and for  $1 \leq e \leq 2m - 3$ ,  $m > 1$ , or  $m = 1$ ,  $j > 1$

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_H}^{E,e} \leq \begin{cases} \Lambda_H^{4-2m-j} \left(\frac{\Lambda_H}{\Lambda}\right)^{e+j-1} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right) & e \text{ odd,} \\ \Lambda_R^{4-2m-j} \left(\frac{\Lambda_H}{\Lambda}\right)^{e+j-2} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right) & e \text{ even,} \end{cases} \quad (3.3)$$

where  $\bar{\Lambda} = \max(E, \Lambda, \Lambda_m)$ . For  $e > 2m - 3$  the bound is simply  $\bar{\Lambda}^{4-2m-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right)$ .<sup>11</sup>

iv) The case  $\Lambda \in [0, \Lambda_H]$  and  $e = 1$ ,  $m = 1$ ,  $j = 0, 1$  is a little different, as in lemma 5. Again, we do not have a restricted norm, but an inequality similar to (2.7),

$$\max_{p \leq E} |\partial_p^j V_2^r(\Lambda)| \leq \bar{\Lambda}^{2-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right). \quad (3.4)$$

<sup>11</sup> We notice that the bound iii) for the case when  $e = 2m - 1$ , and thus when all momenta may become small individually is greater than the bound obtained in [1](lemma 1) for all momenta being of order  $m$ , i.e.  $\|\partial_p^j V_{2m}^r(\Lambda)\| \leq m^{4-2m-j}$  for all  $p_i \sim m$ ,  $\Lambda \leq m$ . This is because the bound in lemma 6 is for the case when all momenta ‘may’, but need not, become small, so the bounds cannot decrease as  $e$  increases. The fact that we have the bound of  $\Lambda_m^{4-2m-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right)$  for  $e = 2m - 3$  means then that the bound for  $e = 2m - 1$  must be at least as large. After proving the bounds in lemma 6 it is possible to go back and investigate the case where all momenta actually are of order  $m$  with an equation similar to (2.6), and verify the expected result. (The bound over all momenta being needed first because of the one term on the right-hand side with an integral over all momenta.) We leave this verification to the more dedicated reader.

Using equations (2.6), (2.8) and (2.9) we may set about proving lemma 6. As for all previous lemmas the proof is by induction, and the induction scheme is the same as for lemma 1. We assume that the lemma is true up to order  $r - 1$  in the expansion coefficient  $g$ , and that at first order in  $g$  it is true down to  $m + 1$ .

a) We first consider the irrelevant vertices for  $\Lambda \in [\Lambda_H, \Lambda_0]$ . Just as in §2.4a, the flow equation and the boundary conditions on the irrelevant vertices are exactly the same as they were for lemma 1 in [1]. The bound is also the same except for the fact that  $\Lambda_R$  is replaced by  $\Lambda_m$ . It is easy to see that this does not change the argument used in the proof of lemma 1 as long as  $\Lambda_H \geq \Lambda_m$ . Thus, we may verify i) for  $\Lambda$  in this range simply by using the flow equation and integrating from  $\Lambda$  up to  $\Lambda_0$ .

b) The step for the irrelevant vertices with  $\Lambda \in [\Lambda_m, \Lambda_H]$  and  $\Lambda \in [0, \Lambda_m]$  is very similar to that in §2.4b. We first consider iii) for  $\Lambda \in [\Lambda, \Lambda_H]$  and  $E \geq \Lambda$ , with  $e$  odd and  $\leq 2m - 3$ . Integrating from  $\Lambda$  up to  $\Lambda_H$ , taking norms, and using the derived boundary condition obtained by evaluating (3.1) at  $\Lambda_H$  we obtain

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_H}^{E,e} \leq \Lambda_H^{4-2m-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right) + \int_{\Lambda}^{\Lambda_H} d\Lambda' \left\| \frac{\partial}{\partial \Lambda'} \left( \partial_p^j V_{2m}^r(\Lambda') \right) \right\|_{\Lambda_H}^{E,e}. \quad (3.5)$$

We are now able to bound the integrand in the second term on the right using the bounds in ii), iii) and iv). Using iv) we can now simplify the last set of terms in (2.6) in a similar way as we did in §2.4b. We thus find that

$$\max_{p_i \geq E} (\Lambda^{-3} |\partial^{j_1} K_{\Lambda}(p_i)| \cdot |\partial_p^{j_2} V_2^*(p_i; \Lambda)|) \leq \bar{\Lambda}^{-1-j_1-j_2} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right). \quad (3.6)$$

So, once again the term on the right-hand side of (2.6) containing the two-point function and its first derivative with exceptional momentum effectively behaves like the other explicitly written terms with two vertices.

