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Abstract

This paper is concerned with the development of a quantitative analysis method that can extract the mixed linear and nonlinear dynamical response of a thermoviscoelastic process in complex materials. A tractable set of simultaneous equations with well behaved coefficients can be generated by taking time series moments of a suitably truncated Volterra series expansion. This moment hierarchy is a set of inhomogeneous nonlinear integral equations, based on a vector multidimensional convolution form of the Volterra functional series expansion. The hierarchy developed is used to analyse the time dependent thermoviscoelastic properties of resin matrix composite materials. Estimates of the temporal response of the measured deformation gradient to the measured mechanical and thermal forces were then used to predict the out of sample stress field values. These predictions demonstrated that the response functions provided a good, locally time invariant, representation of the thermoviscoelastic process. The range of applied loads span a significant region of the phase space for the specimen and the estimated response function values and steady state transport coefficients remained constant over this range. Generally speaking, the response functions estimated from the data are used to determine the dynamic and steady state transport coefficients, which can, be used to develop either an empirical field theory of the phenomena or alternatively be used in the design process.

Introduction

Many materials have viscoelastic properties that are both nonlinear and time dependent. Any complete theory of thermoviscoelasticity should be able to describe the local deformation gradient in terms of the properties of the material and the forces acting on the material. The thermomechanical properties in a local region should be characterised as a multidimensional function of the physical observables that effect that region.

The main difficulty in characterising thermoviscoelastic behaviour, from an experimental point of view, is the accurate simultaneous measurement of all the variables needed to describe the process. A guide to the choice of observables in an experiment, is that the governing equation used should be in a closed form, for example, the conservation laws in a control volume should contain terms for all of the physical processes which significantly contribute to the process. The characterisation obtained from the data can then be related to the theory underpinning the process. Generally speaking, the ability of current data analysis methods to accurately and consistently quantify all of these interactions under general dynamic boundary conditions is severely limited. Thus, there is a need to develop and refine data analysis techniques that can separate and quantify the thermoviscoelastic processes and their interactions and relate them to an appropriate theoretical description of the process.

The notion of inverse equations and their approximate solution by discretisation is commonly employed for linear ordinary differential equations. A coupled differential representation being developed, which is suitable for the analysis of the thermoviscoelastic process, will be discussed elsewhere. In this work a vector multidimensional convolution form of the Volterra series expansion, suitably truncated, is operated on to obtain a tractable moment hierarchy with well behaved coefficients, from which the dynamical response of the thermoviscoelastic process can be determined. The solution of the moment hierarchy yields, directly, the Volterra kernel values. These kernel functions are usually known as the linear and nonlinear response functions and are fundamental properties of the physical system being studied. The convolution form of representation is an extension of the Taylor's expansion to processes which posses a finite memory. The vector multidimensional convolution form of the Volterra series is used in the present paper to analyse experimental data for the mixed linear and nonlinear properties of complex thermoviscoelastic behaviour. The response function values can be extracted at different times of the sample's life cycle and they represent elements of the life history of the thermoviscoelastic material.

There have been previous attempts to develop multiple integral theories of viscoelasticity. Several viscoelastic theories have used the Volterra functionals to represent the relationship between the components of the stress field $\{\sigma_{ij}(t)\}$ and the strain field $\{\varepsilon_{ij}(t)\}$; where the component of stress, $\sigma_{ij}(t)$, is considered to be the i th component of the force per unit area on the surface acting in a direction e_j which is a unit normal and where $\varepsilon_{ij}(t)$ are the components of the local deformation gradient ,$\{\nabla x(t)\}$, the components being $\dfrac{\partial x_i(t)}{\partial X_j(t)}$.

The most comprehensive of these theories is by Green and Rivilin [1], who used a tensor form of the Volterra functional series to develop a three dimensional nonlinear viscoelastic theory. Their theory is based in the assumption that the observed stress field $\{\sigma_{ij}(t)\}$, is a nonlinear function of the history of observed deformation gradient $\{\nabla x(t)\}$. The values of the stress components $\{\sigma_{ij}(t)\}$ can be expressed to any desired degree of approximation as an ascending series of convolution functions with [1]

$$\sigma_{ij}(t) = \sum_{n=1}^{N}\frac{1}{n!}\int_0^t d\tau_1 \ldots \int_0^t d\tau_n E_{ijp_1q_1\cdots p_nq_n}(\tau_1,\ldots,\tau_n)\prod_{k=1}^{n} g_{p_kq_k}(t-\tau_k) \qquad (1)$$

where τ_k denotes delay with respect to the present time t, and where

$g_{p_kq_k}(t-\tau_k)=\sum_{m=1}^{9}\dfrac{\partial x_m(\tau_k)}{\partial X_{p_k}(\tau_k)}\dfrac{\partial x_m(\tau_k)}{\partial X_{q_k}(\tau_k)}$ is the inner product of the deformation vector with itself, at the time $(t-\tau_k)$.

The kernels of the convolution expansion represent the dynamic response functions between the stress and strain fields, and they are called the dynamic relaxation modulus functions. The area under these response functions is the steady state gain between the components [2], and are called the relaxation modulus for each pair of components.

The Volterra series representation is well conditioned for a wide range of loading functions. However, Gradowczyk [3] indicated that the Volterra series is ill conditioned under the step loading case. This result is not surprising, as it has long been recognised that the classical test loadings of the step, the single impulse and the single harmonic driving force are not appropriate for nonlinear processes [4] and render, for example, the Volterra series and its transformations ill conditioned. Indeed, it was this fact that prompted Weiner [4] to propose that Gaussian white noise could be an appropriate loading function for nonlinear systems.

Unfortunately, it can be trivially shown that a Gaussian white noise loading function produces an ill conditioned Volterra series when the system is of a mixed order higher than quadratic. This is perhaps why the Weiner school developed the homogeneous approximation method which uses Gaussian white noise [5], as that method only used the diagonal terms in Gradowczyk's matrix expression. The objective of the homogeneous approximation is to obtain a local single order radial basis transformation type of description which adequately describes the observed behaviour.

Pipkin and Rogers [6] studied a three dimensional Stieltjes form of the Volterra series expansion and represented the components of the stress field, $\{\sigma_{ij}(t)\}$, as a functional expansion with

$$\sigma_{ij}(t) = \sum_{n=1}^{N} \frac{1}{n!} \int_0^t d\varepsilon(\tau_1) \ldots \int_0^t d\varepsilon(\tau_n) R_{ijp_1q_1\cdots p_nq_n}\left(\varepsilon(\tau_1),\ldots,\varepsilon(\tau_n)\right) \qquad (2)$$

where the differentials

$$d\varepsilon(\tau_1)\ldots d\varepsilon(\tau_n) R_{ijp_1q_1\cdots p_nq_n}\left(\varepsilon(\tau_1),\ldots,\varepsilon(\tau_n)\right) = \frac{\partial^n R_{ijp_1q_1\cdots p_nq_n}\left(\varepsilon(\tau_1),\ldots,\varepsilon(\tau_n)\right)}{\partial\varepsilon(\tau_1)\ldots\partial\varepsilon(\tau_n)} \prod_{k=1}^{n} \frac{\partial\varepsilon(\tau_k)}{\partial\tau_k} d\tau_k$$

have the properties of a characteristic function [7].

Pipkin and Rogers undertook a detailed examination of the one dimensional case of equation (2) under an incremental step loading scheme. Although reasonable results were obtained, it was later noticed that equation (2) is ill conditioned under the loading regime used, in the same way that the Green and Rivilin expansion is when a step loading is used.

If local solutions are required then the Volterra series can be truncated to just the first (linear) term. The first term approximation has been used by a series of workers, notably Schapery [8], to develop an approximate constitutive theory for composite materials and then to apply their method to a range of material types with generally reasonable agreement between theory and experiment being observed.

In the Schapery approach, the components of the stress field are related to the temporal differentials of the time series history of deformation gradients. Experimental observations are, by nature, uncertain and their time series are stochastic processes. Most stochastic processes do not posses differentials in the ordinary sense [9], however, a few stochastic processes have differential properties in the mean square, or higher order moment, sense. This indicates that the fundamental relationships between any observed physical quantities should be developed in terms of their time series averaged or convoluted values, and not in terms of the derivatives of the time series values.

More recent work in nonlinear elastic dynamics has concentrated on functional analysis and the solution of nonlinear differential equations [10,11]. The bifurcation and chaotic theories used to describe the nonlinear elastic behaviour do not take into account the fading memory properties [12] of viscoelastic processes and consequently will not be considered further in the present work.

The multidimensional convolution representation developed in the present work relates the components of the observed deformation gradient, $\{\varepsilon_{ij}(t)\}$, to the applied mechanical and local thermal forces and is readily extendible to more complex situations. The formalism assumes that a causal relation exists between the deformation gradients, the forces acting on the body and the fluxes flowing through the body; whether they be mechanical, electrical, thermal or chemical in origin. This indicates that each experimental case should be examined, and a characterisation chosen that represents the causal nature of the interactions between the physical processes.

That is, in the present case the deformation induced by the mechanical forcing and the thermal gradients are characterised by the estimated response function values. The formalism is presented in general terms without specific properties being attributed to the functionals and their coefficients, the response function values. The formalism is then used to analyse specific data and for that case meaning is attributed to the functionals and their coefficients.

The formalism is developed simultaneously characterises the dynamical properties of the thermal and mechanical process and their mutual interactions. The linear and non linear response functions of the formalism are estimated directly from the experimental data [13,2,12]. The formalism is then applied to experimental data to analyse the thermoviscoelastic process in resin matrix composites under stochastic loading conditions. The analysis formalism is developed in terms of physical observables so that experimental data can be analysed and interpreted in terms of derived quantities such as creep compliance, relaxation modulus, elasticity, conductivity and other transport coefficients.

Linear elastic materials

Before details of the multidimensional convolution formalism are given it is of value to outline the underlying methodology with a simple example. The theory of linear perfectly elasticity materials is the cornerstone of the macroscopic treatment of solid mechanics. Such ideal materials deform instantaneously in response to an applied load and have the ability to store energy without dissipation, so that all of its stored energy can be recovered. For these materials Hook's law applies, so that the observed stress field if directly proportional to the applied strain and the behaviour is linear. On the other hand, a perfectly viscous fluid has the ability to dissipate energy but not to store it and the stress depends on the rate of change of the strain field.