So, substituting the bounds in iii) and ii) and the result derived above into (2.6) both when  $E \geq \Lambda'$  and  $E\Lambda'$ , and realizing that the power counting for  $E$  works in exactly the same way as previously, we obtain

$$\|\partial_p^j V_{2m}^r(\Lambda)\|_{\Lambda_H}^{E,e} \leq \Lambda_H^{4-2m-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right) + \Lambda_H^{3-2m+e} \left( \int_{\Lambda}^E d\Lambda' E^{-e-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right) + \int_E^{\Lambda_H} d\Lambda' (\Lambda')^{-e-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right) \right). \quad (3.7)$$

Again, the other terms are sub-dominant. Both integrals are trivial to evaluate, and we immediately verify (3.3) for  $\Lambda \in [\Lambda_m, \Lambda_H]$  and  $e$  odd and  $\leq 2m - 3$ . The extension to  $E < \Lambda$  and then to even  $e$

and  $e > 2m - 3$  is exactly as in §2.4b. The case  $\Lambda \in [0, \Lambda_m]$  is also treated in an exactly analogous manner.

It is then very easy to verify ii) for the irrelevant vertices, using the same method as was used for the  $e = 0$  irrelevant vertices in §2.4b. In fact, it is even more straightforward than this argument because we do not have to eliminate the logarithm. We have therefore verified lemma 6 for all irrelevant vertices at order  $r$  in  $g$ .

c) The proof of the lemma for the relevant vertices is again very similar to that in §2.4c, and is, if anything, easier. Again, we first consider the 4-point vertex. This time the momenta at which the renormalization condition for this vertex was set are exceptional as low as  $\Lambda \sim \Lambda_m$ . We therefore use (2.6) and the bounds already obtained for the vertices at lower order in  $g$  or equal order in  $g$ , but with greater  $m$ , to write

$$\left| \frac{\partial}{\partial \Lambda} V_4^r(P_i; \Lambda) \right| \leq \Lambda^{-1} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right), \quad (3.8)$$

for  $\Lambda \in [\Lambda_m, \Lambda_H]$ , and the same if we replace  $\Lambda^{-1}$  by  $\Lambda_m^{-1}$  for  $\Lambda \in [0, \Lambda_m]$ . But,

$$|V_4^r(P_i; \Lambda_H)| \leq |V_4^r(P_i; 0)| + \int_0^{\Lambda_H} d\Lambda' \left| \frac{\partial}{\partial \Lambda'} V_4^r(P_i; \Lambda') \right|. \quad (3.9)$$

Using the renormalization condition on  $V_4^r(P_i; 0)$ , and splitting the integral into two, one from  $\Lambda = 0$  to  $\Lambda_m$  and the other from  $\Lambda_m$  to  $\Lambda_H$ , we easily obtain  $|V_4^r(P_i; \Lambda_H)| \leq P \log(\frac{\Lambda_H}{\Lambda_m})$ , and thus have a bound on the vertex defined at  $\Lambda_H$  for the particular momenta at which the renormalization condition is set. Using Taylor's formula for  $\Lambda = \Lambda_H$  we can verify (3.2) and (3.3) for  $\Lambda = \Lambda_H$ , and obtain a boundary condition on the vertex at  $\Lambda_H$ ;  $\|V_4^r(\Lambda_H)\| \leq P \log(\frac{\Lambda_H}{\Lambda_m})$ . It is then straightforward to verify ii) and iii) for  $V_4^r(\Lambda)$  for  $\Lambda \in [\Lambda_m, \Lambda_H]$  and  $\Lambda \in [0, \Lambda_m]$  in exactly the same way as these bounds were verified for the irrelevant vertices. Thus, ii) and iii) are verified for the four-point vertex at order  $r$  in  $g$ .

The verification of ii) and iii) for the second momentum derivative of the two-point vertex is performed in exactly the same manner as for the four-point vertex. As in §2.4c, the cases of the two-point vertex and its first momentum derivative are not quite so simple because they obey a different type of bound to all the other vertices if  $\Lambda \leq \Lambda_H$  and  $p \leq \Lambda_H$ . However, these bounds may be verified in a similar, but more straightforward, way as that which we used to verify iv) in §2.4.