Real materials have the capacity to both store and dissipate energy and the response to an applied force will be a fast deformation followed by a slow flow process. In a linear viscoelastic material the strain is directly proportional to the strain field and for a given constant applied stress the strain increases with time. This process is known as creep and when the applied force is reduced, or stopped, there is a period of creep recovery when the material experiences strain decay. This is known as relaxation. The phenomena of relaxation and creep are basic characteristics of viscoelastic materials. Any theory that successfully describes the behaviour of viscoelastic materials should be able to characterise the constitutive relationship between the observed deformation and the forces acting and the fluxes flowing. In addition, the theory should be able to characterise the storage and dissipative processes that simultaneously act in the material. The present work attempts to develop such a theory and to illustrate how the coefficients which characterise these relationships can be estimated from experimental data.

As an example of the basis of the methodology underlying the treatment of complex materials, consider a one dimensional linear elastic material that is submitted to a history of mechanical forces in the absence of other forces and thermodynamic fluxes. Then the local strain, $\{\varepsilon(t)\}$, can be expressed as a convolution between the observed stress, $\{\sigma(t)\}$, and the response function, $J_{\varepsilon\sigma}(\tau_1)$, which is called the creep compliance.

For a discrete process which possesses a local fading memory of duration μ, the convolution can be expressed as

$$\varepsilon(t) = \sum_{\tau_1=0}^{\mu} J_{\varepsilon\sigma}(\tau_1)\sigma(t-\tau_1) \tag{5}$$

where τ_1 denotes delay with respect to the time t. As it stands, it is ill posed because there are $(\mu+1)$ unknowns and only one equation. A set of $(\mu+1)$ equations need to be formed and solved for the response function values, $J_{\varepsilon\sigma}(\tau_1)$ which are the dynamic linear creep compliance values. If the local strain, $\{\varepsilon(t)\}$, and the observed stress, $\{\sigma(t)\}$, are drawn from stochastic processes, then equation (5) can be operated on to yield the moment equation

$$\langle\sigma(t-\varsigma_1)\varepsilon(t)\rangle = \sum_{\tau_1=0}^{\mu} J_{\varepsilon\sigma}(\tau_1)\,\langle\sigma(t-\varsigma_1)\sigma(t-\tau_1)\rangle \tag{6}$$

where $\langle\sigma(t-\varsigma_1)\varepsilon(t)\rangle$ and $\langle\sigma(t-\varsigma_1)\sigma(t-\tau_1)\rangle$ are the cross and auto moments between the strain field $\{\varepsilon(t)\}$ and the stress field $\{\sigma(t)\}$. That is, the average product of each side with the delayed value of stress, $\sigma(t-\tau_1)$, has been obtained using the operator $\langle\sigma(t-\tau_1)*\rangle$ for $0\leq\tau_1\leq\mu$ to give the $(\mu+1)$ equations required. In this form the equations, given by (6), can be readily solved with standard matrix methods.

Under steady state conditions the one dimensional linear stress strain relationship becomes

$$\varepsilon = \sigma \sum_{\tau_1=0}^{\mu} J_{\varepsilon\sigma}(\tau_1) \tag{7}$$

that is $\varepsilon = \sigma J^*_{\varepsilon\sigma}$, where $J^*_{\varepsilon\sigma}$ is the steady state creep compliance. If now the load is incremented by and amount $\Delta\sigma$ at some time t, then after μ units of time the strain field will be given by

$$\varepsilon^* = \sigma J^*_{\varepsilon\sigma} + \sum_{\tau_1=0}^{\mu} J_{\varepsilon\sigma}(\tau_1)\Delta\sigma(t-\tau_1)$$

After μ units of time have elapsed, the strain field will be given by

$$\varepsilon^* = \sigma J^*_{\varepsilon\sigma} + \Delta\sigma J^*_{\varepsilon\sigma} \tag{8}$$

which satisfies the Boltzman superposition principle for a linear viscoelastic process.

Thus, the general linear stress strain expression for an arbitrary sequence of loading forces is the convolution equation

$$\varepsilon(t) = \sum_{\tau_1=0}^{\mu} J_{\varepsilon\sigma}(\tau_1)\sigma(t-\tau_1) \tag{9}$$

The inverse form, which gives the linear one dimensional strain stress relationship is the convolution

$$\sigma(t) = \sum_{\tau_1=0}^{\mu} E_{\sigma\varepsilon}(\tau_1)\varepsilon(t-\tau_1) \tag{10}$$

Under steady state conditions this becomes

$$\sigma = \varepsilon \sum_{\tau_1=0}^{\mu} E_{\sigma\varepsilon}(\tau_1) = \varepsilon\, E^{*}_{\sigma\varepsilon} \tag{11}$$

which is Hook's law for the behaviour of a linear one dimensional elastic material.

Volterra functional series representation of thermoviscoelasticity

There are many physical processes where the form of the differential equations that govern the observed behaviour are not know. In such cases other representations must be used to describe the physical process. For example, in the fields of thermodynamics, fluid dynamics and elasticity use a truncated Taylor's series expansion representations have been used. When the Taylor's series expansion description is used, the physical laws that describe aspects of the observed behaviour can be based on the values of the coefficients of the ascending order terms in the expansion.

Constitutive equations are expressions which characterise the observed behaviour between forces and fluxes and conservation expressions relate a conserved variable to the constituent variables. For example, an observed thermodynamic flux may be characterised in terms of the observed thermodynamic forces and observed properties of the medium. The empirical coefficients of the Taylor's series expansion describe the steady state transport properties of the process. Such empirical coefficients represent the, so called, steady state gains of the independent variable to the dependent variables and cannot be derived from any fundamental theory, but are estimated directly from the experimental data.

If the thermodynamic flux at a given point in space and instant of time depends on a set of local thermodynamic forces, $\{F_i(t)\}$, then the thermodynamic flux can be written as a multidimensional function of the forces, with

$$f_k(t) = f_k(F_1, \ldots, F_I, t) \tag{12}$$

This multidimensional function can be written as an ascending multivariate Taylor's series expansion with

$$f_k(t) = \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{I} \cdots \sum_{i_n=i_{n-1}}^{I} L_{f_k i_1 \ldots i_n} \prod_{j=1}^{n} F_{i_j}(t) \tag{13}$$

where the lowest order transport coefficients are given by

$$L_{J_k i_1} = \left(\frac{\partial f_k}{\partial F_{i1}}\right)_0 \quad \text{and} \quad L_{J_k i_1 i_2} = \left(\frac{\partial f_k^2}{\partial F_{i_1} \partial F_{i_2}}\right)_0$$

In the linear approximation this expansion reduces to the well known Onsager relations.

Equally, the phenomena can be described by an expansion of functionals which characterises the physical process as a mapping between functional spaces. If there is a unique solution to equation then, formally at least, it will be the inverse mapping.

The inverse mapping associates each value of the independent physical variable to the finite history of a set of other dependent physical variables. The emphasis of the inverse problem approach is to identify the form of relationship between the observables and hence establish the laws govern the process. If the process has a finite memory, of duration μ, then the nonlinear nonequlibrium behaviour of a macroscopic thermodynamic process can be described as a functional expansion of physically observable causal time series quantities.

For example, the constitutive equations which describe thermodynamic processes, each thermodynamic flux, $f_k(t)$, can be described as a multidimensional convolution expansion in terms of the local thermodynamic forces acting, and defined as

$$f_k(t) = \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{I} \cdots \sum_{i_n=i_{n-1}}^{I} \int_{t-\mu}^{t} d\sigma_1 \cdots \int_{t-\mu}^{t} d\sigma_n \, J_{f_k F_{i_1} \ldots F_{i_n}}(i_1, \ldots, i_n, \sigma_1, \ldots, \sigma_n) \prod_{j=1}^{n} F_{i_j}(t - \sigma_j) \quad (14)$$

where N is the order of the system, where t denote time and where the σ_i's denotes time delay with respect to the time t.

A discrete approximation to the multidimensional convolution expansion can be defined as

$$f_k(t) = \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{I} \cdots \sum_{i_n=i_{n-1}}^{I} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu} J_{f_k F_{i_1} \ldots F_{i_n}}(i_1, \ldots, i_n, \sigma_1, \ldots, \sigma_n) \prod_{j=1}^{n} F_{i_j}(t - \sigma_j) \quad (15)$$

where N is the order of truncation the system, where t denote time, where I is the number of and where the σ_i's denotes time delay with respect to the time t.

On discretisation, the truncated Volterra series remains ill posed in the sense that there are too many unknown coefficients to solve for. Thus, the approximate method of discretisation used for the linear case cannot by themselves be used to solve the Volterra series. A tractable set of simultaneous equations with well behaved coefficients can be generated by taking time series moments of a suitably truncated Volterra series expansion.

The kernel function values, $J_{f_k F_{i_1} \ldots F_{i_n}}(i_1, \ldots, i_n, \sigma_1, \ldots, \sigma_n)$, characterise the behaviour of $f_k(t)$, in terms of the forces, $\{F_i(t)\}$. Integrating each kernel function yields the linear and nonlinear gain between the dependent and independent variables [2] , with

$$L_{f_k F_{i_1} \ldots F_{i_n}} = \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu} J_{f_k F_{i_1} \ldots F_{i_n}}(i_1, \ldots, i_n, \sigma_1, \ldots, \sigma_n) \quad (16)$$

That is, the integral of the kernel function values yields the steady state gain between the observables and are equivalent to the ascending order transport coefficients of the phenomena being characterised.

Equations (15) is ill posed, in the sense that there are many coefficients to solve for with only one equation. In addition, as thermodynamic processes are stochastic, in general, the equation is also ill conditioned because it has stochastic variables.