Using the result iii) we see that for all  $p \leq \Lambda_H$  that  $|\partial_p^2 V_2^r(p, -p; \Lambda)| \leq P \log(\frac{\Lambda_H}{\Lambda_m})$ . We may construct  $\partial_{p_\mu} V_2^r(p, -p; \Lambda)$  for  $\Lambda \in [0, \Lambda_H]$  using Taylor's formula. Taking the modulus of both sides and again adopting the notation in Appendix B of [1] and letting  $k = \rho p$ , we obtain

$$|\partial_{p_\mu} V_2^r(p, -p; \Lambda)| \leq |\bar{p}| \int_0^1 d\rho |\partial_{\bar{k}} \partial_{k_\mu} V_2^r(k, -k; \Lambda)|. \quad (3.10)$$

If  $p = E$ , the right-hand side of this inequality is clearly  $EP \log(\frac{\Lambda_H}{\Lambda_m})$ , and we have verified iv). In order to verify ii) for  $\|\partial_p V_2^r(p_1; \Lambda)\|_{\Lambda_H}$  we can use the result that  $\|\partial_j^2 V_2^r(p, -p; \Lambda)\|_{\Lambda_H}^{E,1} \leq P \log(\frac{\Lambda_H}{\Lambda_m})$  for  $E$  as low as zero, in conjunction with Taylor's formula and the result in appendix B of [1] to show that  $\|\partial_p V_2^r(p, -p; \Lambda)\|_{\Lambda_H}^{E,1} \leq \Lambda_H P \log(\frac{\Lambda_H}{\Lambda_m})$  for  $E$  as low as zero. This clearly verifies ii) and is consistent with iv). Thus, ii) is verified for  $\partial_p V_{2m}^r(\Lambda)$  at order  $r$  in  $g$ .

Finally we consider the two-point vertex with no momentum derivatives. The proof of iv) for the two-point vertex is practically identical to that of iv) in lemma 5. Again we use (2.8) instead of (2.6), and the argument is exactly the same. iv) is therefore verified for the two-point vertex at order  $r$  in  $g$  for  $\Lambda \in [0, \Lambda_H]$ . The verification of ii) for the two-point vertex is, again, the same as the corresponding verification in the last section, and ii), iii) and iv) are verified for all vertices at order  $r$  in  $g$ .

d) Using the derived boundary conditions for the relevant couplings at  $\Lambda_H$  for momenta with magnitude  $\leq \Lambda_H$ , we can verify i) for the relevant vertices by integrating the coupling constants from  $\Lambda \in [\Lambda_H, \Lambda_0]$  down to  $\Lambda_H$  and using Taylor's formula, as in [1]; as always, working downwards in  $m$  and, for given  $m$ , downwards in number of derivatives. Thus, lemma 6 is verified for the relevant vertices at order  $r$  in  $g$ , and therefore for all vertices at this order. By induction, since it is trivially satisfied at zeroth order in  $g$ , the lemma is therefore true at all orders in  $g$ . ■

As far as the proof of Weinberg's theorem is concerned we are only interested in (3.2) evaluated at  $\Lambda = 0$ . This tells us that for  $e = 0$ , so that all momenta and their partial sums have magnitudes greater than  $\Lambda_H$ , we have

$$\|\partial_p^j \tilde{G}_{2m}^{c,r}(p_1, \dots, p_{2m}; \Lambda_0, \lambda_i)\|_{\Lambda_H} \leq \Lambda_H^{4-2m-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right). \quad (3.11)$$

For all the  $p_i$  with magnitudes of order  $\Lambda_H$  the damping factors of  $K_{\Lambda_H}^{1/4}(p_i)$  are all of order one and we see that

$$\partial_p^j \tilde{G}_{2m}^{c,r}(p_1, \dots, p_{2m}; \Lambda_0, \lambda_i) \leq \Lambda_H^{4-2m-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right), \quad (3.12)$$

and we have proved Weinberg's theorem to all orders in perturbation theory for Euclidean momenta with magnitudes of order  $\Lambda_H$ , not only for the amputated connected Green's functions, but also for all their momentum derivatives.