The conditioning can be improved statistical averaging and the use of operators allows a set of tractable equations with average variable values to be generated. Equation (15) is operated on with a series of averaging operators, one for each permutation of delayed applied forces $\left\langle F_i(t-\tau_j)*\right\rangle$, to give a moment hierarchy of the form

$$\left\langle \prod_{j=1}^{n} F_{r_j}(t-\tau_j) f_k(t) \right\rangle = \sum_{n=1}^{N} \frac{1}{n!} \sum_{r_1=1}^{K} \cdots \sum_{r_n=r_{n-1}}^{K} \sum_{\sigma_1=0}^{\mu} \cdots \sum_{\sigma_n=0}^{\mu} J_{f_k F_{r_1} \ldots F_{r_n}}(r_1,\ldots,r_n,\sigma_1,\ldots,\sigma_n) \left\langle \prod_{j=1}^{n} F_{r_j}(t-\tau_j) \prod_{i=1}^{n} F_{r_i}(t-\sigma_i) \right\rangle \tag{17}$$

where <*> denotes the averaging operation. The moment hierarchy can be rewritten in the obvious matrix form $\underline{C} = \underline{\underline{M}}\underline{h}$ where $\underline{\underline{M}}$ is a square matrix whose elements are the auto-moments of the applied forces $\left\{F_i(t-\tau_j)\right\}$, where $\underline{C}$ is a column vector whose elements are the cross moments between the thermodynamic flux, $\left\{f_k(t)\right\}$ and the applied forces $\left\{F_i(t)\right\}$ and where $\underline{h}$ is a column vector whose elements are the kernel function values of the mapping between $\left\{F_i(t)\right\}$ and $\left\{f_k(t)\right\}$.

If the matrix $\underline{\underline{M}}$ is non-singular then $\underline{h} = \left(\underline{\underline{M}}\right)^{-1}\underline{C}$ has a unique solution. If however $\underline{\underline{M}}$ is singular, then $\underline{\underline{M}}$ is rank deficient and some of its rows will be linearly dependent on the others. If the same relationship holds between the corresponding elements of the column vector $\underline{C}$, the solution will not be unique, indeed an infinity of solutions will exist. If this is not the case then the matrix expression is not consistent and there will not be any solution. Thus, in general, there may be a unique solution, an infinite number of solutions or no solution. However, given the construction of the moment values used in the moment hierarchy, the rows of $\underline{\underline{M}}$ will be linearly independent of each other, thus the matrix will usually be non-singular and have a unique solution. This will be true for many mixed stochastic and deterministic processes.

At this stage an observation can be made. The Volterra functional expansion is not, in general, tractable. However, the application of averaging operators has generated a tractable hierarchy of moment equations which are likely to have well behaved coefficients for many physical processes. The truncated Volterra expansion has been operated on in order to obtain a linear algebraic expression where the elements of the vector and matrix retain all of the information about the complex dynamical nonlinear process being studied.

There are exceptions to this however, for example,

1) when the data $\{x(t)\}$ are composed of delta function, a step function, and for any distribution which has a delta functional form of each member of the ascending order of auto moments, or their transformation, of the data $\{x(t)\}$

2) when the data $\{x(t)\}$at successive time series points are not causally linked

3) when the data $\{x(t)\}$ are composed of a very nonstationary sequence and

4) when the representation of the process is not of a closed form, i.e. some of the contributing variables are not measured or analysed.

Consider representing an observable, for example the components of a deformation gradient, $\{\varepsilon_{ij}(t)\}$, in terms of a set of physical observables, $\{\theta_{r_i}(t)\}$. That is, the deformation gradient, $\{\varepsilon_{ij}(t)\}$, is a function of the set of physical observables which may include the mechanical, thermal, electrical and other properties of the media and external forces that act on the media. These interactions with the deformation gradient, $\{\varepsilon_{ij}(t)\}$, can be represented as a multidimensional convolution expansion in terms of the other observables, $\{\theta_{r_i}(t)\}$, which in discrete form is defined as

$$\varepsilon_{ij}(t) = \sum_{n=1}^{N} \frac{1}{n!} \sum_{r_1=1}^{I} \cdots \sum_{r_n=r_{n-1}}^{I} \sum_{\tau_1=0}^{\mu} \cdots \sum_{\tau_n=0}^{\mu} J_{\varepsilon_{ij}\theta_{r_1}\ldots\theta_{r_n}}(r_1,\ldots,r_n,\tau_1,\ldots,\tau_n) \prod_{k=1}^{n} \theta_{r_k}(t-\tau_k) \tag{18}$$

where $\{\tau_i\}$ denote time delay, where N is the order of the expansion and where I is the number of observables used to describe the field $\sigma(t)$. The estimated response values $J_{\varepsilon_{ij}\theta_{r_1}\ldots\theta_{r_n}}(r_1,\ldots,r_n,\tau_1,\ldots,\tau_n)$ not only characterise the process in terms of the observed material properties and the forces acting and fluxes flowing, but they also represent the solutions to the equations which describe the process [13].

The moment hierarchy in the thermoviscoelastic case is given by

$$\langle \prod_{p=1}^{m} \theta_{s_p}(t - \varsigma_p)\varepsilon_{ij}(t)\rangle = \sum_{n=1}^{N}\frac{1}{n!}\sum_{r_1=1}^{I}\cdots\sum_{r_n=r_{n-1}}^{I}\sum_{\tau_1=0}^{\mu}\cdots\sum_{\tau_n=0}^{\mu} J_{\varepsilon_{ij}\theta_{r_1}\ldots\theta_{r_n}}(r_1,\ldots,r_n,\tau_1,\ldots,\tau_n)\,\langle \prod_{p=1}^{m}\theta_{s_p}(t - \varsigma_p)\prod_{k=1}^{n}\theta_{r_k}(t - \tau_k)\rangle \tag{19}$$

here the averaged values $\langle \prod_{p=1}^{m} \theta_{s_p}(t - \varsigma_p)\varepsilon_{ij}(t)\rangle$ denotes the ascending cross moments between the component of strain, $\{\varepsilon_{ij}(t)\}$, and the other observables; and where $\langle \prod_{p=1}^{m}\theta_{s_p}(t - \varsigma_p)\prod_{k=1}^{n}\theta_{r_k}(t - \tau_k)\rangle$ denote the ascending auto moments between the fluxes, physical properties and forces observed to be acting.

Facilities and provisional analysis of the thermoviscoelasticity experiments

The above formalism is currently being tested in a comprehensive fatigue test of experiments on coupons, plates and a composite turbine blade, the latter being subjected to a stochastic three dimensional loading. The local deformation is being determined by photoelastic and holographic methods. Two dimensional stress field point measurements are being made at many positions on the blade, the thermal gradient within the solid is being measured with thermocouples. A thermal camera is being used to identify regions of critical stress and temperature gradient. Information regarding the state of the microstructure throughout the fatigue test is being determined from acoustic emission measurements. These experiments facilitate a detailed analysis of the macroscopic thermoviscoelastic process in a complex composite material under realistic forcing conditions and provide a relationship between the microstructural damage to the macroscopic response through the life cycle of the composite material. For example, during the fatigue test the blade is clamped at the root, the longitudinal axis horizontal and chord vertical. The load is applied with two actuators, acting in a plane which is perpendicular to the longitudinal axis of the blade. A time varying combination of flapwise and edgewise loading is applied with a maximum load of 75 kN and minimum load of -75 kN. The test has three phases, an initial static load test, 5 million cycles at 1 Hz below the maximum loading and finally about 1 million cycles at 1 Hz at maximum loading until failure. The results from that experiment will be reported in detail when the analysis has been completed early in 1995.

Prior to the full blade test, a series of fatigue tests on 1 dimensional composite coupons and two dimensional blade sections were performed. The samples used in that sequence were glass/polyester coupons which had been manufactured with the lay-up of typical wind turbine blade material were drilled with various depths of small hole (diameter 6mm).

Stress concentrators, in the form if bored holes, were introduced into the coupons, with four hole depths in the range of 50% (5mm from viewed surface) to 95% (0.5mm from viewed surface) being tested in fatigue at frequencies between 0.1 Hz and 10 Hz. The coupons were loaded in uniaxial fatigue with a maximum load of 27 kN and an R-ratio of 0.1. A sequence of thermograms was recorded for the 95% hole, a hot spot due to the influence of the hole became apparent almost immediately on applying the cyclic load.

The provisional analysis has considered the observed stress as a mixed linear and nonlinear dynamical function of the applied load and temperature within the solid. Time series readings were collected each 40ms throughout each experiment, the time interval for data collection being determined by the response time of the sensors used. A total of some 1000 time series points for each sensor were collected. Of these some 400, in sample, points were used to estimate the response factor values of the process and 500, out of sample, points were used to compare with the values of the strain field values predicted using the response factor values estimated in sample. This enabled the accuracy and time invariant nature of the estimated response function values to be determined. These response function values were then used to predict the behaviour of the sample under a different loading regime, thus testing the nature of the solutions determined by the moment hierarchy method.

The time series values of the stress field were considered as 1) a linear function of the applied load and the local temperature gradient 2) a mixed linear and nonlinear function of the applied load and 3) a mixed linear and nonlinear function of the applied load and temperature within the solid. In each case the properties of the process were characterised with 400 data points. The response functions estimated with these 400 points were then used to predict the future, out of sample, behaviour of the stress field for 500 points.

These predicted values of stress, $\{\varepsilon(t)\}$, were then compared statistically with the observed values, $\{\varepsilon_p(t)\}$, it should be stressed that during the prediction phase no use was made of the observed stress values. It should also be noted that the mean value had to be subtracted from the data in order to perform the linear analysis, no such modification was made in the nonlinear analyses. This provides a quantitative measure of the quality of the response function characterisation of the thermoviscoelastic process. The accuracy of the predicting ability was determined by comparing the root mean square differences between the actual, $\{\varepsilon(t)\}$, and predicted $\{\varepsilon_p(t)\}$, time series sequences. An example of the prediction together with the actual values is shown in figure 1. These sample statistics are presented in table 1 and below. This provides a sensitive measure of the quality of the response function characterisation of the thermoviscoelastic process.