In this section  $\Lambda_H$  has no particular relevance, in the sense that it is not linked to the renormalization conditions in any way. The only requirement satisfied by  $\Lambda_H$  is that it be  $\geq m$  and  $\leq \Lambda_0$ . Thus, we have proved Weinberg's theorem for all momenta in this range. Also, writing (3.3) and (3.4) evaluated at  $\Lambda = 0$  we obtain far more comprehensive bounds for the magnitudes of Green's functions and their derivatives for momenta with magnitudes above the mass of the particle, but with various 'exceptional' partial sums with magnitudes below  $\Lambda_H$ : for  $1 \leq e \leq 2m - 2$ ,  $m \geq 2$ ,

$$\|\partial_p^j \tilde{G}_{2m}^c(p_1, \dots, p_{2m}; \Lambda_0, \lambda_i)\|_{\Lambda_H}^{E,e} \leq \begin{cases} \Lambda_H^{4-2m-j} \left(\frac{\Lambda_H}{\Lambda}\right)^{e+j-1} P \log\left(\frac{\Lambda_H}{\Lambda}\right) & e \text{ odd,} \\ \Lambda_H^{4-2m-j} \left(\frac{\Lambda_H}{\Lambda}\right)^{e+j-2} P \log\left(\frac{\Lambda_H}{\Lambda}\right) & e \text{ even,} \end{cases} \quad (3.13)$$

where  $\bar{\Lambda} = \max(E, \Lambda_m)$ ; for  $e = 2m - 1$ , the bound is  $\bar{\Lambda}^{4-2m-j} P \log(\frac{\Lambda_H}{\bar{\Lambda}})$ . In the same way as described at the end of §2.4 we can improve the bounds on the momentum derivatives by extra factors of  $\bar{\Lambda}/\Lambda_H$  if some of them act only on unexceptional momenta.

### 3.2. Systematic Improvement.

If we were now to take the conventional view of quantum field theory and take  $\Lambda_0$  to infinity we would recover the conventional form of Weinberg's theorem. If instead we consider keeping  $\Lambda_0$  finite, it is easy to verify that the bounds (3.11) and (3.13) are not changed if we introduce more general boundary conditions on the irrelevant vertices at  $\Lambda_0$ , as long as they satisfy the conditions outlined in §3.1 of [1]. Indeed, combining the techniques of this section with those in §2.3 and §3.1 of [1], we can, with a little effort, but no further insight, prove the result analogous to lemma 4 for Green's functions with external momenta with magnitudes greater than  $\Lambda_R$ . Assuming that all operators with dimension less than or equal to  $D$  have couplings which are physically relevant, and thus set on Green's functions with momenta of order  $m$ , then for  $e=0$  we can show that

*Lemma 7:*

$$\left\| \left( \frac{\partial}{\partial \Lambda_0} \partial_p^j \tilde{G}_{2m}^c(\Lambda_0, \lambda_i) \right)_{\lambda_i} \right\|_{\Lambda_H} \leq \Lambda_0^{2-D} \Lambda_H^{D+1-2m-j} P \log\left(\frac{\Lambda_0}{\Lambda_H}\right) P \log\left(\frac{\Lambda_H}{\Lambda_m}\right), \quad (3.14)$$

and

$$\left\| \partial_p^j (\tilde{G}_{2m}^c(\Lambda_0, \lambda_i) - \hat{G}_{2m}^c(\Lambda_0, \lambda_i)) \right\| \leq \Lambda_0^{3-D} \Lambda_H^{D+1-2m-j} P \log\left(\frac{\Lambda_0}{\Lambda_H}\right) P \log\left(\frac{\Lambda_H}{\Lambda_m}\right), \quad (3.15)$$

where  $\hat{G}_{2m}^c(\Lambda_0, \lambda_i)$  are the amputated connected Green's functions for the theory with different boundary conditions on irrelevant couplings at  $\Lambda_0$ .

The results for  $e > 0$  are similar: to find the bound on the  $\Lambda_0$ -derivative of a theory where we have set low energy renormalization conditions on all coupling constants corresponding to operators with canonical dimension up to  $D$ , and set all undetermined couplings at  $\lambda_0$  equal to zero, we multiply the right-hand side of (3.13) by  $\Lambda_0^{-1} (\frac{\Lambda_H}{\Lambda_0})^{D-3} P \log(\frac{\Lambda_H}{\Lambda_m})$ ; to find the bound on the difference between two theories, one as described above, the other with the same low energy renormalization conditions and with undetermined couplings at  $\Lambda_0$  all natural, we multiply the right-hand side of (3.13) by  $(\frac{\Lambda_H}{\Lambda_0})^{D-3} P \log(\frac{\Lambda_H}{\Lambda_m})$ . In this we can justify the results quoted in §3.2 of [1]; we may indeed maintain the predictive power of an effective theory as we consider processes of higher and higher energies by specifying more and more low energy renormalization conditions.