Table 1: Root mean square difference between the predicted and actual stress values

Mean applied load	Linear vector analysis	Nonlinear univariate analysis	Nonlinear vector analysis
6.0 kN @ 0.4 Hz	$39. * 10^{-3}$	$6.7 * 10^{-3}$	$13. * 10^{-3}$
8.3 kN @ 0.4 Hz	$38. * 10^{-3}$	$4.3 * 10^{-3}$	$7.7 * 10^{-3}$
12.1 kN @ 0.4 Hz	$40. * 10^{-3}$	$4.0 * 10^{-3}$	$5.6 * 10^{-3}$

Table 2: Fractional mean difference between the predicted and actual stress values

Mean applied load	Linear vector analysis	Nonlinear univariate analysis	Nonlinear vector analysis
6.0 kN @ 0.4 Hz	0.18	0.018	0.025
8.3 kN @ 0.4 Hz	0.025	0.0018	0.0075
12.1 kN @ 0.4 Hz	0.062	0.0080	0.0085

In linear analysis without correcting for the mean value (which itself may be linear or nonlinear) the null hypothesis that the thermoviscoelastic process is linear was not accepted by the sample statistics. However, the sample statistics for both of the nonlinear and mean corrected linear analyses the values of the test statistics for the differences between the measured, $\{\varepsilon(t)\}$, and predicted, $\{\varepsilon_p(t)\}$, output stress field for both modelled and predicted data lay within the acceptance region; thus each representation accurately characterises the observed behaviour of the stress.

Integrating the estimated response function values yields an estimate of the steady state transport coefficients. These transport coefficient values can be compared with the transport coefficient as estimated from the measured mean applied load, stress and cross sectional area of the sample. The estimated creep compliance at the three mean applied fatigue loads are 1.44±0.20, 1.46± 0.21 and 1.46±0.22 respectively. Until the present work, no simultaneous estimates of creep compliance and the thermoviscoelastic transport coefficient have been made, for this reason only estimates of the mechanical transport coefficient are presented here. However, on the basis of the current provisional analysis, the thermal gradient thermodynamic force is approximately the same size the mechanical thermodynamic force, both containing elements of dynamical potential storage with the thermal force also providing a dissipation mechanism.

Table 3: Area under the response functions associated with the mechanical force

Mean applied load	Linear analysis $\sum_{\sigma_1=0}^{\mu} J_{f_k F_{i_1}}(i_1,\sigma_1)$	Linear term univariate analysis $\sum_{\sigma_1=0}^{\mu} J_{f_k F_{i_1}}(i_1,\sigma_1)$	Nonlinear term univariate analysis $\sum_{\sigma_1=0}^{*}\sum_{\sigma_2=0}^{*} J_{f_k F_{i_1} F_{i_2}}(i^1,i^2,\sigma^1,\sigma^2)$	Linear term vector analysis $\sum_{\sigma_1=0}^{\mu} J_{f_k F_{i_1}}(i_1,\sigma_1)$	Nonlinear term vector analysis $\sum_{\sigma_1=0}^{*}\sum_{\sigma_2=0}^{*} J_{f_k F_{i_1} F_{i_2}}(i^1,i^2,\sigma^1,\sigma^2)$
6.0 kN @ 0.4 Hz	$2.59*10^{-4}$ $\pm 0.33*10^{-4}$	$4.61*10^{-4}$ $\pm 0.45*10^{-4}$	$-1.70*10^{-5}$ $\pm 0.38*10^{-5}$	$4.81*10^{-4}$ $\pm 0.49*10^{-4}$	$-1.18*10^{-5}$ $\pm 0.32*10^{-5}$
8.3 kN @ 0.4 Hz	$2.69*10^{-4}$ $\pm 0.33*10^{-4}$	$4.62*10^{-4}$ $\pm 0.47*10^{-4}$	$-1.16*10^{-5}$ $\pm 0.30*10^{-5}$	$4.97*10^{-4}$ $\pm 0.50*10^{-4}$	$-1.25*10^{-5}$ $\pm 0.32*10^{-5}$
12.1 kN @ 0.4 Hz	$2.85*10^{-4}$ $\pm 0.33*10^{-4}$	$4.05*10^{-4}$ $\pm 0.43*10^{-4}$	$-0.52*10^{-5}$ $\pm 0.25*10^{-5}$	$4.21*10^{-4}$ $\pm 0.51*10^{-4}$	$-0.56*10^{-5}$ $\pm 0.27*10^{-5}$

Table 4: Effective linear and nonlinear creep compliance determined from the response function values

Mean applied load	Linear analysis	Linear term univariate analysis	Nonlinear term univariate analysis	Linear term vector analysis	Nonlinear term vector analysis
6.0 kN @ 0.4 Hz	1.04 ± 0.13	1.85 ± 0.18	-0.068 ± 0.015	1.93 ± 0.19	-0.047 ± 0.013
8.3 kN @ 0.4 Hz	1.08 ± 0.13	1.85 ± 0.19	-0.047 ± 0.012	1.98 ± 0.20	-0.054 ± 0.013
12.1 kN @ 0.4 Hz	1.14 ± 0.14	1.62 ± 0.17	-0.021 ± 0.010	1.68 ± 0.20	-0.022 ± 0.011

The sample statistics for the differences between the measured, $\{\varepsilon(t)\}$, and predicted, $\{\varepsilon_p(t)\}$, output stress field for both modelled and predicted data lay well within the acceptance region for the linear and mixed linear with nonlinear representations. Thus the mean corrected linear representation and each of the mixed linear and nonlinear representations can accurately characterise the observed behaviour of the stress. The linear representation where the mean level was included in the analysis could not accurately represent the observed behaviour. However, in the experiments currently underway a more comprehensive set of observables are being measured in a more accurate way, in particular, more accurate measurements of the thermal gradient within the solid body. The results of that work will be presented early in 1995.

Conclusions

The results presented in this paper can be summarised as follows: a hierarchy of moment equations of the Volterra series can used to study nonlinear thermoviscoelastic process in complex materials. The nature of one dimensional thermoviscoelasticity was considered. Linear and mixed linear and nonlinear local constitutive representations were used to characterise the thermoviscoelastic process. The results of the linear and mixed linear and nonlinear analysis of the one dimensional thermoviscoelastic data show that the process is, within the experimental uncertainties, linear with a weak nonlinear component. The analysis has been demonstrated that the moment hierarchy can extract and isolate linear and ascending order nonlinear response functions when the input data are drawn from a stochastic process.

The moment hierarchy was used to analyse the properties a composite solid under a range of applied loads. The first and second order response functions were estimated from the time series data collected from the applied force, strain gauge and temperature gradient values. These estimated response functions were then used to predict the out of sample stress field values. These predictions demonstrated that the response functions provided a good, locally time invariant, representation of the thermoviscoelastic process. The range of applied loads span a significant region of the phase space for the specimen and the estimated response function values and steady state transport coefficients remained constant over this range. The final results from the present project will be reported early in 1995.

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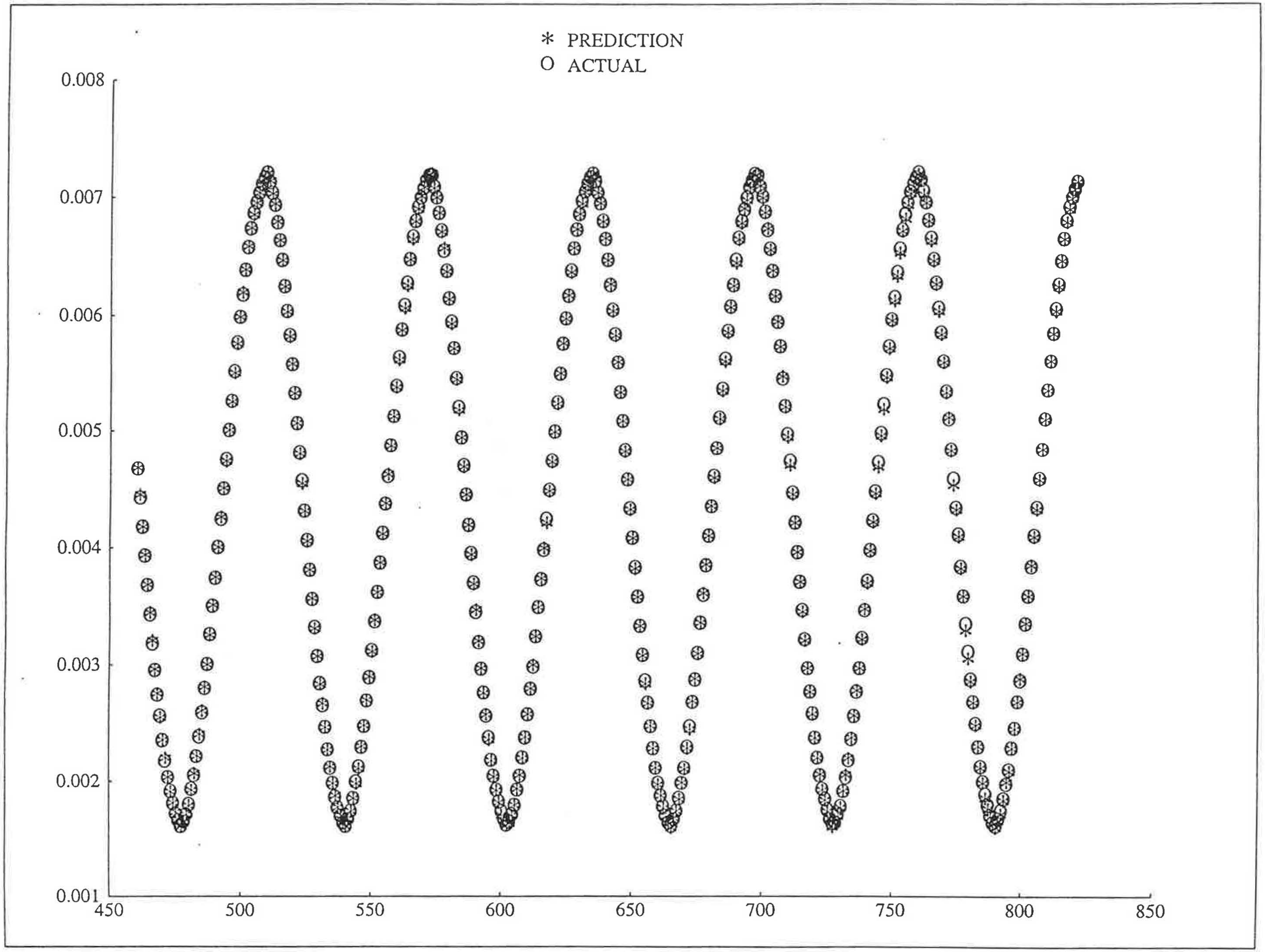

* PREDICTION
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850