Since the effective theory makes sense for arbitrarily large external momenta, we can also consider Green's functions with Euclidean momenta with magnitudes above  $\Lambda_0$ , even though these will require the experimental determination of an infinite number of couplings to specify them completely. Here however the Weinberg bounds will generally break down; the best we can do is

use (3.11) in the limit  $\Lambda_H \rightarrow \Lambda_0$  to show is that for each momentum, and all partial sums, having magnitude greater than or equal to  $\Lambda_0$  then

$$\partial_p^j \tilde{G}_{2m}^{c,r}(p_1, \dots, p_{2m}; \Lambda_0, \lambda_i) \leq \Lambda_0^{4-2m-j} P \log\left(\frac{\Lambda_0}{\Lambda_m}\right) K_{\Lambda_0}^{-1/4}(p_1) \cdots K_{\Lambda_0}^{-1/4}(p_{2m}) : \quad (3.16)$$

the Green's functions are bounded in the same way as the vertex functions defined at  $\Lambda_0$ . Clearly for these very large momenta the Green's functions should indeed depend strongly on the form of the vertices at  $\Lambda_0$ , and we would need more details to provide a stronger bound than that above. However, it is for momenta with magnitudes below  $\Lambda_0$  that our theory is most 'effective', and we have been able to show that below this naturalness scale Weinberg's theorem is true for all effective scalar field theories.

By analytically continuing the Euclidean Green's functions we can also obtain loose bounds on the amputated connected Green's functions on the boundary of the physical region. By making a continuation which avoids any region of exceptional momenta we would naively expect to obtain the bound

$$|\partial_p^j \tilde{G}_{2m}^{c,r}(p_1, \dots, p_{2m}; \Lambda_0, \lambda_i)| \lesssim \Lambda_H^{4-2m-j} P \log\left(\frac{\Lambda_H}{\Lambda_m}\right). \quad (3.17)$$

In general this will be correct. However, the bounds may be violated if we get too close to physical singularities, such as poles or normal branch points, where the connected Green's functions may grow very large. Even in these cases there should be no change in the dependence on  $\Lambda_0$  and the boundary conditions on irrelevant operators there; this dependence will simply be the size of the Green's function multiplied by the same factors of  $(\Lambda_H/\Lambda_0)$  and  $P \log(\Lambda_H/\Lambda_0)$  as in (3.14) and (3.15).

#### 4. Conclusions.

By exploiting the exact renormalization group we have achieved two great savings compared to the normal way of investigating infrared behaviour. Firstly we have not had to worry at all about the ultraviolet behaviour of the theory when examining the infrared behaviour<sup>12</sup>, since the two energy scales are effectively separated. Secondly, even when looking at low energies, we only have to consider the topological properties of diagrams containing either one vertex and one loop or two vertices and no loops, rather than those of arbitrarily complicated graphs. So, as promised in the introduction, the exact renormalization group makes the investigation of the scale dependence of Green's functions much more simple than conventional methods.

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<sup>12</sup> In particular, we note that we have a non-zero mass term in the bare Lagrangian. Usually arguments for the infrared behaviour of field theories assume mass counterterms are absent. If they are present they must be finely tuned in order that the infrared behaviour for non-exceptional momenta not be spoiled. In our case this fine tuning is achieved simply by setting the required low energy renormalization condition on the mass.

One may feel, however, that the description of the infrared behaviour of a quantum field theory is not particularly interesting for a purely scalar theory, since the massless case is simply due to a rather extreme fine-tuning of the renormalization condition on the mass. However the results of this paper will become very useful when considering the extension of the methods in [1] to an examination of the renormalizability of effective gauge theories in later articles, as well as to the theories containing particles of significantly different masses discussed in an accompanying paper[4].

Finally, it is important to realize that we have proven Weinberg's theorem rather late in the day, and in particular after the boundedness, convergence, universality, unitarity and causality of the effective theory have already been established (in [1]). It was thus clearly not necessary to know anything of Weinberg's theorem in order to perform these other proofs. Thus from the point of view of the exact renormalization group Weinberg's theorem is no longer to be regarded as an essential ingredient when proving formal results in quantum field theory<sup>13</sup>; rather it should be regarded as a powerful constraint on the complete amputated connected Green's functions in the deep Euclidean region, as well as a good indicator, when combined with the Landau rules, of the high energy behaviour of S-matrix elements.

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<sup>13</sup> In fact, we do not even prove it for individual Feynman diagrams, only for the complete amputated connected Green's functions.

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